Geometric scaling violation:
[1] DIS with nuclei
[2] running QCD coupling

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B. D. Saez and E. L., *"Violation of the geometric scaling behaviour of the amplitude for running QCD coupling in the saturation region,"* **[arXiv:1106.6257 [hep-ph]]**

A. Kormilitzin, E. L., S. Tapia, *"Geometric scaling behavior of the scattering amplitude for DIS with nuclei,"* **[arXiv:1106.3268 [hep-ph]].**

Why we believe in GS ?

1. A natural consequence of one dimensional scale: Q_s;

2. For $\tau = r^2 Q_s \geq 1$ it follows from BFKL and GLAP evolution;

3. Deeply in the saturation region follows from B-K equation at at fixed α_S;

4. Two models with simplified BFKL kernel with fixed α_S reproduce GS:

   - kernel in diffusion approximation, B-K reduces to F-KPP equation with traveling wave solution;
   - only leading twist contribution to the kernel, B-K reduces to wave equation with traveling wave solution;

Will GS survive if there is an additional dimensional scale?

New scales:

- Running α_S: Λ_{QCD};
- DIS with nuclei: Q_s at low energy (initial condition);

Intuition:

- Running α_S: violation of GS since conformal symmetry of the kernel is broken;
- DIS with nuclei: no violation since we expect that deeply inside the saturation region the solution does not depend on initial conditions;

Simplified BFKL kernel

$$\frac{\partial N_A\left(r,Y;b\right)}{\partial\,Y} = \frac{C_F\,\alpha_S}{2\,\pi^2}\int d^2r'\,K\left(r;r'\right)\times\Big\{2N_A\left(r',Y;\vec{b}\,-\,\frac{1}{2}\,(\vec{r}-\vec{r}\,')\right)$$

$$-\quad N_A\left(r,Y;\vec{b}\right)\;-\;N_A\left(r',Y;\vec{b}-\frac{1}{2}\,(\vec{r}-\vec{r}\,')\right)\,N_A\left(\vec{r}-\vec{r}\,',Y;b-\frac{1}{2}\vec{r}\,'\right)\Big\}$$

For $\tau\;=\;r^2\,Q_s^2\;<\;1$

$$\int d^2r'\,K\,(r,r')\;=\int d^2r'\frac{r^2}{r'^2\,(\vec{r}-\vec{r}')^2}\;\rightarrow\;\pi\,r^2\,\int_{r^2}^{\frac{1}{\Lambda^2_{QCD}}}\frac{dr'^2}{r'^4}$$

For $\tau\;=\;r^2\,Q_s^2\;>\;1$

$$\int d^2r'\,K\,(r,r')\;\rightarrow\;\pi\,\int_{1/Q_s^2(Y,b)}^{r^2}\frac{dr'^2}{r'^2}\;+\;\pi\,\int_{1/Q_s^2(Y,b)}^{r^2}\frac{d|\vec{r}-\vec{r}'|^2}{|\vec{r}-\vec{r}'|^2}$$

- $$\chi(\gamma) = \begin{cases} \frac{1}{\gamma} \quad \text{for } \tau = r^2 Q_s^2 \geq 1\,; & \text{logs:} \left(\alpha_S \ln(1/r^2 \Lambda^2_{QCD})\right)^n \\ \frac{1}{1-\gamma} \quad \text{for } \tau = r^2 Q_s^2 \leq 1\,; & \text{logs:} \left(\alpha_S \ln(r^2 Q_s^2)\right)^n \end{cases}$$

B-K equation inside the saturation region:

- $$\frac{\partial^2 \tilde{N}(r, Y; b)}{\partial Y\, \partial \ln r^2} = \bar{\alpha}_S \left\{ \left(1 - \frac{\partial \tilde{N}(r, Y; b)}{\partial \ln r^2}\right) \tilde{N}(r, Y; b) \right\}$$

- $$\tilde{N}(r, Y; t = 0) = \int^{r^2} dr^2\, N(r, Y; b) / r^2$$

Solution

- $$N = 1 - e^{-\phi(\xi,Y)}; \qquad \phi'_Y\, e^{-\phi} = \bar{\alpha}_S \widetilde{N}\, e^{-\phi};$$
- $$\frac{\partial^2 \phi}{\partial Y\,\partial \xi} = \bar{\alpha}_S \left(1 - e^{-\phi(Y;\xi)}\right)$$

where

$$\xi_s = \ln\left(Q_s^2\left(Y\right)/Q_s^2\left(Y = Y_0\right)\right) = 4\bar{\alpha}_S\left(Y - Y_0\right),$$

