

Gravitational Anomaly and Transport

Eugenio Megías*

In Collaboration with:

K. Landsteiner, L. Melgar, F. Peña-Benítez

**Instituto de Física Teórica, IFT-CSIC/UAM, Universidad Autónoma de Madrid, Spain
and Department de Física and IFAE, Universitat Autònoma de Barcelona, Spain.**
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E. Megías et al. Phys. Rev. Lett. 107:021601(2011),
JHEP 1109:121(2011) and arXiv:1110.3615[hep-ph].

Issues

- 1 Anomalous Effects and Kubo Formulas
 - The Chiral Magnetic Effect
 - Kubo Formulas
- 2 Hydrodynamics of Relativistic Fluids
 - Weak coupling
 - Strong coupling
 - Fluid/Gravity Correspondence
- 3 Chemical Potentials for Anomalous Symmetries

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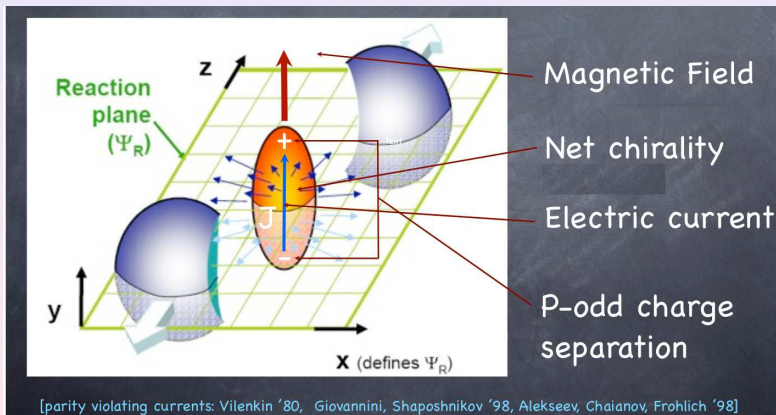
The Chiral Magnetic Effect

Please click below for an introductory movie

RHIC's hot quark soup

The Chiral Magnetic Effect

[Kharzeev, McLerran, Warringa '07]



Strong Magnetic field induces a P-odd charge separation \implies
 \implies Electric current: $\vec{J} = \sigma^B \vec{B}$.

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Kubo Formulas

- Chiral Magnetic Conductivity [**Kharzeev, Warringa '09**]

$$\begin{aligned} \vec{J} &= \sigma^B \vec{B}, \\ \text{In momentum space} &\implies J_i = \sigma^B \epsilon_{ijk} i p_j A_k. \end{aligned}$$

- Kubo formula for a general symmetry group (**A. Gynther et al., JHEP1102(2011)110.**)

$$\begin{aligned} [T_A, T_B] &= if_{AB}^C T_C, \\ (\sigma^B)_{AB} &= \lim_{p_j \rightarrow 0} \sum_{i,k} \frac{i}{2p_j} \epsilon_{ijk} \langle J_A^i J_B^k \rangle \Big|_{\omega=0}. \end{aligned}$$

Retarded correlation function of currents.

Kubo Formulas

- Chiral fermions

$$J_i^A = \sum_{f,g=1}^N (T^A)^g_f \bar{\Psi}_g \gamma_i \mathcal{P}_+ \Psi^f, \quad \mathcal{P}_+ = \frac{1}{2}(1 + \gamma_5),$$

- Chemical potentials and generators in the Cartan subalgebra

$$\mu^f = \sum_A q_A^f \mu_A, \quad H_A = q_A^f \delta^f_g.$$

Chemical potentials break group $G \rightarrow \hat{G}$ fulfilling $T_{A g}^f \mu^g = \mu^f T_{A g}^f$.
 Only currents that lie in \hat{G} are conserved and participate in the hydrodynamics.

