Transfer of gauge invariant operators from complete to Möbius representation and vice versa

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- Comparison BFKL and BK approaches in NLO
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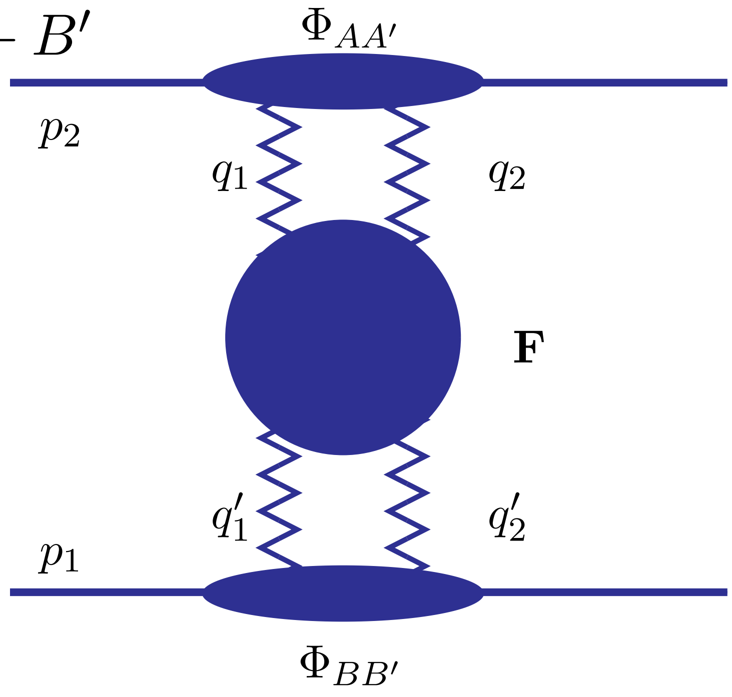
Introduction

- The sum of all ladder diagrams - Green function G satisfies the BFKL equation $\frac{d\hat{G}}{dY} = \hat{\mathcal{K}}\hat{G}$, where $Y = \ln\frac{s}{Q^2}$, $\hat{\mathcal{K}}$ — is the BFKL kernel.

- The kernel was built in the momentum representation $\langle \vec{q}_1, \vec{q}_2 | \hat{\mathcal{K}} | \vec{q}_1{}', \vec{q}_2{}' \rangle$, $\vec{q}_i$ - transverse momenta of the incoming and $\vec{q'}_i$ - outgoing reggeons.

- The amplitude for the process $A + B \to A' + B'$ is $\mathcal{A}_{AB}^{A'B'} = \Phi_{AA'} \otimes \hat{G} \otimes \Phi_{BB'}$,
Φ — impact-factors,
describing external particles.



- The BFKL kernel in NLO is known in the momentum representation for forward (1998 Fadin, Lipatov; Ciafaloni, Camici) and nonforward (2005 Fadin, Fiore) scattering.
It is complicated.

Introduction

BFKL kernel in the operator form looks

$$\hat{\mathcal{K}} = \hat{\omega}_1 + \hat{\omega}_2 + \hat{\mathcal{K}}_r \ ,$$

ω — is gluon Regge tragectory,
$\hat{\mathcal{K}}_r$ — real part of the kernel. It describes real particle production in Reggeon collisions.

s-channel discontinuity for the process $A + B \to A' + B'$ has the form

$$-4i(2\pi)^{D-2}\delta(\vec{q}_A - \vec{q}_B)\text{disc}_s\mathcal{A}_{AB}^{A'B'} = \langle A'\bar{A}|e^{Y\hat{\mathcal{K}}}|\bar{B}'B\rangle \ .$$

$Y = \ln(s/Q^2)$, Q^2 — is energy scale for transverse momenta,
$q_A = p_{A'} - p_A, \ \ q_B = p_B - p_{B'}$.
$\langle A'\bar{A}|, |\bar{B}'B\rangle$ — impact factors.

