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Rapidity RGE FAQs (that you may have never asked)

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Outline

- What are rapidity divergences?
  In what problems do they arise? In what kinematical region do they arise? Do the IR divergences cancel properly in presence of rap. divs.? Are these divergences a problem?
- TMD PDFs! What's the commotion?
  Why the plethora of definitions? Why did Collins come up with his strange definition? What happens to TMD PDFs in SCET?
- Is there a simple example that illustrates this formalism?
  Yes! Sudakov log resummation for massive gauge boson.
- What is one lesson that I should take from this talk?
  Be vigilant about the choice of regulators you make.

What are rapidity divergences?

- What are rapidity divergences indeed? Is there a simple way to analyze that an integral I am dealing with has rapidity divergences?
  Rapidity divergences are divergences coming from small rapidity region in collinear integrals (like those that contribute to TMP PDFs or even the ordinary jet function for thrust that we are familiar with). They also arise from large rapidity region in the soft integrals. The simple answer is: if an integral in SCET has an eikonal propagator it likely has a rapidity divergence.
- What kinematical region in the momentum integral do they arise from?
  In SCET integrals, they arise from the region when $k^+ k^-$ is held fixed but $k^+ \to \infty$ and $k^- \to 0$ or vise versa. They appear in form of the un-regulated integral of type, where at least one of the limits is either 0 or ∞: $I_{\text{rap. div.}} = \int \frac{dk^+}{k^+}$.
- Why do SCET_I matrix element do not have rapidity divergences?
  They will cancel in matrix elements that arise in SCET_I theory after zero-bin subtractions. Often in this case rapidity divergences arise from the zero-bin region corresponding to ultrasoft modes, when $k^+ \sim k^- \sim Q\,\lambda^2$. They can also come from $k^+ \sim k^- \sim Q\,\lambda$, but this is usually excluded due to a measurement on residual k^+. After a zero-bin subtraction that removes contribution from this entire

region these divergences diappear. For explicit cancellation of rapidity divergences in SCET_I matrix elements, see [AJ, Procura, Waalewijn 2011] where we show this for fragmenting jet functions and beams functions.

- Why do they show in SCET_{II}?

  Here we probe p_T of the final state radiation (or mass for Sudakov FF). This restricts $k^- k^+ \sim p_T^2 \sim Q^2 \lambda^2$, so rapidity divergences arise when $k^+ \sim k^- \sim Q\lambda$. For collinear region $\left(1 \sim k^- >> k^+ \sim \lambda^2\right)$ this limit corresponds to $k^- \to 0$ and $k^+ \to \infty$. These integrals are insensitive to the ultrasoft zero-bin region. A soft-bin subtraction will not remove them either because rapidity divergences really arise due of the boundary between the soft and collinear region [Manohar, Stewart 2006] or the end-point region of each sector. Note that unlike SCET_I, all modes live on the same hyperbola $k^+ k^- \sim \lambda^2$ and in order to factorize these modes from each other we need to choose a boundary that clearly seperates them. This boundary choice effectively provides a rapidity cutoff and matrix element in each sector depend on this cutoff. In absence of a rapidity cutoff they become unregulated divergences, as in effective theory SCET_{II}. The example below illustrates this:

Jet broadening in QCD:

In full QCD we get an integral like:

$$\frac{B}{\alpha_s}\frac{d\sigma}{dB} \sim \int_{QB}^{Q}\frac{1}{k^-}\,dk^- + \int_{QB}^{Q}\frac{1}{k^+}\,dk^+$$

$$= \left(\int_{QB}^{\Lambda}\frac{1}{k^-}\,dk^- + \int_{\Lambda}^{Q}\frac{1}{k^-}\,dk^-\right) + \left(\int_{QB}^{\Lambda}\frac{1}{k^+}\,dk^+ + \int_{\Lambda}^{Q}\frac{1}{k^+}\,dk^+\right)$$

which we have split into two pieces, each of which is sensitive to cutoff Λ as Log[Λ]. For factorization into soft and collinear regions, an appropriate choice is $\Lambda = \sqrt{QB}$. With Factorization in $\overline{\text{MS}}$ these integrals show up as (due to power counting, $\Lambda << Q$ and $\Lambda >> B$):

$$s \sim \int_{QB}^{\infty}\frac{dk^-}{k^-} + \int_{QB}^{\infty}\frac{dk^+}{k^+} = \int_0^{\infty}\frac{dk^+}{k^+}$$

$$j_n \sim \int_0^{Q}\frac{dk^-}{k^-}$$

$$j_{\bar{n}} \sim \int_0^{Q}\frac{dk^+}{k^+}$$

where for the soft case integrals can be effectively combined into a single variable due to restriction on $p_t = QB$, at 1-loop. Each integral is ill-defined due to unregulated rapidity divergence.