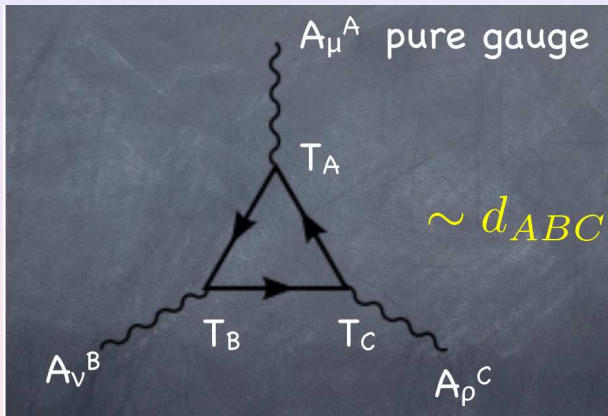
- 1-loop calculation [Kharzeev, Warringa '09], [Megías et al. '11]

$$(\sigma^B)_{AB} = \frac{1}{8\pi^2} \sum_C \text{tr}(T_A \{T_B, H_C\}) \mu^C = \frac{1}{4\pi^2} \underbrace{d_{ABC}}_{\text{Anomaly Coeff.}} \mu^C.$$

Chiral Magnetic Conductivity induced by the Chiral Anomaly.

Kubo Formulas

Triangle Diagram



Energy Transport and Chiral Vortical Effect

[I. Amado, K. Landsteiner, F. Pena-Benitez, JHEP1105 '11]

- finite density: charge transport \implies energy transport

$$\delta T_{0i} = \mu \delta J_i \stackrel{\delta J_i = \delta \sigma^B B_i}{=} \mu \delta \sigma^B B_i.$$

- Energy flux sourced by magnetic fields

$$\sigma^{\mathcal{V}} = \frac{i}{2\rho_j} \sum_{i,k} \epsilon^{ijk} \langle J_i T_{0k} \rangle \Big|_{\omega=0} = \int \mu d\sigma^B + \text{const}$$

- $T_{\mu\nu}$ sourced by metric

$$ds^2 = -(1 - 2\Phi)dt^2 + 2\vec{A}_g dt d\vec{x} + (1 + 2\Phi)d\vec{x}^2,$$

- A_g "gravitomagnetic field" \implies chiral gravitomagnetic effect

$$\vec{J} = \sigma^{\mathcal{V}} \vec{B}_g, \quad B_g^i = \epsilon^{ijk} \partial_j A_{g,k}.$$

- Chiral vortical effect: local fluid velocities

$$u^\mu = (1, 0, 0, 0), \quad u_\mu = (-1, \vec{A}_g),$$

$$J^i = \sigma^{\mathcal{V}} \epsilon^{ijk} \partial_j u_k, \quad \sigma^{\mathcal{V}} \equiv \text{Vorticity coeff.}$$

Hydrodynamics of Relativistic Fluids

[Son,Surowka], [Eling,Neiman,Oz], [Erdmenger et al.], [Banerjee et al.], [Loganayagam],
 [Kharzeev, Yee], [Sadovyyev et al.]

$$T^{\mu\nu} = \underbrace{(\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu}}_{\text{Ideal Hydro}} + \underbrace{\tau^{\mu\nu}}_{\text{Dissipative}},$$

$$J^\mu = \underbrace{nu^\mu}_{\text{Ideal Hydro}} + \underbrace{\nu^\mu}_{\text{Dissipative}}.$$

- Landau frame: $T^{0i} = (\epsilon + P)u^i$

$$\tau^{\mu\nu} = -\eta P^{\mu\alpha} P^{\nu\beta} \left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta} \nabla^\lambda u_\lambda \right) - \zeta P^{\mu\nu} \nabla^\alpha u_\alpha + \dots$$

$$\nu^\mu = -\sigma T P^{\mu\nu} \nabla_\nu \left(\frac{\mu}{T} \right) + \sigma E^\mu + \xi^B B^\mu + \xi^V \omega^\mu + \dots$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, and **vorticity**: $\omega^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\lambda} u_\nu \nabla_\rho u_\lambda$.

Hydrodynamics of Relativistic Fluids

- Laboratory rest frame:

$$\sigma^B = \lim_{k_c \rightarrow 0} \frac{i}{2k_c} \sum_{a,b} \epsilon_{abc} \langle J^a J^b \rangle |_{\omega=0},$$

$$\sigma^V = \lim_{k_c \rightarrow 0} \frac{i}{2k_c} \sum_{a,b} \epsilon_{abc} \langle J^a T^{0b} \rangle |_{\omega=0}.$$

- Landau frame:

ξ 's coefficients are different from σ 's, but they are related to them:

$$\xi^B = \lim_{k_c \rightarrow 0} \frac{i}{2k_c} \sum_{a,b} \epsilon_{abc} \left(\langle J^a J^b \rangle - \frac{n}{\epsilon + P} \langle T^{0a} J^b \rangle \right) |_{\omega=0},$$

$$\xi^V = \lim_{k_c \rightarrow 0} \frac{i}{2k_c} \sum_{a,b} \epsilon_{nkl} \left(\langle J^a T^{0b} \rangle - \frac{n}{\epsilon + P} \langle T^{0a} T^{0b} \rangle \right) |_{\omega=0}.$$