Möbius form of BFKL kernel

— is the kernel in the coordinate representation simplified for scattering of colorless particles. These simplifications are possible because

- impact factors for colorless particles have the following property

$$\langle A'\bar{A}|\psi\rangle = 0, \quad \text{if} \quad \langle \vec{q}_1, \vec{q}_2|\psi\rangle \sim \delta(\vec{q}_1) \text{ or } \delta(\vec{q}_2),$$

  i.e. in the coordinate space $\langle \vec{r}_1\vec{r}_2|\psi\rangle$ does not depend on $\vec{r}_1$ or on $\vec{r}_2$.
- real part of the kernel vanishes if one of the incoming reggeon momenta is equal to 0

$$\langle \vec{q}_1, \vec{q}_2|\hat{\mathcal{K}}_r|\vec{q}_1{}', \vec{q}_2{}'\rangle|_{\vec{q}_i' \to 0} \to 0.$$

- As a result

$$\langle A'\bar{A}|\hat{\mathcal{K}}|\psi\rangle = 0, \quad \text{if} \quad \langle \vec{q}_1, \vec{q}_2|\psi\rangle \sim \delta(\vec{q}_1) \text{ or } \delta(\vec{q}_2)$$

  ($\langle \vec{r}_1\vec{r}_2|\psi\rangle$ does not depend on $\vec{r}_1$ or on $\vec{r}_2$), i.e. the kernel conserves the properties of the projectile impact factor.

Möbius form of the kernel

- It means that the second impact factor can be changed without changing the discontinuity adding to it some terms independent of one of the coordinates $\vec{r}_1$ or $\vec{r}_2$. To simplify the kernel one should use such transformations to convert the second impact factor into dipole form, i.e. make it satisfy the condition

$$\langle \vec{r}\,', \vec{r}\,' | \bar{B}' B \rangle_d = 0 \ .$$

One can do it via the transformation

$$\langle \vec{r}_1{}', \vec{r}_2{}' | \bar{B}\,' B \rangle \to$$

$$\langle \vec{r}_1{}', \vec{r}_2{}' | \bar{B}' B \rangle_d = (\langle \vec{r}_1{}', \vec{r}_2{}' | - 1/2 \langle \vec{r}_1{}', \vec{r}_1{}' | - 1/2 \langle \vec{r}_2{}', \vec{r}_2{}' |) | \bar{B}' B \rangle.$$

- The kernel in the coordinate representation has the form

$$\langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}} | \vec{r}_1{}' \vec{r}_2{}' \rangle = A(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}') + \delta\left(\vec{r}_{1'2'}\right) D(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}'),$$

where does not have $\delta\left(\vec{r}_{1'2'}\right)$. Adding to this matrix element some terms independent of $\vec{r}_1$ or of $\vec{r}_2$ one can make vanish when $\vec{r}_1 = \vec{r}_2$.

Möbius form of the kernel

- Indeed, $\langle A'\bar{A}|\hat{\mathcal{K}}^n \to \langle A'\bar{A}|(\hat{\mathcal{K}}+\hat{\mathcal{C}})^n$, where $\hat{\mathcal{C}}$ — is the operator with matrix element independent of $\vec{r}_1$ or of $\vec{r}_2$. Expanding we get all terms with $\hat{\mathcal{C}}$ have $\langle A'\bar{A}|\hat{\mathcal{C}} = 0$ or $\langle A'\bar{A}|\hat{\mathcal{K}}^m\hat{\mathcal{C}} = 0$.
- After this the kernel can be rewritten as

  $$\hat{\mathcal{K}} \to \hat{\mathcal{K}}_m + \hat{\mathcal{D}},$$

  where the matrix element $\hat{\mathcal{K}}_m$ is equal to 0 when $\vec{r}_1 = \vec{r}_2$, and the matrix element $\hat{\mathcal{D}}$ has $\delta\left(\vec{r}_{1'2'}\right)$.
- the operator $\hat{\mathcal{D}}$ can be dropped without changing the discontinuity because in $\left(\hat{\mathcal{K}}_m + \hat{\mathcal{D}}\right)^n|\bar{B}'B\rangle_d$ all terms with $\hat{\mathcal{D}}$ have

  $$\hat{\mathcal{D}}|\bar{B}'B\rangle_d = 0 \quad \text{or} \quad \hat{\mathcal{D}}(\hat{\mathcal{K}}_m)^k|\bar{B}'B\rangle_d = 0.$$

After all these manipulations we get the kernel $\hat{\mathcal{K}}_m$, which is called dipole or Möbius.