- Do IR and rapidity divergences have any overlap? Do UV and rapidity divergences have any overlap?

  No they don't! Best regulator to analyze this is to use, gluon mass for IR, eikonal regulator δ [Chiu *et. al*. 2009] for rapidity, and dimensional regularization for UV. For examples see [AJ, Procura, Waalewijn 2011]. Since δ-regulator is so beautiful, I decided to have a special name for it, *i.e.* iekonal regulator.

- Do the IR divergences cancel properly in presence of rap. divs.?

  Yes they do. Of course this can be messed up by an improper choice of a rapidity regulator. An improper choice will be a regulator that mixes various divergences into each other.

- Are rapidity divergences a problem?

  Divergences appear for a reason. There are logs in full theory result that are associated with these divergences. These logs are yearning for resummation, much like logs associated with UV divergences. Just like dim. reg. and $\overline{\text{MS}}$ provide a method to resum these logs, I see rapidity divergences as an opportunity to resum logs aasociated with rapidity appearing in the jet sector and soft sector. Futhermore, a choice of a good rapidity regulator that is along the line of dim. reg. will not make only this task straightforward but also can lead to proper definitions of TMD PDFs and quantities a like that suffer from rapidity divergences.

TMD PDFs! What's the commotion?

Plethora of definitions:

- [Collins 2003] : goes off the light cone. has an effective rapidity cutoff ζ and an running in ζ – Collin-Soper equation [Collins-Soper 1982]
- [Hautmann 2007] : implements subtractions by Wilson lines and requires auxialiary non-light like directions. Have an additional parameter regularization parameter ζ
- [Cherednikov, Stefanis 2008]: analyzes at 1-loop in axial gauges; have Collins-Soper like equation.
- [Collins 2011]: Includes soft subtractions. has non-light like Wilson lines in soft subtractions. Effective rapidity cut-off. No rapidity divergences. Simpler power corrections and Simpler RG equations.
- So far no satisfying definition in SCET!! Except for the one that you heard last week in Mannie's talk which I will review here.

Collins latest definition:

- unexpectedly complicated defintion
- (still) has non-light like Wilson line in Soft factors
- rapidity divergences cancel
- two effective rapidity cutoffs ζ_A and ζ_B, one for each proton: coming from rapidity of the non-light like Wilson lines. Constraint: $\zeta_A\,\zeta_B = Q^4$.
- **So what's the gain of doing all this?** Simpler RG equations and freedom from rapidity divergence, I guess!!

Collins new RG equations:

$$\frac{\partial \ln f_\perp(x, b_T; \mu, \zeta)}{\partial \ln \zeta} = K(b_T; \mu), \quad \text{where } \frac{dK}{d\ln\mu} = -\gamma_K[\alpha_s(\mu)] \tag{1.1}$$

$$\frac{\partial \ln f_\perp(x, b_T; \mu, \zeta)}{\partial \ln \mu} = \gamma_{\text{non-cusp}}[\alpha_s(\mu)] - \gamma_K[\alpha_s(\mu)] \ln \frac{\zeta}{\mu^2} \tag{1.2}$$

They are indeed simple enough and related to each other.

- **But do we really need to go through all that to attain this?**
  Often it is easier to study difficult problems in a convinient choice of co-ordinate system. In a good choice of co-ordinate system irrelevant co-ordinates drop out thus simplifying the picture of the problem. For QCD with light-like directions that co-ordinate system is SCET, where irrelevant modes drop out and relevant ones factorize to give simple results. Afterall SCET in one lightcone is boosted copy of QCD. Therefore SCET is perfect theory to analyze TMD PDFs.

