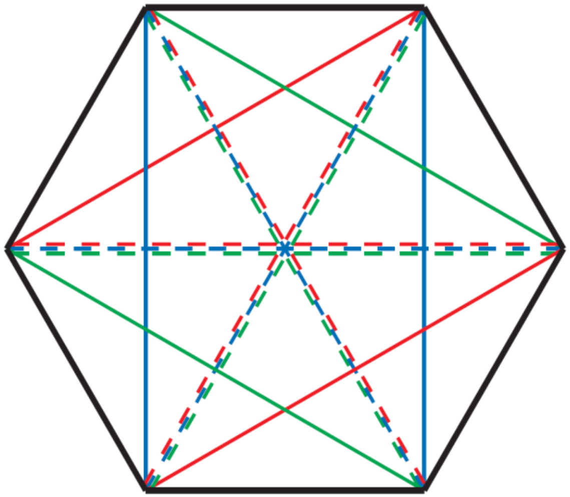
Bootstrapping the 3-loop Hexagon

or

The Lost Symbol



Lance Dixon (SLAC)

with J. Drummond and J. Henn

arXiv:1108.4461 [hep-th]

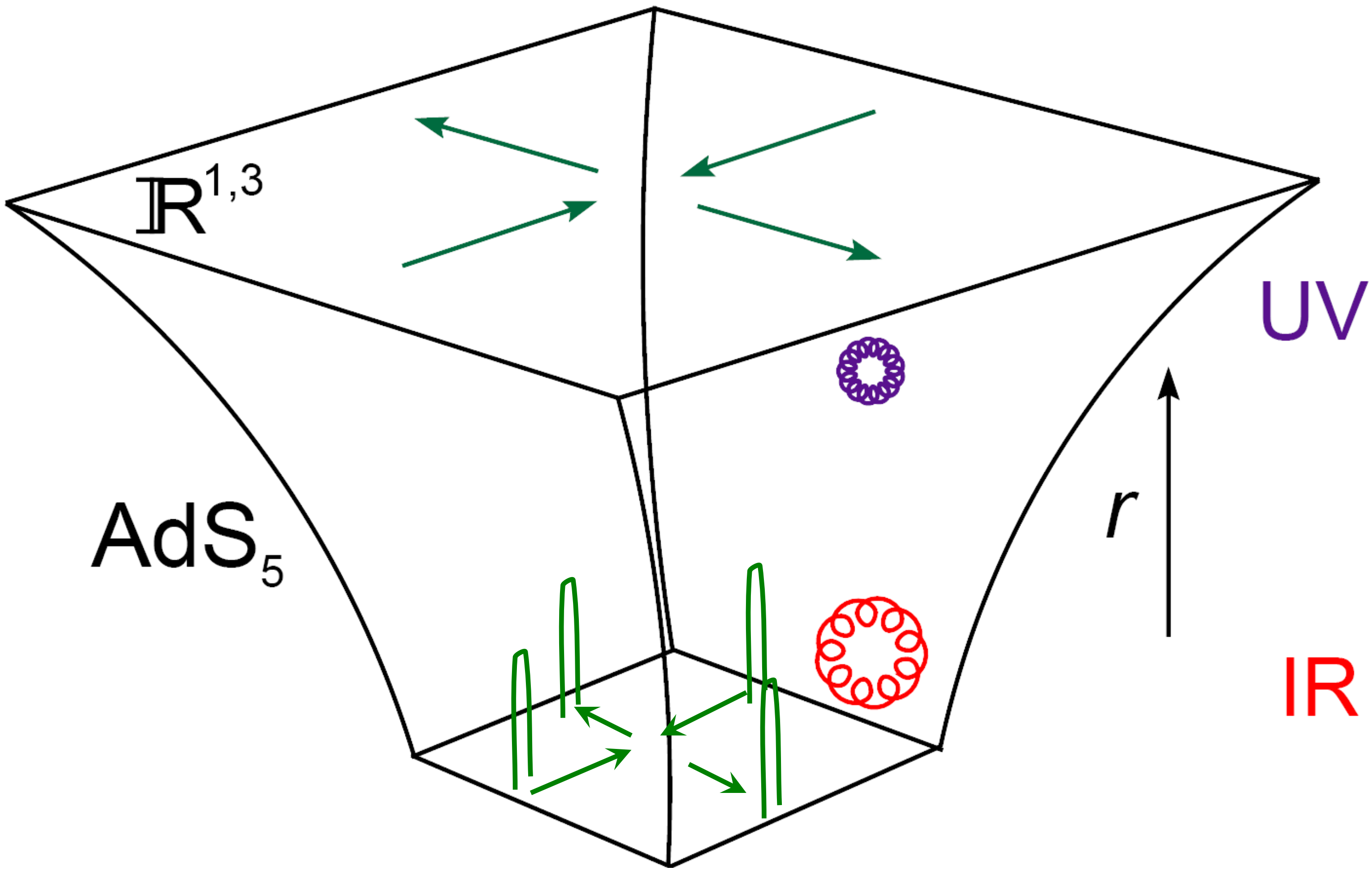
Frontiers in QCD Workshop

INT, Seattle, Sept. 29, 2011

Scattering amplitudes in planar N=4 Super-Yang-Mills

- Planar (large N_c) N=4 SYM is a 4-dimensional analog of QCD, (potentially) solvable to all orders in $\lambda = g^2 N_c$
- It can teach us what types of mathematical structures will enter multi-loop QCD amplitudes
- Its amplitudes have remarkable hidden symmetries
- In strong-coupling, large λ limit, AdS/CFT duality maps the problem into weakly-coupled gravity/semi-classical strings moving on $AdS_5 \times S^5$

AdS/CFT in one picture



Remarkable, related structures recently unveiled in planar N=4 SYM scattering

- Exact exponentiation of 4 & 5 gluon amplitudes
- Dual (super)conformal invariance
- Amplitudes equivalent to Wilson loops
- Strong coupling and “soap bubbles”