and

$$\xi = \ln\left(r^2 Q_s^2\left(Y = Y_0\right)\right)$$

$$\boxed{\frac{\partial^2 \phi}{\partial \xi_s\,\partial \xi} = \frac{1}{4}\left(1 - e^{-\phi(Y;\xi)}\right)}$$

$$\frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \phi}{\partial x^2} = \frac{1}{4}\left(1 - e^{-\phi(Y;\xi)}\right)$$

with $z = \xi_s + \xi$ and $x = \xi_s - \xi$

1. **Traveling wave solution:**

- $$\int_{\phi_0}^{\phi} \frac{d\phi'}{\sqrt{c + \frac{1}{2(\lambda^2 - \kappa^2)}\left(\phi' - 1 + e^{-\phi'}\right)}} = \kappa\, x + \lambda\, z$$

2. **Self-similar solution:** $\phi = \phi(\zeta)$ with $\zeta = \xi_s\, \xi$

$$\zeta \frac{d^2 \phi(\zeta)}{d\zeta^2} + \frac{d\phi(\zeta)}{d\zeta} = \frac{1}{4}\left(1 - e^{-\phi(\zeta)}\right)$$

- A.D. Polyanin and V. F. Zaitsev, “ *Handbook of nonlinear Partial Differential Equations*”, Chapman & Hall/CRC, 2004

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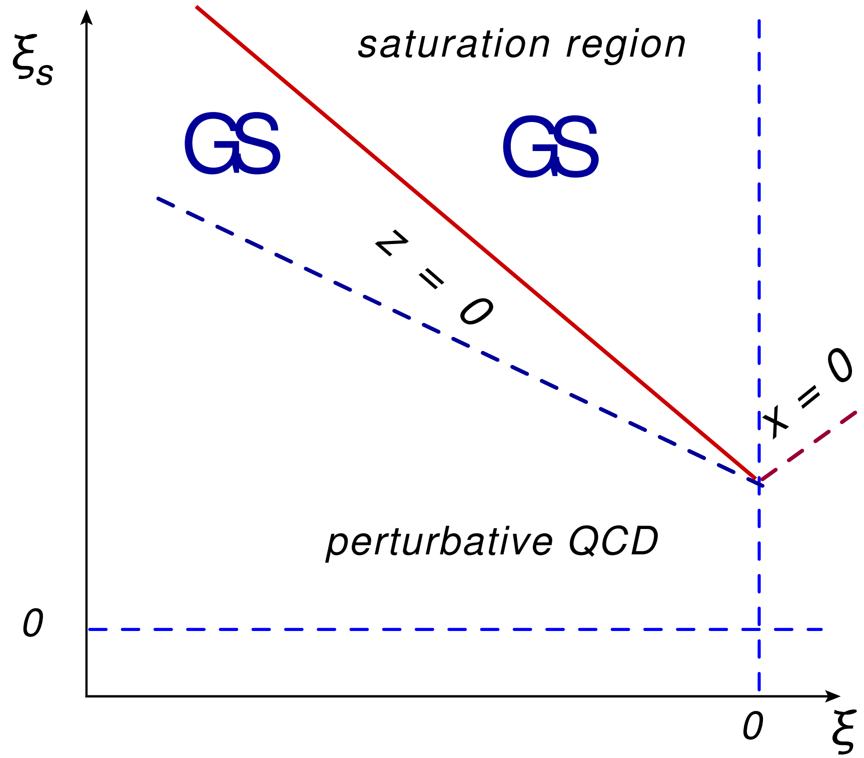
Initial conditions from $\tau = r^2 Q_s^2 > 1$:

$$\phi\,(t \equiv z = 0, z) \;=\; \phi_0; \quad \phi'_z\,(t \equiv z = 0, z) \;=\; \frac{1}{2}\,\phi_0;$$

Finally: **traveling wave solution with $\kappa = 0$**



Running α_S

BFKL Kernel:

$$K\left(r;r_1,r_2\right)\;=\;\bar{\alpha}_S\left(r^2\right)\left\{\frac{r^2}{r_1^2\,r_2^2}\;+\;\frac{1}{r_1^2}\left(\frac{\bar{\alpha}_S\left(r_1^2\right)}{\bar{\alpha}_S\left(r_2^2\right)}\;-\;1\right)\;+\;\frac{1}{r_2^2}\left(\frac{\bar{\alpha}_S\left(r_2^2\right)}{\bar{\alpha}_S\left(r_1^2\right)}\;-\;1\right)\right\}$$