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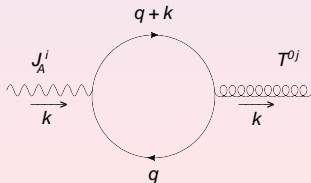
Field Theory Computation

- As before: general symmetry group

$$T^{0i} = \frac{i}{2} \sum_{f=1}^N \bar{\Psi}_f (\gamma^0 \partial^i + \gamma^i \partial^0) \mathcal{P}_+ \Psi^f.$$

- Chiral Vortical Conductivity:** defined from the retarded correlation function of $J_A^i(x)$ and $T^{0j}(x')$, i.e.

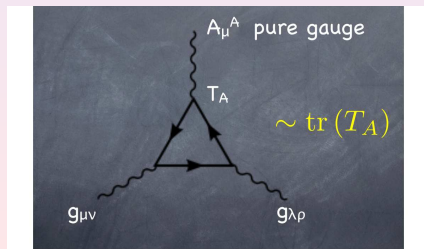
$$G_A^{\nu}(\mathbf{x} - \mathbf{x}') = \frac{i}{2} \epsilon_{ijn} \theta(t - t') \langle [J_A^i(\mathbf{x}), T^{0j}(\mathbf{x}')] \rangle.$$



Field Theory Computation

- 1 loop calculation [Landsteiner, Megías, Pena-Benitez, PRL107 '11]:

$$\begin{aligned}
 (\sigma^\nu)_A &= \lim_{k_n \rightarrow 0} \sum_{ij} \epsilon_{ijn} \frac{-i}{2k_n} \langle J_A^i T^{0j} \rangle |_{\omega=0} = \frac{1}{8\pi^2} \sum_{f=1}^N (T_A)^f_f \left[(\mu^f)^2 + \frac{\pi^2}{3} T^2 \right] \\
 &= \frac{1}{8\pi^2} \sum_{B,C} d_{ABC} \mu^B \mu^C + \underbrace{\frac{T^2}{24} \text{tr}(T_A)}_{\text{Gravitational Anomaly}}
 \end{aligned}$$



Field Theory Computation

- Anomaly:

$$\nabla_{\mu} \mathbf{J}_A^{\mu} = \epsilon^{\mu\nu\rho\lambda} \left(\frac{d_{ABC}}{32\pi^2} F_{\mu\nu}^B F_{\rho\lambda}^C + \frac{b_A}{768\pi^2} R^{\alpha}{}_{\beta\mu\nu} R^{\beta}{}_{\alpha\rho\lambda} \right),$$

$$d_{ABC} = \frac{1}{2} [\text{tr}(T_A \{T_B, T_C\})_R - \text{tr}(T_A \{T_B, T_C\})_L],$$

$$b_A = \text{tr}(T_A)_R - \text{tr}(T_A)_L.$$

- Special case: vector and axial current for one Dirac fermion
 - Vector vortical conductivity $\sigma_V^{\mathcal{V}} = \frac{\mu\mu_A}{2\pi^2}$ (no grav. anom.)
 - Axial vortical conductivity

$$\sigma_A^{\mathcal{V}} = \frac{\mu^2 + \mu_A^2}{4\pi^2} + \frac{T^2}{12}$$

See also [Vilenkin '80].

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Holography

[K. Landsteiner, E. Megías, L. Melgar, F. Pena-Benitez, 1107.0368[hep-th].]

- Mixed gauge-gravitational Chern Simons term

$$S = \frac{1}{16\pi G} \int d^5x \sqrt{-g} \left[R + 2\Lambda - \frac{1}{4} F_{MN} F^{MN} + \epsilon^{MNPQR} A_M \left(\frac{\kappa}{3} F_{NP} F_{QR} + \lambda R^A{}_{BNP} R^B{}_{AQR} \right) \right] + S_{GH} + S_{CSK}$$

- To reproduce gravitational anomaly at general hypersurface (not only $r \rightarrow \infty$)

$$S_{CSK} = -\frac{1}{2\pi G} \int_{\partial} d^4x \sqrt{-h} \lambda n_M \epsilon^{MNPQR} A_N K_{PL} D_Q K_R^L.$$