Mb̈ius form of the kernel

So, to find the Möbius form of the kernel one has to pass the following steps:

- Furier transform the kernel into the coordinate space

$$\langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}} | \vec{r}_1{}' \vec{r}_2{}' \rangle = \int \frac{d^2 q_1}{2\pi} \frac{d^2 q_2}{2\pi} \frac{d^2 q_1'}{2\pi} \frac{d^2 q_2'}{2\pi} \langle \vec{q}_1, \vec{q}_2 | \hat{\mathcal{K}} | \vec{q}_1{}', \vec{q}_2{}' \rangle e^{i[\vec{q}_1 \vec{r}_1 + \vec{q}_2 \vec{r}_2 - \vec{q}_1{}' \vec{r}_1{}' - \vec{q}_2{}' \vec{r}_2{}']} \, .$$

- Drop all terms proportional to $\delta\left(\vec{r}_{1'2'}\right)$.
- Add to the kernel some terms independent of $\vec{r}_1$ or of $\vec{r}_2$ so that the kernel acquires the “dipole” property $\langle \vec{r}\ \vec{r} | \hat{\mathcal{K}} | \vec{r}_1{}' \vec{r}_2{}' \rangle = 0$.

After all these transformations in LO one gets the dipole evolution kernel

$$\langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}}_m^{LO} | \vec{r}_1{}' \vec{r}_2{}' \rangle = \frac{\alpha_s(\mu) N_c}{2\pi^2} \int d\vec{\rho} \frac{\vec{r}_{12}{}^2}{\vec{r}_{1\rho}^{\,2} \vec{r}_{2\rho}^{\,2}} \Bigg[\delta(\vec{r}_{11'}) \delta(\vec{r}_{2'\rho}) + \delta(\vec{r}_{1'\rho}) \delta(\vec{r}_{22'}) - \delta(\vec{r}_{11'}) \delta(r_{22'}) \Bigg]$$

Here $\vec{r}_{i\rho} = \vec{r}_i - \vec{\rho}$.

Möbius form of the kernel in NLO

has the form

$$\langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}}_m^{NLO} | \vec{r}_1{}' \vec{r}_2{}' \rangle = \frac{\alpha_s^2(\mu) N_c^2}{4\pi^3} \Bigg[\delta(\vec{r}_{11'}) \delta(\vec{r}_{22'}) \int d\vec{\rho}\, g^0(\vec{r}_1, \vec{r}_2; \rho)$$

$$+ \delta(\vec{r}_{11'}) g(\vec{r}_1, \vec{r}_2; \vec{r}_2{}') + \delta(\vec{r}_{22'}) g(\vec{r}_2, \vec{r}_1; \vec{r}_1{}') + \frac{1}{\pi} g(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}') \Bigg] .$$

The functions g were calculated.

Möbius kernel

$$g^0(\vec{r}_1,\vec{r}_2;\vec{\rho}) = 2\pi\zeta(3)\delta\left(\vec{\rho}\right) - g(\vec{r}_1,\vec{r}_2;\vec{\rho})\ ,$$

$$g(\vec{r}_1,\vec{r}_2;\vec{r}_2^{\,\prime}) \ = \frac{11}{6}\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}\vec{r}_{12'}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{r_\mu^2}\right) + \frac{11}{6}\left(\frac{1}{\vec{r}_{22'}^{\,2}} - \frac{1}{\vec{r}_{12'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{22'}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)$$