For the region:

$r\;\approx\;r_2\;\gg\;r_1\;\gg\;1/Q_s^2$ **and** $r\;\approx\;r_1\;\gg\;r_2\;\gg\;1/Q_s^2$

$$\int d^2r'\,K\left(r,r'\right)\;\rightarrow\;\pi\int_{1/Q_s^2(Y,b)}^{r^2}\frac{\bar{\alpha}_S\left(r_1^2\right)dr_1^2}{r_1^2}\;+\;\pi\int_{1/Q_s^2(Y,b)}^{r^2}\frac{\bar{\alpha}_S\left(r_2^2\right)dr_2^2}{r_2^2}$$

- $\tilde{N}\left(r,Y;b\right)\;=\;\int_{1/Q_s^2}^{r^2}dr'^2\,\frac{\bar{\alpha}_S\left(r'^2\right)}{r'^2}\,N\left(r',Y;b\right)$

- $\dfrac{\partial N\left(r,Y;b\right)}{\partial Y}\;=\;\tilde{N}\left(r,Y;b\right)\left(1\;-\;N\left(r,Y;b\right)\right)$

Introducing

$$l \;=\; \int^{r^2} dr'^2 \frac{\bar{\alpha}_S\left(r'^2\right)}{r'^2} \;=\; \frac{4N_c}{b}\,\ln\left(1/\bar{\alpha}_S\left(r^2\right)\right) \;=\; \frac{4N_c}{b}\,\ln\left(\bar{\xi}\right)$$

with $\bar{\xi} \;=\; -\ln\left(r^2\,\Lambda^2_{QCD}\right) \;\equiv\; -\xi$

We obtain:

$$\frac{\partial^2 \phi(r,Y;b)}{\partial Y\,\partial l} \;=\; 1 \;-\; e^{-\phi(r,Y;b)}$$

Equation for saturation scale:

$$\frac{32N_c}{b}\,Y \;=\; \bar{\xi}^2$$

Solutions:

1. Traveling wave solution contradicts initial conditions

2. Self-similar solution?!

$$\phi \gg 1$$

Equation:

$$\frac{\partial^2 \phi(Y,l)}{\partial Y\, \partial l} = 1$$

Obvious solution:

$$\phi_\infty(Y,l;b) = Y\,l + F(Y) + G(l)$$
$$= Y\,(l - l_s) - \frac{1}{2}\phi_0\left(e^{l} - e^{l_s}\right) - \frac{1}{2}\left(e^{2l} - e^{2l_s}\right) + \phi_0$$

- **Main problem with IC;**
- **No indication on GS;**
  - **However for $\bar{\xi} \to \xi_s$ solution:**

$$\tilde{\phi}_\infty(Y,l;b) \xrightarrow{\bar{\xi}-\xi_s \ll \xi_s} \phi_0 - \frac{3}{4}\phi_0\left(\bar{\xi} - \xi_s\right) = \phi_0 + \frac{1}{2}\phi_0\, z$$

- **Solution leads to $\phi_\infty < 0$, contradicts unitarity?!**

3. Searching self-similar solution:

$\phi\left(\zeta\right)$ **with** $\zeta = Y\ (l - l_s)$

Equation:

$$(\zeta - 2)\ \frac{d^2\phi\left(\zeta\right)}{d\zeta^2} + \frac{\phi\left(\zeta\right)}{d\zeta} = 1 - e^{-\phi(\zeta)}$$

IC:

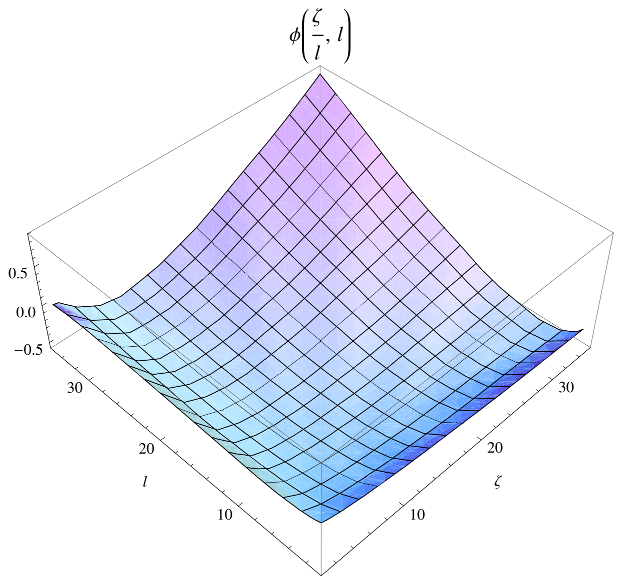
$$\phi\left(\zeta = 0\right) = \phi_0;\ \ \frac{d\phi\left(\zeta = 0\right)}{d\zeta} = -\frac{1}{2}\phi_0/\xi_s = -\frac{1}{2}\phi_0\Big/\sqrt{\frac{32N_c}{b}Y}$$