- Equations of motion

$$G_{MN} - \Lambda g_{MN} = \frac{1}{2} F_{ML} F_N{}^L - \frac{1}{8} F^2 g_{MN} + 2\lambda \epsilon_{LPQR} (M \nabla_B (F^{PL} R^B{}_N)^{QR})$$

$$\nabla_N F^{NM} = -\epsilon^{MNPQR} \left(\kappa F_{NP} F_{QR} + \lambda R^A{}_{BNP} R^B{}_{AQR} \right)$$

Holography

- Current:

$$J^\mu = -\frac{\sqrt{-\gamma}}{16\pi G} \left[F^{r\mu} + \frac{4}{3} \kappa \epsilon^{\mu\nu\rho\lambda} A_\nu F_{\rho\lambda} \right]_{\partial}.$$

- On-shell we recover the anomaly

$$D_\mu J^\mu = -\frac{1}{16\pi G} \epsilon^{\mu\nu\rho\lambda} \left(\kappa F_{\mu\nu} F_{\rho\lambda} + \lambda R_{(4)\beta\mu\nu} R_{(4)\alpha\rho\lambda}^\beta \right)$$

$$\implies -\frac{\kappa}{16\pi G} = \frac{1}{32\pi^2}, \quad -\frac{\lambda}{16\pi G} = \frac{1}{768\pi^2}.$$

- Energy-momentum tensor

$$T^{\mu\nu} = \frac{\sqrt{-\gamma}}{8\pi G} \left[K_{(4)}^{\mu\nu} - K_{(4)} \gamma^{\mu\nu} + 4\lambda \epsilon^{(\mu\rho\lambda\alpha} \left(\frac{1}{2} F_{\rho\lambda} R_\alpha^\nu \right) + D_\beta (A_\rho R^{\beta\nu})_{\lambda\alpha} \right]_{\partial}.$$

- Energy-momentum (non)-conservation relation (see also [Neiman,Oz '11])

$$D_\nu T^{\nu\mu} = F^{\mu\nu} J_\nu - A^\mu D_\alpha J^\alpha.$$

Holographic Renormalization

[Martelli, Mueck '03], [Papadimitriou, Skenderis '04], [Clark, Love, Veldhuis '10],
 [Landsteiner, Megías, Melgar, Pena-Benitez '11]

- Consider gauge $A_r = 0$, $ds^2 = dr^2 + \gamma_{\mu\nu} dx^\mu dx^\nu$.
- Off shell action in terms of transverse quantities

$$S^0 = \frac{1}{16\pi G} \int d^5x \sqrt{-\gamma} \left[R_{(4)} + 2\Lambda + K^2 - K_{\mu\nu} K^{\mu\nu} - \frac{1}{2} E_\mu E^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right],$$

$$S_{CS}^1 = -\frac{\kappa}{12\pi G} \int d^5x \sqrt{-\gamma} \epsilon^{\mu\nu\rho\lambda} A_\mu E_\nu F_{\rho\lambda},$$

$$S_{CS}^2 = -\frac{8\lambda}{16\pi G} \int d^5x \sqrt{-\gamma} \epsilon^{\mu\nu\rho\lambda} \left[A_\mu R_{(4)\beta\rho\lambda}^\alpha D_\alpha K_\nu^\beta + E_\mu K_{\nu\alpha} D_\rho K_\lambda^\alpha + \frac{1}{2} F_{\mu\rho} K_{\nu\alpha} \partial_r K_\lambda^\alpha \right]$$

where $K_{\mu\nu} = \frac{1}{2} \partial_r \gamma_{\mu\nu}$.

- Problem: The term $\frac{1}{2} F_{\mu\rho} K_{\nu\alpha} \partial_r K_\lambda^\alpha \sim \partial_r^3 \gamma_\lambda^\alpha \implies$
 \implies Field equations will be generically third order in r -derivatives
 \implies We need to fix $\gamma_{\mu\nu}$, $K_{\mu\nu}$ and also $\partial_r K_{\mu\nu}$!!!