Outstanding question:

Can these structures be used to solve exactly for all planar N=4 SYM amplitudes?

Exact exponentiation

Bern, LD, Smirnov, hep-th/0505205

Inspired by IR structure of QCD, Mueller, Collins, Sen, Magnea, Sterman,…
based on evidence collected at 2 and 3 loops for *n=4,5* using
generalized unitarity and factorization, **we proposed an ansatz**:

$$\mathcal{A}_n^{\rm BDS} = \mathcal{A}_n^{\rm tree} \times \exp\Big[\sum_{l=1}^{\infty} \big[\frac{\lambda}{8\pi^2}\big]^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon; s_{ij}) + C^{(l)} + \mathcal{O}(\epsilon)\Big)\Big]$$

constants, indep.of kinematics

all kinematic dependence in known 1-loop amplitude (normalized by tree)

n=4 $\Rightarrow$ $$\mathcal{M}_4|_{\rm finite} = \exp\Big[\frac{1}{8}\gamma_K(\lambda)\, \ln^2\Big(\frac{s}{t}\Big) + {\rm const.}\Big]$$

Alday Maldacena 0705.0303 0710.1060

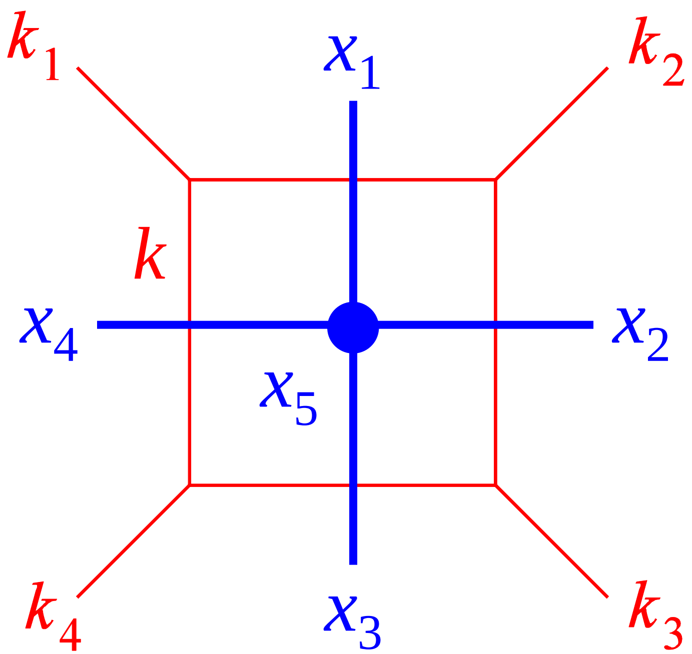
Confirmed at strong coupling using AdS/CFT, directly at *n=4*, indirectly at *n=5*. Fails for *n > 5*.