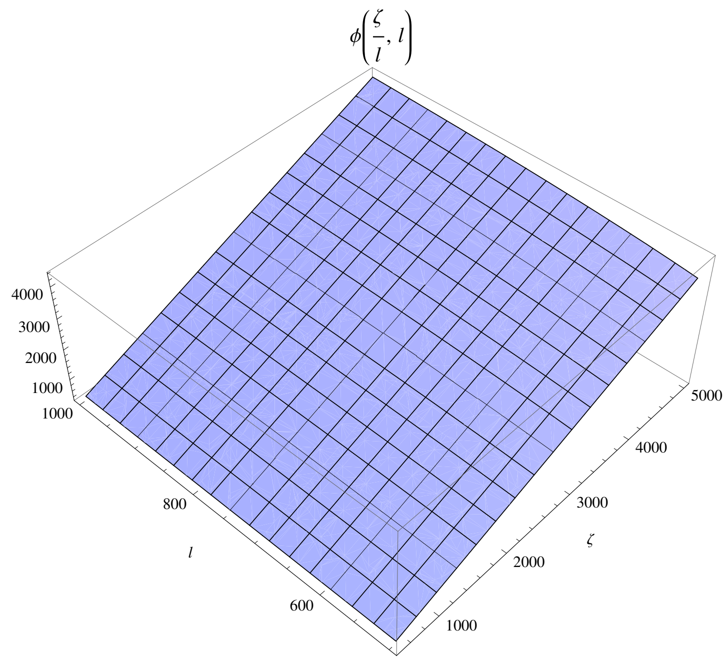
IC at large Y:

$$\phi\left(\zeta = 0\right) = \phi_0;\ \ \frac{d\phi\left(\zeta = 0\right)}{d\zeta} = -\frac{1}{2}\phi_0/\xi_s = 0$$

Good chance for self-similar solution at $Y \gg 1$

4. Searching self-similar solution: numerical solution.





No SSS **Approaching SSS**

Self- similar solution but at unreasonable large ζ

5. GS solution in vicinity of the saturation scale.

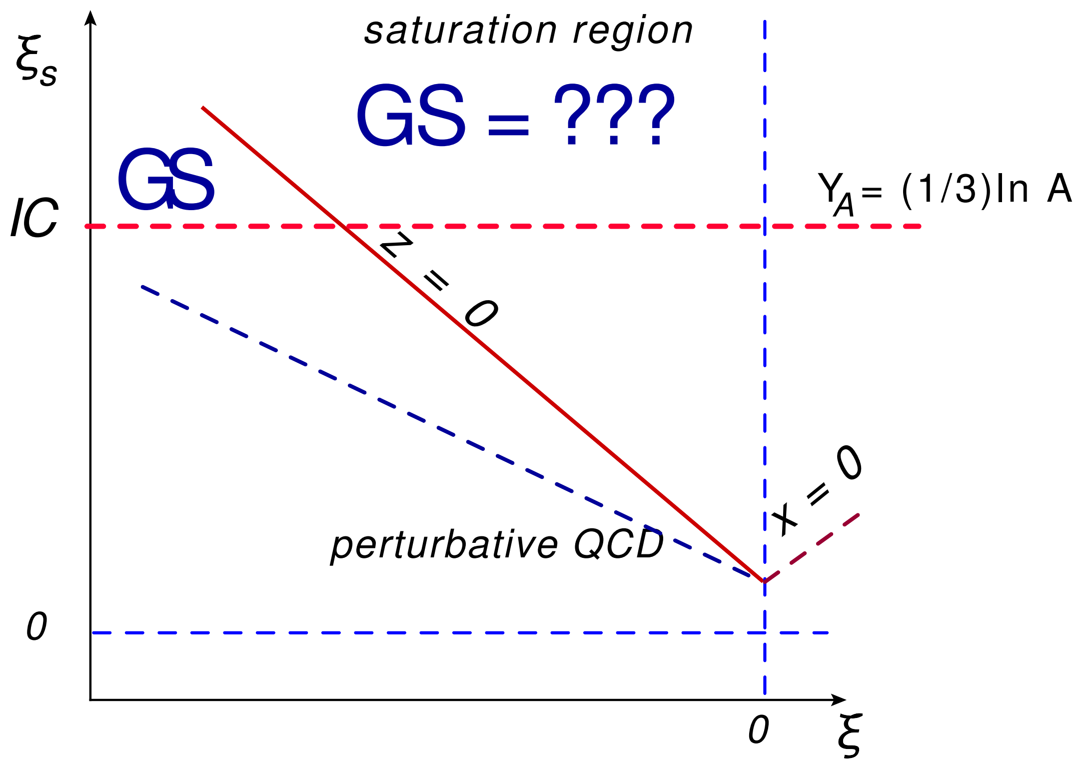
In terms of $\tilde{z} = \sqrt{\frac{16N_c}{b}}\,z$ equation looks as:

$$\sqrt{\frac{16N_c}{b}}\frac{\tilde{z}}{\sqrt{2\,Y}}\,\frac{d^2\phi\left(\tilde{z};b\right)}{d\,\tilde{z}^2} + \frac{d^2\phi\left(\tilde{z};b\right)}{d\,\tilde{z}^2} = 1 - e^{-\phi(\tilde{z};b)}$$

GS for $\sqrt{\frac{16N_c}{b}}\tilde{z} \ll 2\,Y$

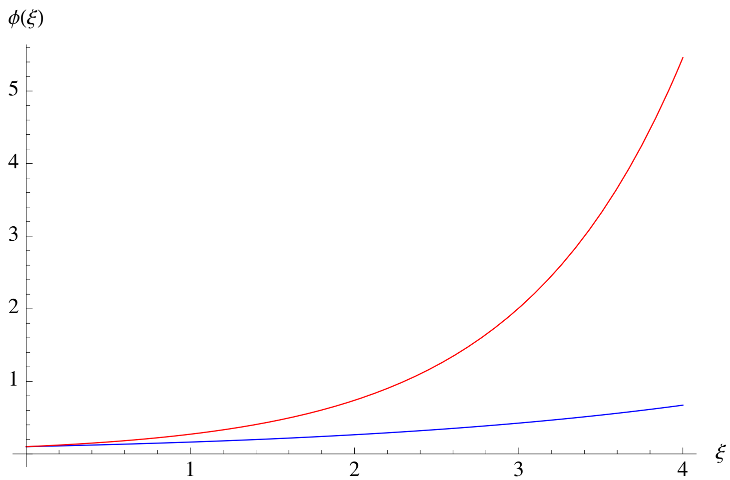
GS for $\alpha_S(Q_s)\ \ln\left(r^2\,Q_s^2\right) \leq 1$

DIS with nuclei



IC: McLerran-Venugopalan formula $\alpha_S ln(1/x) \ll 1; \alpha_S^2 A^{1/3} \approx 1$

$$N_A\left(r^2; Y; b\right) \;=\; 1 - \exp\left(-\,r^2\, Q_s^2\left(A; Y = Y_A; b\right)\right)$$



- **No GS**

Solution: $\phi \gg 1$

$$\phi_{\xi_s,\xi} = \frac{1}{4}; \qquad \text{or}$$

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = \frac{1}{4};$$

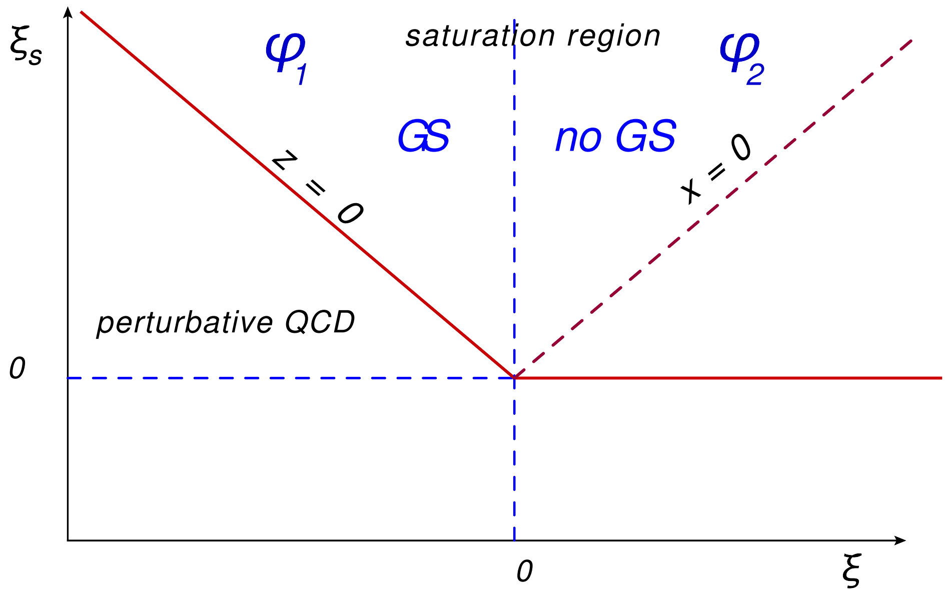
General solution:

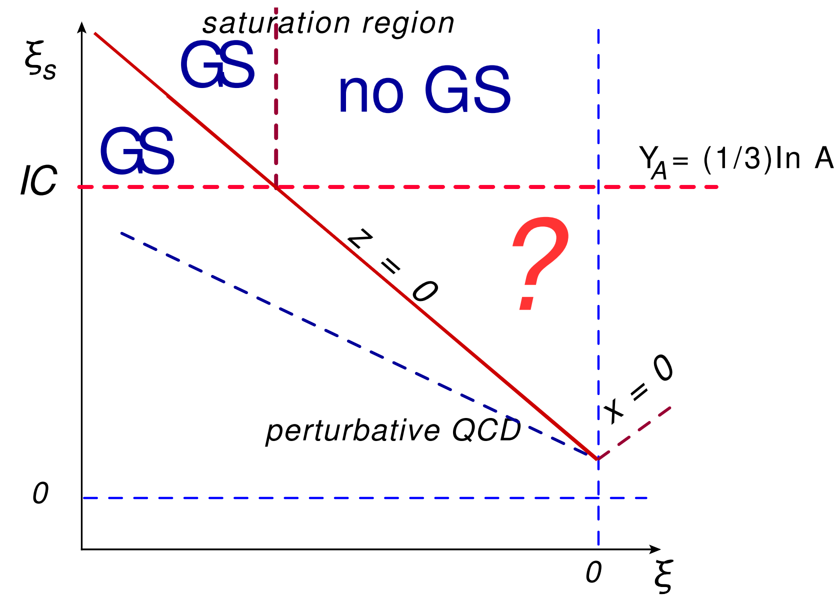
$$\phi\left(\xi_s, \xi\right) = \frac{1}{4}\xi_s\,\xi + F_1\left(\xi_s\right) + F_2\left(\xi\right)$$

Solution for $t = z < x$ $(\xi < 0)$: $\phi_1\left(z\right) = \frac{1}{8}z^2 + \frac{\phi_0}{2}\,z + \phi_0$

Solution for $t = z > x$ $(\xi > 0)$:

$$\phi_2\left(z, \xi\right) = z^2/8 - \xi^2/8 + \phi_0\,e^{\xi} + \frac{1}{2}\phi_0\,\xi_s$$





For CGC (McL-V formula) :

GS for $\tau = r^2 Q_s^2 > 1$ **and** $r^2 < 1/Q^2(Y = Y_A)$

no GS for $\tau = r^2 Q_s^2 > 1$ **and** $r^2 > 1/Q^2(Y = Y_A)$

BFKL Pomeron Calculus for DIS with nuclei

Goal: discuss $Y_A > Y \; > \; 1$

Simplified B-K equation

$$N_A^{BFKL} \;=\; \int d^2b'\, N_N^{BFKL}\left(r,\, Y;\, \vec{b} - \vec{b}'\right) T_A(b')$$
$$=\; T_A(b) \int d^2b''\, N_N^{BFKL}(r, Y;\, b'')$$
$$=\; N_N^{BFKL}\,(r, Y; t = 0)\, T_A(b)$$

Assumption: $\vec{b} - \vec{b}' \equiv \vec{b}''$.