Holographic Renormalization

- Boundary condition \implies metric has an asymptotically AdS expansion of the form:

$$\gamma_{\mu\nu} = e^{2r} \left(g_{\mu\nu}^{(0)} + e^{-2r} g_{\mu\nu}^{(2)} + e^{-4r} (g_{\mu\nu}^{(4)} + 2r \tilde{g}_{\mu\nu}^{(4)}) + \dots \right), \quad r \rightarrow \infty.$$

Then the problematic term $\frac{1}{2} F_{\mu\rho} K_{\nu\alpha} \partial_r K_\lambda^\alpha \xrightarrow{r \rightarrow \infty} 0$, and the boundary action depends only on $\gamma_{\mu\nu}$ and not on $\partial_r \gamma_{\mu\nu}$ ☺

- Expansion in eigenfunctions of the dilatation operator:

$$\delta_D = 2 \int d^4x \gamma_{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}},$$

then

$$\begin{aligned} K_\nu^\mu &= K_{(0)\nu}^\mu + K_{(2)\nu}^\mu + K_{(4)\nu}^\mu + \tilde{K}_{(4)\nu}^\mu \log e^{-2r} + \dots, & \delta_D K_{(n)} &= -n K_{(n)} \\ A_\mu &= A_{(0)\mu} + A_{(2)\mu} + \tilde{A}_{(2)\mu} \log e^{-2r} + \dots, & \delta_D A_{(n)\mu} &= -n A_{(n)\mu} \end{aligned}$$

Holographic Renormalization

- Divergent terms in the action are:

$$S_{div} = S_{(0)} + S_{(2)} + \tilde{S}_{(4)}.$$

- The counterterm for the on-shell action is:

$$S_{ct} = -\frac{(d-1)}{8\pi G} \int_{\partial} d^4x \sqrt{-\gamma} \left[1 + \frac{1}{(d-2)} P - \frac{1}{4(d-1)} \left(P_{\nu}^{\mu} P_{\mu}^{\nu} - P^2 - \frac{1}{4} F_{(0)\mu\nu} F_{(0)}^{\mu\nu} \right) \log e^{-2r} \right],$$

where $P = \frac{R_{(4)}}{2(d-1)}$, $P_{\nu}^{\mu} = \frac{1}{2} \left[R_{(4)\nu}^{\mu} - P \delta_{\nu}^{\mu} \right]$.

Gauge-gravitational Chern Simons term does not induce new divergences, and the renormalization is not modified by it ☺

(See also Clark, Love & Veldhuis '10).

Transport coefficients

- Background: charged AdS black hole

$$ds^2 = \frac{r^2}{L^2} (-f(r) dt^2 + d\vec{x}^2) + \frac{L^2}{r^2 f(r)} dr^2, \quad f(r) = 1 - \frac{ML^2}{r^4} + \frac{Q^2 L^2}{r^6},$$

$$A_{(0)} = \phi(r) dt = \left(\alpha - \frac{\mu r_h^2}{r^2} \right) dt.$$

- Mass, Charge and Temperature of the Reissner-Nordström black hole

$$M = \frac{r_h^4}{L^2} + \frac{Q^2}{r_h^2}, \quad Q = \frac{\mu r_h^2}{\sqrt{3}}, \quad T = \frac{r_h^2}{4\pi L^2} f'(r_h) = \frac{(2 r_h^2 M - 3 Q^2)}{2\pi r_h^5}$$

- Fluctuations in shear sector

$$A_M = A_M^{(0)} + \epsilon a_M, \quad g_{MN} = g_{MN}^{(0)} + \epsilon h_{MN}.$$

Gauge field: a_x, a_z .

Metric: $h_t^x, h_t^z, h_y^x, h_y^z$.

Transport coefficients

[Kaminski, Landsteiner, Mas, Shock, Tarrío, JHEP02 '10]

- Second order action on-shell (expansion $\mathcal{O}(\epsilon^2)$)

$$\delta S_{ren}^{(2)} = \int \frac{d^d k}{(2\pi)^d} \left\{ \Phi'_{-k} \mathcal{A}_{IJ} \Phi_k'^J + \Phi'_{-k} \mathcal{B}_{IJ} \Phi_k^J \right\} \Big|_{r \rightarrow \infty},$$

where $\Phi_k^I \equiv \left(\frac{a_x}{\mu}, h_t^x, \frac{a_z}{\mu}, h_t^z \right)$, and

$$\mathcal{A} = \frac{r_h^4}{16\pi GL^5} \text{Diag} \left(-3af, \frac{1}{u}, -3af, \frac{1}{u} \right), \quad u = \frac{r^2}{r^2}, \quad a = \frac{\mu^2 L^2}{3r_h^2},$$