Dual conformal invariance

Broadhurst (1993); Lipatov (1999); Drummond, Henn, Smirnov, Sokatchev, hep-th/0607160

Conformal symmetry acting in momentum space, on dual or sector variables x_i

First seen in N=4 SYM planar amplitudes in the loop integrals



$$I = \int d^4k \frac{(k_1+k_2)^2(k_2+k_3)^2}{k^2(k-k_1)^2(k-k_1-k_2)^2(k+k_4)^2}$$

$$I = \int d^4x_5 \frac{x_{24}^2 x_{13}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

$$k_1 = x_{41}$$
$$k_2 = x_{12}$$
$$k_3 = x_{23}$$
$$k_4 = x_{34}$$
$$k = x_{45}$$

invariant under inversion:

$$x_i^\mu \rightarrow \frac{x_i^\mu}{x_i^2}$$

$$x_{ij}^2 \rightarrow \frac{x_{ij}^2}{x_i^2 x_j^2}, \qquad \mathrm{d}^4 x_i \rightarrow \frac{\mathrm{d}^4 x_i}{x_i^8}$$

Dual conformal constraints

• Symmetry fixes form of amplitude, up to functions of dual conformally invariant cross ratios:

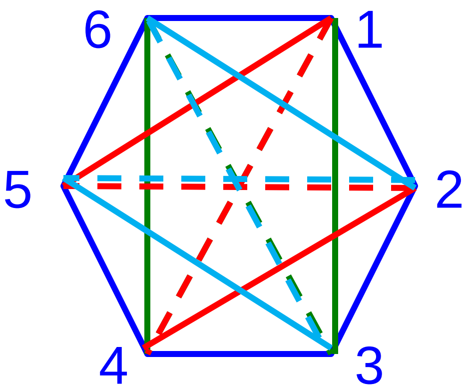
$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

• Because $x_{i-1,i}^2 = k_i^2 = 0$ there are no such variables for n=4,5 (after all loop integrations are performed).

• For n=6, there are precisely 3 ratios:

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

+ 2 cyclic perm's

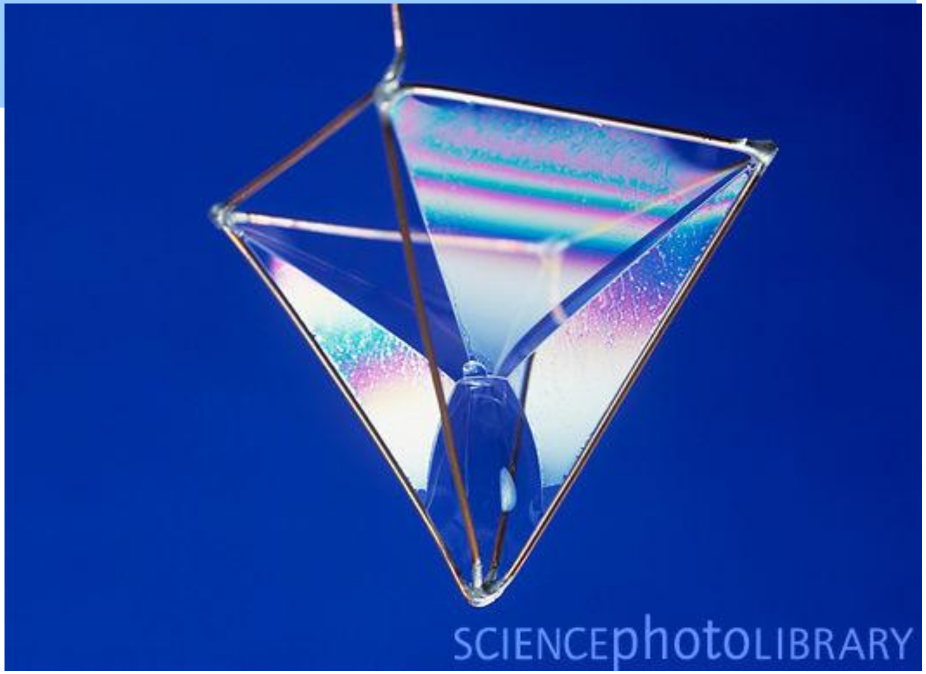


Strong coupling and soap bubbles

Alday, Maldacena, 0705.0303

- Use AdS/CFT to compute scattering amplitude
- High energy scattering in string theory semi-classical: two-dimensional string world-sheet stretches a long distance, classical solution minimizes area

Gross, Mende (1987,1988)



Classical action imaginary
→ exponentially suppressed tunnelling configuration

$$A_n \sim \exp[-\sqrt{\lambda} S_{\rm cl}^{\rm E}]$$

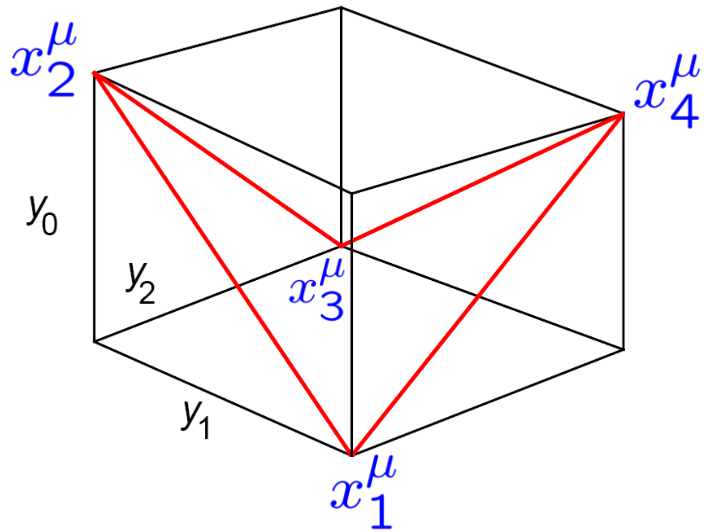
Dual variables and strong coupling

• **Soap bubble boundaries:** polygons composed of light-like segments with length equal to the gluon momenta k_i^μ

• Corners (cusps) at x_i^μ – same variables used to describe dual conformal invariance.
• Strong-coupling problem lives in AdS_5 – isometries include conformal group.
→ answer automatically dual conformal invariant!

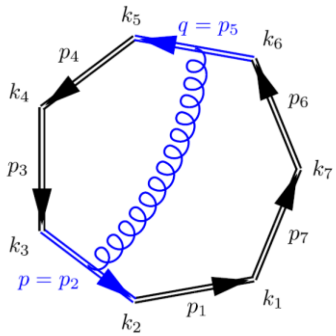
gg → *gg* boundary :

$$k_1 = x_{41}$$
$$k_2 = x_{12}$$
$$k_3 = x_{23}$$
$$k_4 = x_{34}$$

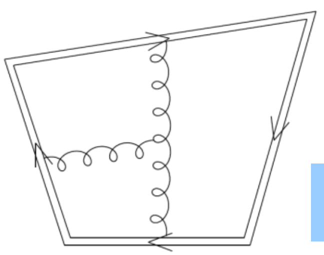


Wilson loops at weak coupling

Computed for same “soap bubble” boundary conditions as scattering amplitude:



- One loop, *n=4* — Drummond, Korchemsky, Sokatchev, 0707.0243
- One loop, any *n* — Brandhuber, Heslop, Travaglini, 0707.1153
- Two loops, *n=4,5,6* — Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466; Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465



Wilson-loop VEV **always matches** [MHV] scattering amplitude!