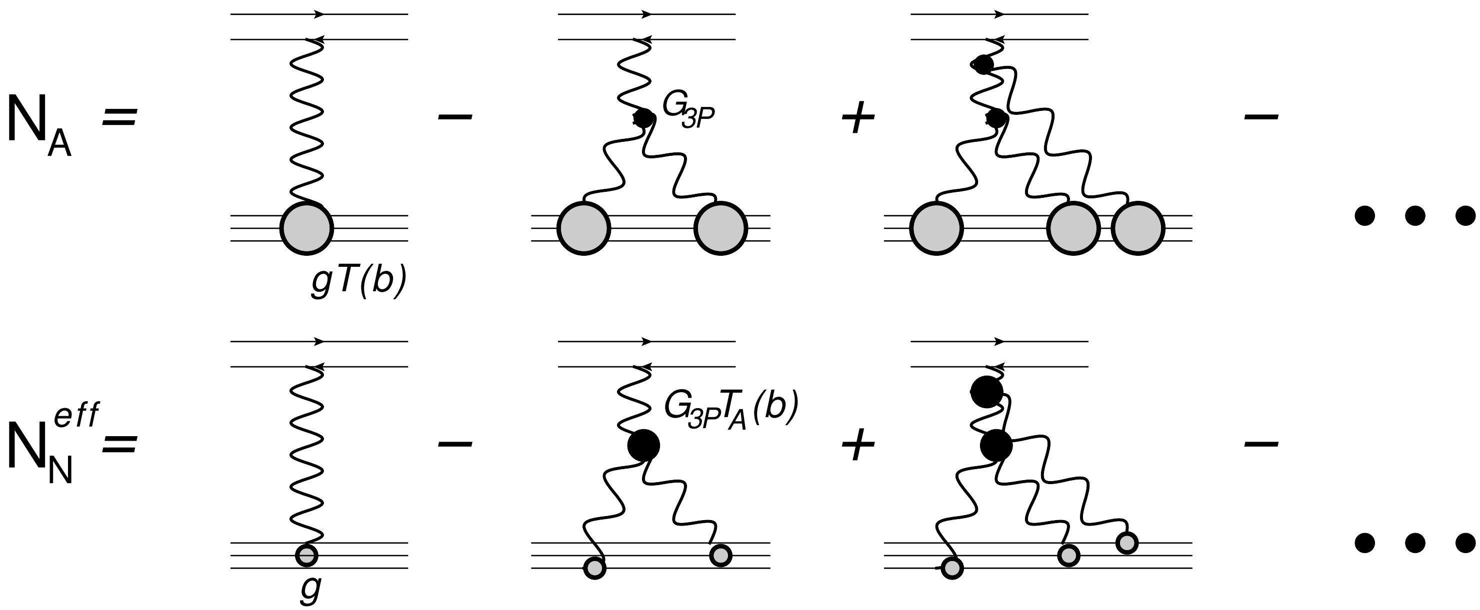
R^2 (dipole - nucleon) $= R_0^2 + \alpha'_P(0)\, Y \ll R_A^2$

$$T_A(b) \;=\; \int_{-\infty}^{+\infty} dz\, \rho\,(b, z)$$

$$\frac{\partial N^{eff(2)}_N(r,Y;t=0)}{\partial Y} = \frac{\bar{\alpha}_S}{2\pi}\int \frac{d^2r'\,r^2}{(\vec{r}\,-\,\vec{r}\,')^2\,r'^2}$$

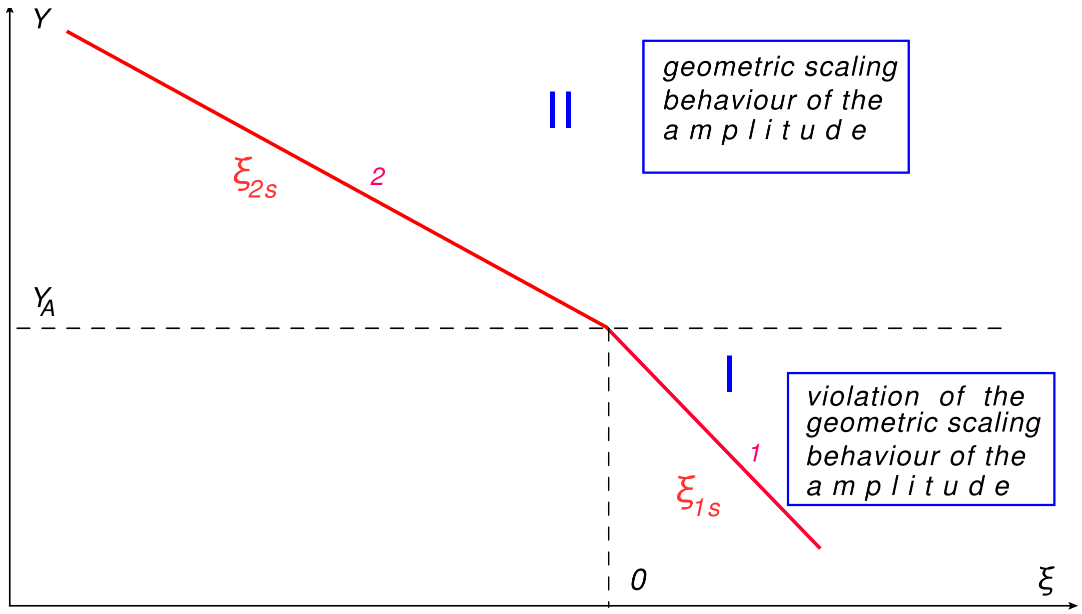
$$\times\Big\{2\,N^{eff(1)}_N(r',Y;t=0) - N^{eff(1)}_N(r,Y;t=0)$$

$$-\,T_A(b)\,N^{eff(1)}_N(r',Y;t=0)\,N^{eff(1)}_N(\vec{r}-\vec{r}\,',Y;t=0)\Big\}$$