$$\mathcal{B}_{AdS+\partial} = \frac{r_h^4}{16\pi GL^5} \begin{pmatrix} 0 & -3a & \frac{4\kappa i k \mu^2 \phi L^5}{3r_h^4} & 0 \\ 0 & -\frac{3}{u^2} & 0 & 0 \\ \frac{-4\kappa i k \mu^2 \phi L^5}{3r_h^4} & 0 & 0 & -3a \\ 0 & 0 & 0 & -\frac{3}{u^2} \end{pmatrix},$$

$$\mathcal{B}_{CT} = \frac{r_h^4}{16\pi GL^5} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{3}{u^2 \sqrt{f}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{u^2 \sqrt{f}} \end{pmatrix}, \quad \mathcal{B} = \mathcal{B}_{AdS+\partial} + \mathcal{B}_{CT}.$$

Transport coefficients

- Bulk solutions ($\varphi_k^J \equiv$ boundary values)

$$\Phi_k^I(r) = F^I{}_J(k, r) \varphi_k^J, \quad F^I{}_J(k, r) \xrightarrow{r \rightarrow \infty} 1.$$

- Retarded Green functions

[Son, Starinets '02], [Herzog, Son '03], [Kaminski et al '10], [Amado et al '09]

$$G_{IJ}(k) = -2 \lim_{r \rightarrow \infty} \left(\mathcal{A}_{IM}(F^M{}_J(k, r))' + \mathcal{B}_{IJ} \right).$$

- Correlators are

$$\langle JJ \rangle = -ip_z \left(\frac{\kappa}{2\pi G} \mu - \frac{\kappa}{6\pi G} \alpha \right) \implies \sigma^B = \frac{\mu}{4\pi^2}$$

$$\langle JT \rangle = -ip_z \left(\frac{\kappa}{4\pi G} \mu^2 + \frac{2\pi\lambda}{G} T^2 \right) \implies \sigma^V = \frac{\mu^2}{8\pi^2} + \frac{T^2}{24}$$

$$\langle TT \rangle = -ip_z \left(\frac{\kappa}{6\pi G} \mu^3 + \frac{4\pi\lambda}{G} \mu T^2 \right) \implies \sigma^{\epsilon, V} = \frac{\mu^3}{12\pi^2} + \frac{\mu T^2}{12}$$

Coefficients consistent with weak coupling.

No T^3 terms! (CPT invariance)

See also [Neiman, Oz], [Loganayagam].

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Fluid/Gravity Correspondence

[Erdmenger et al.], [Bhattacharyya et al.], [Banerjee et al.].

- Boosted black branes:

$$ds^2 = -r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu - 2u_\mu dx^\mu dr$$

where

$$f(r) = 1 - \frac{M}{r^4} + \frac{Q^2}{r^6}, \quad P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu,$$

$$A_r = 0, \quad A_\mu = -\frac{\sqrt{3}Q}{r^2} u_\mu.$$

- $u_\mu(x)$, $M(x)$, $Q(x)$ are slowly varying. This means $\partial X \ll X/\ell_{mfp}$
- Then one can compute the fields in a derivative expansion:

$$g_{AB} = g_{AB}^{(0)} + g_{AB}^{(1)} + g_{AB}^{(2)} + \dots$$

$$A_M = A_M^{(0)} + A_M^{(1)} + A_M^{(2)} + \dots$$

Fluid/Gravity Correspondence

- Decompose the fields into **scalar**, **vector** and **tensor sectors**:

$$\begin{aligned}
 ds^2 &= r^2 k(r) u_\mu u_\nu dx^\mu dx^\nu + r^2 h(r) P_{\mu\nu} dx^\mu dx^\nu + r^2 \pi_{\mu\nu}(r) dx^\mu dx^\nu \\
 &\quad + r^2 j_\sigma(r) (P_\mu^\sigma u_\nu + P_\nu^\sigma u_\mu) dx^\mu dx^\nu - 2S(r) u_\mu dx^\mu dr, \\
 A_r &= 0, \quad A_\mu = a_\nu(r) P_\mu^\nu + c(r) u_\mu.
 \end{aligned}$$

- Compute $k(r)$, $h(r)$, ... in a derivative expansion of $Q(x)$, $M(x)$ and $u_\mu(x)$.
- Zeroth order is trivial:

$$k^{(0)}(r) = -f(r), \quad h^{(0)}(r) = 1, \quad S^{(0)}(r) = 1, \quad c^{(0)}(r) = -\frac{\sqrt{3}Q}{r^2}$$

and $= 0$ the rest.