Weak-coupling properties linked to superconformal invariance for strings in $AdS_5 \times S^5$ under combined bosonic and fermionic T duality symmetry
Berkovits, Maldacena, 0807.3196; Beisert, Ricci, Tseytlin, Wolf, 0807.3228

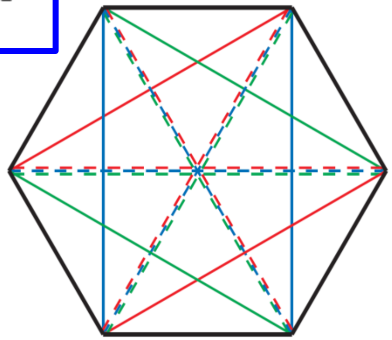
Beyond five gluons

- BDS ansatz correct for $n = 4,5$ to all loops, as a consequence of dual conformal invariance
- $n = 6$ first place it must be modified, due to cross ratios

$$u = u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \qquad v = u_2 = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2} \qquad w = u_3 = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2}$$

$$\mathcal{A}_6(\epsilon; s_{ij}) \;=\; \mathcal{A}_6^{\text{BDS}}(\epsilon; s_{ij}) \exp[R_6(u_1, u_2, u_3)]$$

"Remainder function", first appears at 2 loops, n=6.
Obstruction to solving (MHV sector of) N=4 SYM.



Need for $R_6^{(2)}(u_1,u_2,u_3)$

- Modification of BDS ansatz for $n = 6$ was suspected, based on:
- A large *n*, strong-coupling limit Alday, Maldacena, 0710.1060
- A 2-loop Wilson-loop calculation

  Drummond, Henn, Korchemsky, Sokatchev, 0712.4138
- A high-energy/Regge limit

  Bartels, Lipatov, Sabio Vera, 0802.2065

- Confirmed by a direct amplitude calculation

  Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

that matched the Wilson loop numerically

Drummond, Henn, Korchemsky, Sokatchev, 0803.1466

Formula for $R_6^{(2)}(u_1,u_2,u_3)$

- First worked out analytically from Wilson loop integrals

Del Duca, Duhr, Smirnov, 0911.5332, 1003.1702

17 pages of Goncharov polylogarithms.

- Simplified to just a few classical polylogarithms using symbology

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\,\mathrm{Li}_4(1-1/u_i)\right)$$
$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

$$L_4(x^+,x^-) = \frac{1}{8!!}\log(x^+x^-)^4$$
$$+\sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+)+\ell_{4-m}(x^-))$$

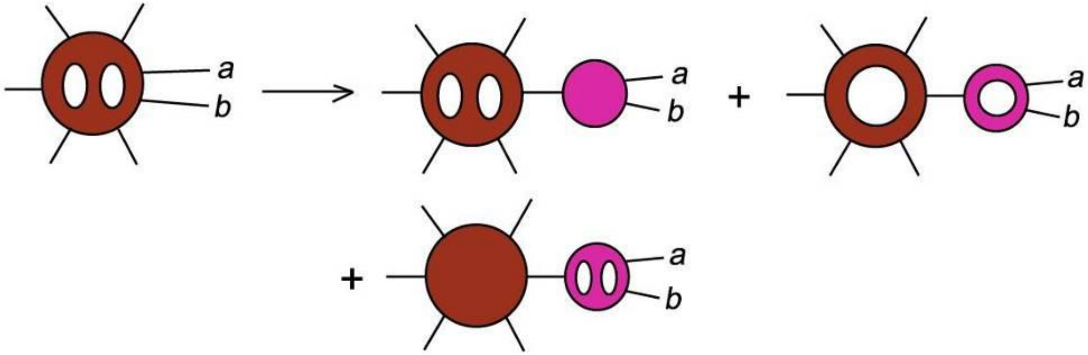
$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n\,\mathrm{Li}_n(1/x)\right)$$
$$J = \sum_{i=1}^{3}(\ell_1(x_i^+) - \ell_1(x_i^-))$$

$$x_i^\pm = u_i x^\pm, \qquad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3} \qquad \Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3$$