$$+\frac{1}{2\vec{r}_{22'}^{\,2}}\ln\left(\frac{\vec{r}_{12'}^{\,2}}{\vec{r}_{22'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{2\,\vec{r}_{22'}^{\,2}\vec{r}_{12'}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right),$$

where

$$\ln r_\mu^2 = 2\psi\left(1\right) - \ln\frac{\mu^2}{4} - \frac{3}{11}\left(\frac{67}{9} - 2\zeta(2)\right),$$

and μ — is $\overline{MS}$ renormalization scale.
The function $g(\vec{r}_1,\vec{r}_2;\vec{r}_2^{\,\prime})$ is 0 when $\vec{r}_1 = \vec{r}_2$.
It has conformally noninvariant terms not proportional to beta-function.
In this form they do not coincide with NLO BK kernel.

Möbius kernel

$$g(\vec{r}_1,\vec{r}_2;\vec{r}_1{}',\vec{r}_2{}') =$$

$$= \frac{1}{2\vec{r}^{\,4}_{1'2'}}\left(\frac{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'} - 2\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{1'2'}}{d}\ln\left(\frac{\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\right) - 1\right) + \frac{\vec{r}^{\,2}_{12}\ln\left(\frac{\vec{r}^{\,2}_{11'}}{\vec{r}^{\,2}_{1'2'}}\right)}{2\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{12'}\,\vec{r}^{\,2}_{22'}}$$
$$+ \frac{\ln\left(\frac{\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\right)}{4\vec{r}^{\,2}_{11'}\,\vec{r}^{\,2}_{22'}}\left(\frac{\vec{r}^{\,4}_{12}}{d} - \frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{1'2'}}\right) + \frac{\ln\left(\frac{\vec{r}^{\,2}_{12}\,\vec{r}^{\,2}_{1'2'}}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\right)}{2\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}}\left(\frac{\vec{r}^{\,2}_{12}}{2\vec{r}^{\,2}_{1'2'}} + \frac{1}{2} - \frac{\vec{r}^{\,2}_{22'}}{\vec{r}^{\,2}_{1'2'}}\right)$$
$$+ \frac{\vec{r}^{\,2}_{21'}\ln\left(\frac{\vec{r}^{\,2}_{21'}\vec{r}^{\,2}_{1'2'}}{\vec{r}^{\,2}_{12}\,\vec{r}^{\,2}_{11'}}\right)}{2\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}\vec{r}^{\,2}_{1'2'}} + \frac{\ln\left(\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{1'2'}}\right)}{4\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}} + \frac{\ln\left(\frac{\vec{r}^{\,2}_{22'}}{\vec{r}^{\,2}_{12}}\right)}{2\vec{r}^{\,2}_{11'}\,\vec{r}^{\,2}_{12'}} + \frac{\vec{r}^{\,2}_{12}\ln\left(\frac{\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{1'2'}}{\vec{r}^{\,2}_{12'}\,\vec{r}^{\,2}_{21'}}\right)}{4\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}\vec{r}^{\,2}_{1'2'}}$$
$$+ \frac{\ln\left(\frac{\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{1'2'}}{\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{22'}}\right)}{2\vec{r}^{\,2}_{11'}\,\vec{r}^{\,2}_{1'2'}} + \frac{\ln\left(\frac{\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{11'}}{\vec{r}^{\,2}_{22'}\,\vec{r}^{\,2}_{1'2'}}\right)}{2\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{1'2'}} + (1 \leftrightarrow 2, 1' \leftrightarrow 2'),$$

where

$$d = \vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'} - \vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}.$$

This function is 0 when $\vec{r}_1 = \vec{r}_2$ and it also has nonconformal terms.