Fluid/Gravity Correspondence

- At n-th order the Einstein-Maxwell equations write (vector sector)

$$\mathbf{J}_i^{(n)}(r) = \partial_r \left(r^5 \partial_r j_i^{(n)}(r) + 2\sqrt{3} Q_0 a_i^{(n)}(r) \right),$$

$$\underbrace{\mathbf{A}_i^{(n)}(r)}_{\text{Sources}} = \underbrace{\partial_r \left(r^3 f(r) \partial_r a_i^{(n)}(r) + 2\sqrt{3} Q_0 j_i^{(n)}(r) \right)}_{\text{Kinetic terms}}.$$

- Sources at 1st order:

$$\mathbf{J}_\nu^{(1)}(r) = 3r^2 u^\mu \partial_\mu u_\nu + \left(\frac{96M}{r^3} - \frac{480Q^2}{r^5} \right) \lambda B_\nu$$

$$- \frac{16\sqrt{3}Q_0}{r^7} (20Mr^2 - 63Q^2) \lambda \omega_\nu$$

$$\mathbf{A}_\nu^{(1)}(r) = -\frac{\sqrt{3}}{r^2} (P_\nu^\alpha \partial_\alpha Q + Q u^\mu \partial_\mu u_\nu) - \frac{16\sqrt{3}Q}{r^3} \kappa B_\nu$$

$$+ \frac{48Q^2}{r^5} \kappa \omega_\nu + \frac{48}{r^{11}} (4M^2 r^4 - 16MQ^2 r^2 + 15Q^4) \lambda \omega_\nu.$$

Fluid/Gravity Correspondence

Boundary conditions when solving Einstein-Maxwell equations:

- Choice (A):

- $j_\mu^{(\bar{4},1)} = 0$ in near boundary expansion $j_\mu^{(1)} \simeq \dots + \frac{j_\mu^{(\bar{4},1)}}{r^4} + \dots$
- $a_\mu =$ finite at the horizon.
- $j_\mu = 0$ at the boundary.
- $a_\mu = 0$ at the boundary.

(1) is the Landau frame condition.

\implies Transport coefficients in Landau frame.

- Choice (B):

- $j_\mu = 0$ at the horizon.
- $a_\mu =$ finite at the horizon.
- $j_\mu = 0$ at the boundary.
- $a_\mu = 0$ at the boundary.

\implies Transport coefficients in Laboratory rest frame.

Fluid/Gravity Correspondence

- In Landau frame (Choice (A)):

$$a_{\mu}^{(1)}(r) = \dots - \left[\frac{6Q^2 \kappa}{M r^2} - \frac{4(Q^2 - 2r_+^6)^2(2Q^2 - r_+^6)}{Mr_+^{12}} \frac{\lambda}{r^2} + \mathcal{O}(r^{-6}) \right] \omega_{\mu} + \dots$$

- Current:

$$\begin{aligned} \langle J_{\mu} \rangle &= \frac{1}{\sqrt{-g^{(\bar{0})}}} \frac{\delta}{\delta A^{\mu}(\bar{0})} S_{\text{ren}}[A_{\mu}^{(\bar{0})}, g_{\mu\nu}^{(\bar{0})}] = -\frac{1}{8\pi G} a_{\mu}^{(\bar{2})} \\ &= \dots + \xi^{\mathcal{B}} B_{\mu} + \xi^{\mathcal{V}} \omega_{\mu} + \dots \end{aligned}$$

- Transport coefficients in Landau frame:

$$\begin{aligned} \xi^{\mathcal{B}} &= \frac{1}{4\pi^2} \left(\mu - \frac{1}{2} \frac{n(\mu^2 + \frac{\pi^2 T^2}{3})}{\epsilon + P} \right), \\ \xi^{\mathcal{V}} &= \frac{\mu^2}{8\pi^2} \left(1 - \frac{2}{3} \frac{n\mu}{\epsilon + P} \right) + \frac{T^2}{24} \left(1 - \frac{2n\mu}{\epsilon + P} \right). \end{aligned}$$