Wilson loop OPEs

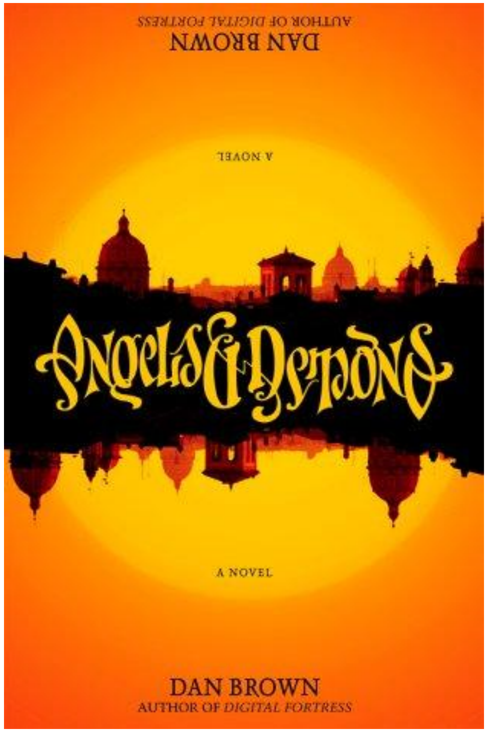
Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788; GMSV, 1010.5009, 1102.0062

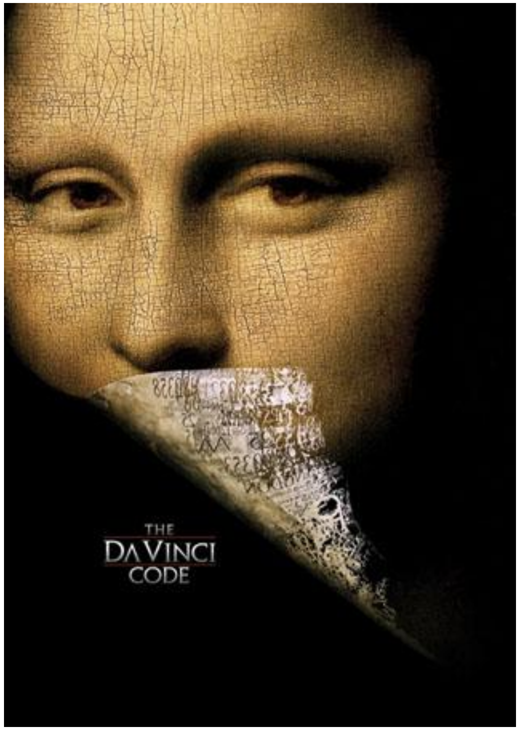
- Remarkably, $R_6^{(2)}(u_1, u_2, u_3)$ can be recovered directly from analytic properties, using "near collinear limits"



- Wilson-loop equivalence → this limit is controlled by an operator product expansion (OPE)
- Here, show how to go to 3 loops, by combining the OPE expansion with symbology

Professor of symbology at Harvard University, has used these techniques to make a series of important advances:





What is symbology?

• Multi-loop integrals generate complicated transcendental functions, iterated integrals that are generalizations of the ordinary polylogarithm:

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) \qquad \mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t)$$

• The symbol $S[f]$ of a function f remembers "important" properties of f, like derivatives and locations of branch cuts, while forgetting other properties, like precise integration contours and numerical values, that can be reconstructed later.

• It trivializes complicated polylogarithmic identities.

Iterated differentiation

- A pure function $f^{(k)}$ of transcendental degree k is a linear combination of k-fold iterated integrals, with constant (rational) coefficients.
- We can also add terms like $\zeta(p) \times f^{(k-p)}$
- Derivatives of $f^{(k)}$ can be written as

$$d\, f^{(k)} = \sum_r f_r^{(k-1)}\, d \log \phi_r$$

for a finite set of algebraic functions ϕ_r
- Define the symbol $\mathcal{S}$ [Goncharov, 0908.2238] recursively in k:

$$\mathcal{S}(f^{(k)}) = \sum_r \mathcal{S}(f_r^{(k-1)}) \otimes \phi_r$$

Polylog examples

- By definition, $\mathcal{S}[\ln x] = x \qquad \mathcal{S}[\ln(1-x)] = 1-x$
- If derivative is known, symbol is known:

$$\frac{d}{dx}\text{Li}_2(x) = -\frac{\ln(1-x)}{x} \quad \Rightarrow \quad \mathcal{S}[\text{Li}_2(x)] = -[(1-x)\otimes x]$$

$$\frac{d}{dx}\text{Li}_n(x) = \frac{\text{Li}_{n-1}(x)}{x} \quad \Rightarrow \quad \mathcal{S}[\text{Li}_n(x)] = -[(1-x)\otimes \underbrace{x\otimes\ldots\otimes x}_{n-1}]$$