Freedom in the definition of the kernel

The discontinuity $\mathrm{disc}_s \mathcal{A}_{AB}^{A'B'}$

$$-4i(2\pi)^{D-2}\delta(\vec{q}_A - \vec{q}_B)\mathrm{disc}_s \mathcal{A}_{AB}^{A'B'} = \langle A'\bar{A}| \left(\frac{s}{s_0}\right)^{\hat{\mathcal{K}}} |\bar{B}'B\rangle$$

does not change if one changes the kernel via a nonsingular operator $\hat{\mathcal{O}}$,

$$\hat{\mathcal{K}} \to \hat{\mathcal{O}}^{-1}\hat{\mathcal{K}}\hat{\mathcal{O}} \ , \ \ \langle A'\bar{A}| \to \langle A'\bar{A}|\hat{\mathcal{O}} \ , \ \ |\bar{B}'B\rangle \to \hat{\mathcal{O}}^{-1}|\bar{B}'B\rangle \ .$$

In LO this operator is fixed by the requirement that LO BFKL kernel equals LO BK kernel. After fixing $\hat{\mathcal{O}}$ in the leading order, there is residual freedom $\hat{\mathcal{O}} = 1 - \hat{O}$, where $\hat{O} \sim g^2$. In NLO after these transformations we get

$$\hat{\mathcal{K}} \to \hat{\mathcal{K}} - [\hat{\mathcal{K}}^{(B)}, \hat{O}] \ ,$$

where $\hat{\mathcal{K}}^{(B)}$ — is LO kernel.

Operator to eliminate the difference of BFKL and BK kernels

- $$\langle \vec{q}_1, \vec{q}_2 | \hat{O} | \vec{q}_1{}', \vec{q}_2{}' \rangle = \langle \vec{q}_1, \vec{q}_2 | - \frac{1}{2} \hat{\mathcal{K}}_r^B \ln \hat{q}_{11'}^{\,2} | \vec{q}_1{}', \vec{q}_2{}' \rangle$$
$$+ \frac{\alpha_s N_c}{4\pi^2} \delta(\vec{q}_{22'}) \delta(\vec{q}_{11'}) \int d^{2+2\epsilon} k \ln \vec{k}^{\,2} \left(\frac{2}{\vec{k}^{\,2}} - \frac{\vec{k}(\vec{k}-\vec{q}_1)}{\vec{k}^{\,2}(\vec{k}-\vec{q}_1)^2} - \frac{\vec{k}(\vec{k}-\vec{q}_2)}{\vec{k}^{\,2}(\vec{k}-\vec{q}_2)^2} \right).$$

- With this operator the Möbius form of the transformed kernel
$$\hat{\mathcal{K}} - [\hat{\mathcal{K}}^{\,B}, \hat{O}],$$
coincides with the BK kernel (2010).

Quasi-conformal kernel

Transition to the composite dipole operators of Balitsky and Chirilli equivalent to the transformation of the kernel with the operator

$$\hat{\mathcal{K}} \to \hat{\mathcal{K}}^{QC} = \hat{\mathcal{K}} - [\hat{\mathcal{K}}^{B}, O_1],$$

where

$$\langle \vec{r}_1 \vec{r}_2 | \hat{O}_{1M} | \vec{r}_1{}' \vec{r}_2{}' \rangle =$$

$$= \frac{\alpha_s(\mu) N_c}{4\pi^2} \int d\vec{\rho} \frac{\vec{r}_{12}{}^2}{\vec{r}_{1\rho}^{\,2} \vec{r}_{2\rho}^{\,2}} \ln\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{1\rho}^{\,2} \vec{r}_{2\rho}^{\,2}}\right) \left[\delta(\vec{r}_{11'})\delta(\vec{r}_{2'\rho}) + \delta(\vec{r}_{1'\rho})\delta(\vec{r}_{22'}) - \delta(\vec{r}_{11'})\delta(r_{22'})\right],$$

It kills all nonconformal terms in the kernel which are not related to renormalization.