TMD PDFs in SCET:

The most simplest and natural definition that comes about is:

$$f_\perp^{\mu\nu}\left(z, \vec{p}_T; \mu\right) = 2\pi \left\langle p(Q) \,\middle|\, B_{n\perp}^{\mu a}(0)\, \delta\!\left(1 - z - \frac{\overline{n}\cdot\hat{P}}{Q}\right) \delta^{(2)}\!\left(\vec{p}_T - \hat{\boldsymbol{P}}_\perp\right) [\text{tr. link}]\, B_{n\perp}^{\nu a}(0) \,\middle|\, p(Q) \right\rangle \tag{1.3}$$

except that it has unregulated rapidity divergences that we need to regulate. This requires η-regulator in light-like Wilson lines that are present in the collinear gauge invariant SCET fields $B_{n\perp}^{\mu a}$,

$$W_n = \sum_{\text{perm}} \exp\left(\frac{-g}{\bar{n}\cdot\hat{P}}\left[w^2\,\nu^\eta\,\left|\bar{n}\cdot\hat{P}\right|^{-\eta}\,\bar{n}\cdot A_{n,q}(0)\right]\right). \tag{1.4}$$

Here w is a book-keeping parameter such that $w^{\text{bare}} = w\,\nu^{\eta/2}$ is independent of η. Bare w is dimensionful, and renormalized w is fixed to 1 at the end of the calculation. Consequently w only has no finite running. In some sense this is like α_s except that it has no finite running. This simply aids in getting RG equations in the much familiar way. After modifying the Wilson lines in the $B_{n\perp}$ fields our renormalized TMD PDF depends upon $\frac{\nu}{Q}$, $f_\perp^{\mu\nu}\left(z, \vec{p}_T; \mu, \frac{\nu}{Q}\right)$.

I will contrast this definition with the state-of-art definition in QCD

- most simple and natural definition
- there are no non-light like directions
- there are no soft modes in this definition
- it is gauge invariant
- we have rapidity divergences
- transverse link at infinity vanishes in covariant gauges
- bare TMD PDF doesn't depend on μ or ν
- renormalized TMD PDF depends on both μ and ν and their dependence is governed by corresponding RG equations
- The newly introduced scale is the same for both the TMD PDFs in a factorization theorem

What are the RG equations we have and how we get them:

1. Calculate in perturbation theory with your favorite IR regulator. Remeber to include the wave function renormalization as always
2. Take limit $\eta \to 0$ and then $\epsilon \to 0$ keeping all powers of ϵ in $\frac{1}{\eta}$ divergences.
3. Seperate divergences from finite parts absorbing divergences in an operator renormalization constant $Z\left(\frac{1}{\epsilon}, \frac{1}{\eta};\ b_T, \mu, \frac{\nu}{Q}\right)$.

$$f_\perp^{\text{bare}}(z, b_T) = Z\left(\frac{1}{\epsilon}, \frac{1}{\eta};\ b_T, \mu, \frac{\nu}{Q}\right) f_\perp^{\text{ren}}\left(z, b_T; \mu, \frac{\nu}{Q}\right) \tag{1.5}$$

4. Calculate anomalous dimensions as usual except there are two renormalization scales now (remeber to use $d w / d \ln\nu = -\eta\, w/2$):

$$\gamma_\mu = Z^{-1}\frac{\partial Z}{\partial \ln\mu} \qquad \gamma_\nu = Z^{-1}\frac{\partial Z}{\partial \ln\nu} \tag{1.6}$$

5. Multiplicative renormalization implies two multiplicative RGEs. Run via standard approach:

$$\frac{\partial f_\perp}{\partial \ln\mu} = \gamma_\mu\, f_\perp \qquad \frac{\partial f_\perp}{\partial \ln\nu} = \gamma_\nu\, f_\perp \tag{1.7}$$

This calculation yields similar result as Collins latest defn.:

$$\frac{\partial \ln f_\perp\left(x, b_T; \mu, \frac{\nu}{Q}\right)}{\partial \ln\nu} = \gamma_\nu, \quad \text{where} \quad \frac{\partial\gamma_\nu}{\partial\ln\mu} = \Gamma_{\text{cusp}}[\alpha_s(\mu)] \tag{1.8}$$

$$\frac{\partial \ln f_\perp\left(x, b_T; \mu, \frac{\nu}{Q}\right)}{\partial \ln \mu} = \gamma_{\text{non-cusp}}[\alpha_s(\mu)] + \Gamma_{\text{cusp}}[\alpha_s(\mu)] \ln \frac{\nu}{Q} \tag{1.9}$$