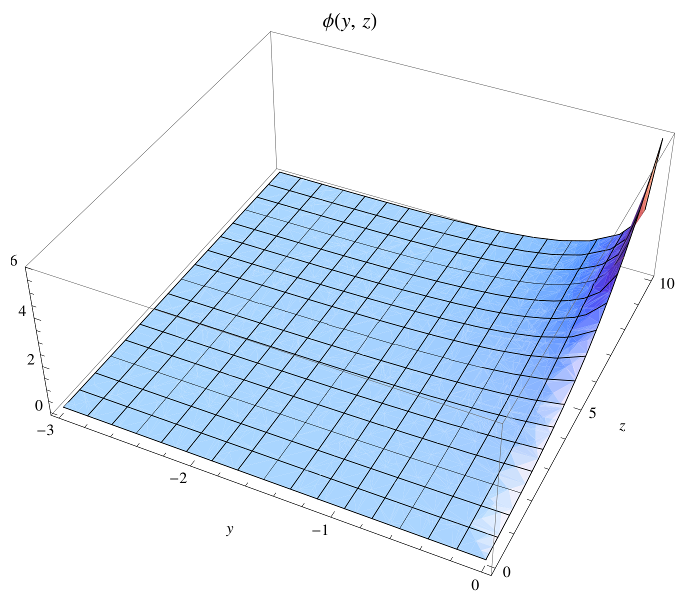


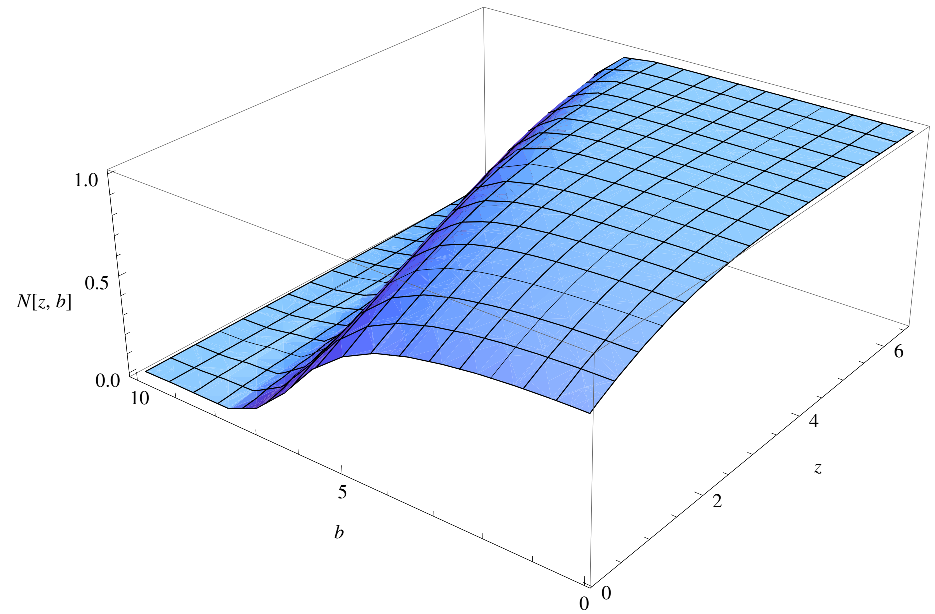
Equation for $Y < Y_A$

$$T_A(b;Y) = \int_{-1/mx}^{+1/mx} dz\,\rho(b,z) = \begin{cases} \rho\,2R_A \propto A^{1/3} & \text{for } Y \geq Y_A\,; \\ e^{Y}\rho/m & \text{for } Y < Y_A\,; \end{cases}$$



$$\xi_{2s} = -4\bar{\alpha}_S(Y - Y_A) \text{ and } \xi = -\xi_{1s} = \left(1 + 2\sqrt{\bar{\alpha}_S}\right)(Y_A - Y)$$





- ϕ **for** $Y < Y_A$
- N_A **for** $Y > Y_A$