Quasi-conformal kernel in QCD

$$g_0(\vec{r}_1, \vec{r}_2; \vec{\rho}) = 6\pi\zeta\,(3)\,\delta\,(\vec{\rho}) - g_1(\vec{r}_1, \vec{r}_2; \vec{\rho})\ ,$$

$$g_1(\vec{r}_1, \vec{r}_2; \vec{r}_2^{\,\prime})\ =$$

$$= \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}\vec{r}_{12'}^{\,2}}\left[\frac{67}{18} - \zeta(2) - \frac{5n_f}{9N_c} + \frac{\beta_0}{2N_c}\ln\left(\frac{\vec{r}_{12}^{\,2}\mu^2}{4e^{2\psi(1)}}\right) + \frac{\beta_0}{2N_c}\frac{\vec{r}_{12'}^{\,2} - \vec{r}_{22'}^{\,2}}{\vec{r}_{12}^{\,2}}\ln\left(\frac{\vec{r}_{22'}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)\right],$$

$$g(\vec{r}_1, \vec{r}_2; \vec{r}_1^{\,\prime}, \vec{r}_2^{\,\prime}) = \frac{1}{\vec{r}_{1'2'}^{\,4}}\left(\frac{\vec{r}_{11'}^{\,2}\,\vec{r}_{22'}^{\,2} - 2\vec{r}_{12}^{\,2}\,\vec{r}_{1'2'}^{\,2}}{d}\ln\left(\frac{\vec{r}_{12'}^{\,2}\,\vec{r}_{21'}^{\,2}}{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}}\right) - 1\right)\left(1 + \frac{n_f}{N_c^3}\right)$$

$$+\left(\frac{3n_f}{2N_c^3}\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1'2'}^{\,2}d} + \frac{1}{2\vec{r}_{11'}^{\,2}\,\vec{r}_{22'}^{\,2}}\left(\frac{\vec{r}_{12}^{\,4}}{d} - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1'2'}^{\,2}}\right)\right)\ln\left(\frac{\vec{r}_{12'}^{\,2}\,\vec{r}_{21'}^{\,2}}{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}}\right)$$

$$+\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}\vec{r}_{1'2'}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}\vec{r}_{1'2'}^{\,2}}{\vec{r}_{12'}^{\,2}\,\vec{r}_{21'}^{\,2}}\right), \quad d = \vec{r}_{12'}^{\,2}\vec{r}_{21'}^{\,2} - \vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}.$$

Here $\beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f$.

Conformally invariant kernel in SUSY N=4

$$\langle \vec{r}_1\vec{r}_2|\hat{\mathcal{K}}_M|\vec{r}_1{}'\vec{r}_2{}'\rangle_{N=4} =$$

$$= \frac{\alpha_s N_c}{2\pi^2}\int d\vec{\rho}\frac{\vec{r}_{12}{}^2}{\vec{r}_{1\rho}^{\,2}\vec{r}_{2\rho}^{\,2}}\left[\delta(\vec{r}_{11'})\delta(\vec{r}_{2'\rho})+\delta(\vec{r}_{1'\rho})\delta(\vec{r}_{22'})-\delta(\vec{r}_{11'})\delta(r_{22'})\right]\left(1-\frac{\alpha_s N_c\zeta(2)}{2\pi}\right)$$

$$+\frac{\alpha_s^2 N_c^2}{4\pi^4}\left[\frac{1}{2\vec{r}^{\;2}_{\;11'}\,\vec{r}^{\;2}_{\;22'}}\left(\frac{\vec{r}^{\;4}_{\;12}}{\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}-\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}-\frac{\vec{r}^{\;2}_{\;12}}{\vec{r}^{\;2}_{\;1'2'}}\right)\ln\left(\frac{\vec{r}^{\,2}_{12'}\,\vec{r}^{\,2}_{21'}}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\right)\right.$$

$$\left.+\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\;2}_{\;11'}\vec{r}^{\;2}_{\;22'}\vec{r}^{\,2}_{1'2'}}\ln\left(\frac{\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{1'2'}}{\vec{r}^{\,2}_{12'}\,\vec{r}^{\,2}_{21'}}\right)+6\pi^2\zeta\left(3\right)\delta(\vec{r}_{11'})\delta(r_{22'})\right].$$

Restoration of full form from Möbius form

- Is it possible to restore the full kernel in the momentum space from its Möbius form in the coordinate space? Will it be unique?