Based on the structure of the factorization theorem μ anomalous dimension has the expected structure. Hard running has $\ln \frac{\mu^2}{Q^2}$ whose coefficient has Γ_{cusp}. Since soft function has no Q dependence, this can only come about if γ_μ for $f_\perp$ has a $\ln Q$ with coefficient as Γ_{cusp}. But Q only comes in fraction $\frac{\nu}{Q}$. Therefore, cusp piece of γ_μ has the structure shown in preceding eqn.

Theorem:

Renormalization group evolution in μ and ν commute.

Proof:

UV and rapidity divergences come from different kinematical regions and are therefore independent. Consequently μ and ν are independent scales. Therefore, μ and ν derivatives commute:

$$\partial_\mu \partial_\nu \ln f_\perp = \partial_\nu \partial_\mu \ln f_\perp$$

$$\Rightarrow \partial_\mu \gamma_\nu = \partial_\nu \gamma_\mu$$

Corollary: $\partial_\mu \gamma_\nu = \Gamma_{cusp}$.

And the best part is that these features are just not restricted to TMD PDFs but they are universal to many many quantities that have rapidity divergences!!

Is there a simple example that illusrates this formalism?

Sudakov form factor is a simple enough example. It factorizes into three sectors: collinear, anti-collinear and soft. Resummations for massive gauge bosons was first carried out in SCET by UCSD grp. [Chiu et. al. 2007] using analytical regulator. By year 2009 they rejected analytical regulator for its obvious problems and introduced the beautiful δ regulator, that I like to call the eikonal regulator. To remind the reader, analytical regulator breaks non-abelian exponentiation and gauge invariance, two novel properties of QCD. Moreover it destroys soft-collinear factorization.

To keep things simple we will work in a scalar world where quarks, gluons, photons, massive gauge bosons etc., all are scalar particles. Full theory integral is given by:

$$I^{\text{full theory}} = i\, g^2\left(p^- \ell^+\right) \int \frac{d^4 k}{(2\pi)^4} \frac{1}{\left[(k-\ell)^2 + i\,\epsilon\right]\left[k^2 - m^2 + i\,\epsilon\right]\left[(k-p)^2 + i\,\epsilon\right]} = \frac{g^2}{16\pi^2}\left(\frac{1}{2}\log^2\left(\frac{\ell^+ p^-}{m^2}\right) + \frac{\pi^2}{3}\right), \tag{1.10}$$

where m is gauge boson mass, $p^- = \bar{n}\cdot p$ and $\ell^+ = n\cdot \ell$ are large lightcone momenta of the two external quarks, which are both onshell and moving back to back. We have power counting $p^- \sim \ell^+ \sim Q >>$ m. Power counting parameter is $\lambda^2 \sim \frac{m^2}{\ell^+ p^-} \sim \frac{m^2}{Q^2}$. Internal gluon is generically offshell by order m^2 and therefore can either be soft, collinear or anti-collinear. This is an SCET_{II} situation where all the modes live on the same hyperbola. This gives rise to factorization into three sectors where each sector has only one integral at one loop, thus giving a simple situation to analyze (I will ignore wave function renormalization for this analysis):

$$I_n = i\, g^2(p^-) \int \frac{d^d k}{(2\pi)^d} \frac{w^2\, \nu^\eta\, |k^-|^{-\eta}\, \mu^{2\epsilon}\, e^{\epsilon\gamma_E}(4\pi)^{-\epsilon}}{[-k^- + i\,\epsilon]\left[k^2 - m^2 + i\,\epsilon\right]\left[k^2 - p^- k^+ + i\,\epsilon\right]} \tag{1.11}$$

$$I_s = i\, g^2 \int \frac{d^d k}{(2\pi)^d} \frac{w^2\, \nu^\eta\, \left|2\,k^3\right|^{-\eta}\, \mu^{2\epsilon}\, e^{\epsilon\gamma_E}(4\pi)^{-\epsilon}}{[-k^- + i\,\epsilon]\left[k^2 - m^2 + i\,\epsilon\right][-k^+ + i\,\epsilon]} \tag{1.12}$$

$$I_{\bar{n}} = i\, g^2\left(\ell^+\right) \int \frac{d^d k}{(2\pi)^d} \frac{w^2\, \nu^\eta\, |k^+|^{-\eta}\, \mu^{2\epsilon}\, e^{\epsilon\gamma_E}(4\pi)^{-\epsilon}}{\left[k^2 - \ell^+ k^- + i\,\epsilon\right]\left[k^2 - m^2 + i\,\epsilon\right][-k^+ + i\,\epsilon]}\ . \tag{1.13}$$