Fluid/Gravity Correspondence

- In Laboratory rest frame (Choice (B)):

$$a_{\mu}^{(1)}(r) = \dots - \left[\frac{6Q^2}{r_+^4} \frac{\kappa}{r^2} + \frac{4(Q^2 - 2r_+^6)^2}{r_+^{10}} \frac{\lambda}{r^2} + \mathcal{O}(r^{-6}) \right] \omega_{\mu} + \dots$$

- Current:

$$\langle J_{\mu} \rangle = -\frac{1}{8\pi G} a_{\mu}^{(\bar{2})} = \dots + \sigma^{\mathcal{B}} B_{\mu} + \sigma^{\mathcal{V}} \omega_{\mu} + \dots$$

- Transport coefficients in Laboratory rest frame:

$$\sigma^{\mathcal{B}} = \frac{\mu}{4\pi^2},$$

$$\sigma^{\mathcal{V}} = \frac{\mu^2}{8\pi^2} + \frac{T^2}{24}$$

Chemical Potentials for Anomalous Symmetries

Two formalisms to chemical potentials [T.Evans '95]:

Formalism	Hamiltonian	Boundary conditions
(A)	$H - \mu Q$	$\Psi(\tau) = -\Psi(\tau - \beta)$
(B)	H	$\Psi(\tau) = -e^{\beta\mu}\Psi(\tau - \beta)$

- If Q is **non anomalous** \implies (A) and (B) are connected by a large gauge transformation: $A_0 \rightarrow A_0 + \partial_0 \chi$ with $\chi = -\mu t$
- If Q is **anomalous**, the action transforms (in formalism (B)):

$$S[A + \partial\chi] = S[A] + \int d^4x \chi \epsilon^{\mu\nu\rho\lambda} (C_1 F_{\mu\nu} F_{\rho\lambda} + C_2 R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\lambda}) .$$

The gauge transformation leads to an inequivalent theory!!!

Chemical Potentials for Anomalous Symmetries

- Introduce a non-dynamical “axion” field:

$$S_{tot}[A, \theta] = S[A] + \underbrace{\int d^4x \theta \epsilon^{\mu\nu\rho\lambda} (C_1 F_{\mu\nu} F_{\rho\lambda} + C_2 R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\lambda})}_{\equiv S_\theta[A, \theta]} .$$

- Demand the axion transform as $\theta \rightarrow \theta - \chi \implies S_{tot}[A, \theta]$ is gauge invariant.
- Under a gauge transformation $\chi = \mu t$:

Formalism (B) \longrightarrow Formalism (A)

$$A_0 = 0 \longrightarrow A_0 + \partial_0 \chi = \mu$$

$$\theta = 0 \longrightarrow \theta - \chi = -\mu t$$

$$H \longrightarrow H - \mu Q - 4\mu C_1 \int d^3x \epsilon^{0ijk} A_i \partial_j A_k$$

This Hamiltonian is similar to the one advocated by **[Rubakov '10]**.

Chemical Potentials for Anomalous Symmetries

- AdS Reissner-Nordström black-brane solution of the Holographic model:

$$A_0 = \phi(r) dt = \left(\alpha - \frac{\mu r_h^2}{r^2} \right) dt.$$

- Choice (A): $\alpha = \mu \implies A_0|_{r=r_h} = 0$
- Choice (B): $\alpha = 0$.
- Choice (A) corresponds to Formalism (A), choice (B) to Form (B).
- (A) and (B) are physically equivalent, but we need to take into account the pure boundary action $S_\theta[A, \theta]$.
- θ does not extend into the bulk \implies non dynamical.
- Ex: Choice (A) with no S_θ gives a vanishing chiral magnetic conductivity. But with S_θ it gives the correct result 😊

Conclusions

- We have studied the effects of external magnetic fields and vortices in a relativistic fluid.
- Anomalies \implies parity violating transport.
- We have derived Kubo formulas.
- Surprise: mixed gauge-gravitational anomaly contributes!!!
- Holography with gauge-gravitational Chern Simons term. Two methods:
 - Kubo Formulas.
 - Fluid/Gravity Correspondence.
- (Non)-renormalization of anomalous conductivities.
- Observable effects in heavy ion physics and cosmology?:
 - Enhanced production of high spin hadrons, especially Ω^- baryons, due to chiral separation effect [**Keren-Zur, Oz '10**].
 - Lepton number separation in the early universe due to gravitational anomaly [**Alexander, Peskin, Sheikh-Jabbari '06**].

[Kharzeev, Son '11], [Kharzeev, Yee '11].