- Yes, if we demand
  - gauge invariance

$$\mathcal{K}_r(\vec{q}_1, \vec{q}_2; \vec{k} = \vec{q}_1) = \mathcal{K}_r(\vec{q}_1, \vec{q}_2; \vec{k} = -\vec{q}_2) = 0.$$

  - absence of terms proportional to $\delta(\vec{q_1})$ or $\delta(\vec{q_2})$ in the kernel. It fixes the residual freedom connected with such terms.

Restoration of full form from Möbius form

- Fourier transform into momentum space

$$\langle \vec{q}_1, \vec{q}_2 | \hat{\mathcal{K}}_M | \vec{q}_1{}', \vec{q}_2{}' \rangle = \int \frac{d\vec{r}_1}{2\pi} \frac{d\vec{r}_2}{2\pi} \frac{d\vec{r}_1{}'}{2\pi} \frac{d\vec{r}_2{}'}{2\pi} e^{-i\vec{q}_1\vec{r}_1 - i\vec{q}_2\vec{r}_2 + i\vec{q}_1{}'\vec{r}_1{}' + i\vec{q}_2{}'\vec{r}_2{}'} \langle \vec{r}_1, \vec{r}_2 | \hat{\mathcal{K}}_M | \vec{r}_1{}', \vec{r}_2{}' \rangle$$

$$= \delta(\vec{q}_1 + \vec{q}_2 - \vec{q}_1{}' - \vec{q}_2{}') \mathcal{K}_M(\vec{q}_1, \vec{q}_2; \vec{k}),$$

where $\vec{k} = \vec{q}_1 - \vec{q}_1{}' = \vec{q}_2{}' - \vec{q}_2$ and

$$\mathcal{K}_M(\vec{q}_1, \vec{q}_2; \vec{k}) = \int \frac{d\vec{r}_{11'}}{2\pi} \frac{d\vec{r}_{22'}}{2\pi} d\vec{r}_{1'2'} e^{-i\vec{q}_1\vec{r}_{11'} - i\vec{q}_2\vec{r}_{22'} - i\vec{k}\vec{r}_{1'2'}} \langle \vec{r}_1, \vec{r}_2 | \hat{\mathcal{K}}_M | \vec{r}_1{}', \vec{r}_2{}' \rangle$$

- subtract singularity at $\vec{r}_{1'2'} = 0$.

$$\mathcal{K}_M(\vec{q}_1, \vec{q}_2; \vec{k})_- = \int \frac{d\vec{r}_{11'}}{2\pi} \frac{d\vec{r}_{22'}}{2\pi} d\vec{r}_{1'2'} e^{-i\vec{q}_1\vec{r}_{11'} - i\vec{q}_2\vec{r}_{22'} - i\vec{k}\vec{r}_{1'2'}} \langle \vec{r}_1, \vec{r}_2 | \hat{\mathcal{K}}_M^{NS} | \vec{r}_1{}', \vec{r}_2{}' \rangle$$

$$+ \int \frac{d\vec{r}_{11'}}{2\pi} \frac{d\vec{r}_{22'}}{2\pi} d\vec{r}_{1'2'} e^{-i\vec{q}_1\vec{r}_{11'} - i\vec{q}_2\vec{r}_{22'}} (e^{-i\vec{k}\vec{r}_{1'2'}} - 1) \langle \vec{r}_1, \vec{r}_2 | \hat{\mathcal{K}}_M^{S} | \vec{r}_1{}', \vec{r}_2{}' \rangle.$$

Restoration of full form from Möbius form

- since

$$\mathcal{K}_M(\vec{q}_1, \vec{q}_2; \vec{k})_- = \mathcal{K}(\vec{q}_1, \vec{q}_2; \vec{k})_-$$

$$-\delta(\vec{q}_2) f_1(\vec{q}_1, \vec{q}_2, \vec{k}) - \delta(\vec{q}_1)) f_2(\vec{q}_1, \vec{q}_2, \vec{k}),$$

  we should drop all terms proportional to $\delta(\vec{q}_1), \delta(\vec{q}_2)$ to get $\mathcal{K}(\vec{q}_1, \vec{q}_2; \vec{k})_-$.