Eikonal propagators arise from Wilson lines. Here we have put in the rapidity regulator η according to the regulated Wilson line [Chiu, AJ, Neill, Rothstein 2011]. For soft integral we get:

$$I_s^{\text{bare}} = \frac{g^2 w^2}{16\pi^2}\left(\frac{2\, e^{\epsilon\gamma_E}\,\Gamma(\epsilon)}{\eta}\left(\frac{\mu^2}{m^2}\right)^{\epsilon} - \frac{1}{\epsilon^2} - \frac{1}{\epsilon}\log\left(\frac{\mu^2}{\nu^2}\right) + \frac{1}{2}\log^2\left(\frac{\mu^2}{m^2}\right) + \log\left(\frac{\mu^2}{m^2}\right)\log\left(\frac{\nu^2}{\mu^2}\right) + \frac{\pi^2}{12}\right) \tag{1.14}$$

Similarly one can calculate other two integrals. It can be checked that rapidity divergences cancel out when three sectors are added at one loop. Renormalizing them yields:

$$Z_s = 1 + \frac{g^2 w^2}{16\pi^2}\left(\frac{2\, e^{\epsilon\gamma_E}\,\Gamma(\epsilon)}{\eta}\left(\frac{\mu^2}{m^2}\right)^{\epsilon} - \frac{1}{\epsilon^2} - \frac{1}{\epsilon}\log\left(\frac{\mu^2}{\nu^2}\right)\right) \tag{1.15}$$

and

$$I_s^{\text{ren}} = 1 + \frac{g^2}{16\pi^2}\left(\frac{1}{2}\log^2\left(\frac{\mu^2}{m^2}\right) + \log\left(\frac{\mu^2}{m^2}\right)\log\left(\frac{\nu^2}{\mu^2}\right) + \frac{\pi^2}{12}\right) \tag{1.16}$$

where we set $w = 1$. Similarly for collinear sector. With some algebra it is seen that ν cancels out and IR divergences agree with full theory result. Corresponding matching will depend only on $\ln Q^2/\mu^2$. Consistence requires that for Z factors we have:

$$Z_h = (Z_n\, Z_s\, Z_{\bar{n}})^{-1} = 1 - \frac{\alpha_s(\mu)}{4\pi}\left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\log\frac{\mu^2}{\ell^+ p^-}\right) \tag{1.17}$$

and this gives running for the matching that only depends on Q^2/μ^2.

$$\frac{d\ln I_{\text{matching}}}{d\ln\mu} = Z_h^{-1}\frac{dZ_h}{d\ln\mu} = -\frac{\alpha_s(\mu)}{2\pi}\log\frac{\mu^2}{\ell^+ p^-} \tag{1.18}$$

Finally, logs in each renormalized function can be minimized by appropriate choice of μ and ν. Logs can be resummed by running the hard function in μ and soft function in ν. This is enough to resum all logs. RGEs for soft function are:

$$\frac{d\ln I_s^{\text{ren}}}{d\ln\mu} = Z_s^{-1}\frac{dZ_s}{d\ln\mu} = \frac{\alpha_s(\mu)}{2\pi}\log\frac{\mu^2}{\nu^2} \tag{1.19}$$

$$\frac{d\ln I_s^{\text{ren}}}{d\ln\nu} = Z_s^{-1}\frac{dZ_s}{d\ln\nu} = -\frac{\alpha_s(\mu)}{2\pi}\log\frac{\mu^2}{m^2}, \tag{1.20}$$

which satisfy the corollary shown earlier at one loop. Non-cusp pieces were zero here because we turned off wave function renormalization, which is the only source of $\frac{\text{const.}}{\epsilon}$ divergences in this calculation. Including them is a straightforward task.