- finally, to get the full kernel we should add to $\mathcal{K}(\vec{q}_1, \vec{q}_2; \vec{k})_-$ some terms independent of $\vec{k}$ so that

$$\mathcal{K}_r(\vec{q}_1, \vec{q}_2; \vec{k} = \vec{q}_1) = \mathcal{K}_r(\vec{q}_1, \vec{q}_2; \vec{k} = -\vec{q}_2) = 0.$$

Restoration of full form from Möbius form

- Suppose we have two full kernels with the same Möbius form. Then their difference will have zero Möbius form and

$$\langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}}^{(1)} | \vec{r}_1{}' \vec{r}_2{}' \rangle - \langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}}^{(2)} | \vec{r}_1{}' \vec{r}_2{}' \rangle \sim \delta(\vec{r}_{1\,'2\,'})$$

  i.e. the full kernels can differ only in terms independent of $\vec{k}$ in the real part.

- On the other hand, gauge invariance requires turning the difference $\mathcal{K}^{(1)}(\vec{q}_1, \vec{q}_2; \vec{k}) - \mathcal{K}^{(2)}(\vec{q}_1, \vec{q}_2; \vec{k})$ into zero at $\vec{k} = \vec{q}_1$ and at $\vec{k} = -\vec{q}_2$. Therefore, it is zero identically.

Full form for O_1

- Using this procedure we restored the full form for matrix element of O_1 in the momentum space

$$\langle\vec{q}_1,\vec{q}_2|\alpha_s\hat{O}_1|\vec{q}_1\,',\vec{q}_2\,'\rangle = \delta(\vec{q}_{11'}+\vec{q}_{22'})\frac{\alpha_s N_c}{4\pi^2}\left[\frac{2}{\vec{k}^{\,2}}\ln(\vec{k}^{\,2})+\frac{1}{\vec{q}_1^{\,2}}\ln\left(\frac{\vec{q}_1\,'^{\,2}\vec{q}_2^{\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}}\right)\right.$$

$$+\frac{1}{\vec{q}_2^{\,2}}\ln\left(\frac{\vec{q}_2\,'^{\,2}\vec{q}_1^{\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}}\right)+\frac{1}{\vec{k}^{\,2}}\ln\left(\frac{\vec{q}_1\,'^{\,2}\vec{q}_2\,'^{\,2}}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\right)-2\frac{\vec{q}_1\vec{k}}{\vec{k}^{\,2}\vec{q}_1^{\,2}}\ln\left(\vec{q}_1\,'^{\,2}\right)+2\frac{\vec{q}_2\vec{k}}{\vec{k}^{\,2}\vec{q}_2^{\,2}}\ln\left(\vec{q}_2\,'^{\,2}\right)$$

$$\left.-2\frac{\vec{q}_1\vec{q}_2}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\ln(\vec{q}^{\,2})\right]-\frac{\alpha_s N_c}{4\pi^2}\delta(\vec{q}_{22'})\delta\left(\vec{q}_{11'}\right)\int d\vec{l}\ln\vec{l}^{\,2}\left(\frac{2}{\vec{l}^{\,2}}-\frac{\vec{l}(\vec{l}-\vec{q}_1)}{\vec{l}^{\,2}(\vec{l}-\vec{q}_1)^2}\right.$$

$$\left.-\frac{\vec{l}(\vec{l}-\vec{q}_2)}{\vec{l}^{\,2}(\vec{l}-\vec{q}_2)^2}\right)-(\psi(1)+\ln 2)\left\langle\vec{q}_1,\vec{q}_2|\hat{\mathcal{K}}^B|\vec{q}_1\,',\vec{q}_2\,'\right\rangle\ ,$$

Summary

- We found quasi-conformal Möbius BFKL kernel in NLO in coordinate representation
- We showed that the NLO BFKL kernel coincides with the linearized NLO BK kernel
- We showed that it is possible to restore full kernel in momentum space from its Möbius form