- Symbols of products are mergings of symbols of factors:

$$\mathcal{S}[\ln(x)\ \ln(1-x)] = x\otimes(1-x) \ + \ (1-x)\otimes x$$

$$\begin{aligned}
&\mathcal{S}[\text{Li}_2(x)\,\text{Li}_2(y)] \\
&= (1-x)\otimes x\otimes(1-y)\otimes y \ + \ (1-x)\otimes(1-y)\otimes x\otimes y \\
&+ (1-x)\otimes(1-y)\otimes y\otimes x \ + \ (1-y)\otimes(1-x)\otimes x\otimes y \\
&+ (1-y)\otimes(1-x)\otimes y\otimes x \ + \ (1-y)\otimes y\otimes(1-x)\otimes x
\end{aligned}$$

Polylog identities at symbol level

- A well-known identity:

$$\text{Li}_2(1-x) = \frac{\pi^2}{6} - \ln x \ln(1-x) - \text{Li}_2(x)$$

- Take symbol of it:

$$\mathcal{S}[\text{Li}_2(1-x)] = \mathcal{S}[\pi^2/6] - \mathcal{S}[\ln(x)\ \ln(1-x)] - \mathcal{S}[\text{Li}_2(x)]$$

$$-x \otimes (1-x) = 0 \quad - \ x \otimes (1-x) - (1-x) \otimes x \quad + \ (1-x) \otimes x$$

- Biggest virtue of the symbol: It transforms all identities between multi-variable transcendental functions into simple algebraic identities

Elementary symbol properties

- **Factorization:**

$$\ldots \otimes xy \otimes \ldots \;=\; \ldots \otimes x \otimes \ldots \;+\; \ldots \otimes y \otimes \ldots$$

- **Integrability:**

Not every (multi-variable) symbol is a function

$$\mathcal{S}[\ln(x)\ln(y)] = x \otimes y \;+\; y \otimes x$$

but no function has symbol $x \otimes y \;-\; y \otimes x$

- Integrability test [Goncharov; GMSV, 1102.0062] :

$$\phi_1 \otimes \ldots \otimes \phi_i \otimes \phi_{i+1} \otimes \ldots \otimes \phi_k$$

$$\to \; d\ln\phi_i \wedge d\ln\phi_{i+1} \;\; \phi_1 \otimes \ldots \otimes \ldots \otimes \phi_k$$

$\Rightarrow$ 0 for symbols of functions

What entries should the symbol have?

- For the hexagon problem, we assume the entries can all be drawn from the set:

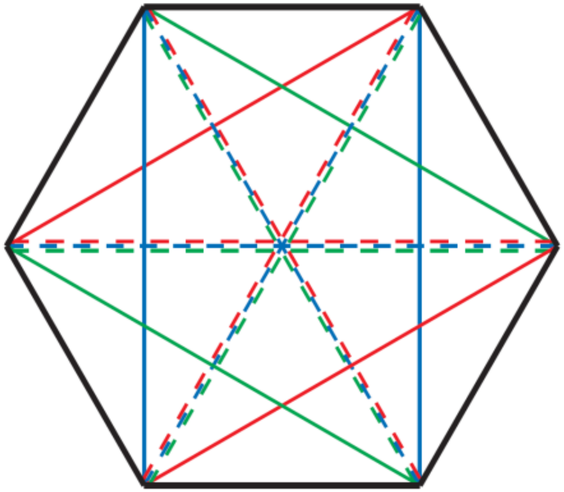
$$\{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

with

$$y_u \equiv \frac{u - z_+}{u - z_-} \qquad + \quad \text{perms}$$

$$z_\pm = \frac{1}{2}\left[-1 + u + v + w \pm \sqrt{\Delta}\right]$$

$$\Delta = (1 - u - v - w)^2 - 4uvw$$



$\mathcal{S}[\, R_6^{(2)}(u,v,w)\,]$ in these variables

GSVV, 1006.5703

$$
\begin{aligned}
-8\,\mathcal{S}[R_6^{(2)}] \;=\;& u \otimes (1-u) \otimes \frac{u}{(1-u)^2} \otimes \frac{u}{1-u} \\
&+ 2\,(u \otimes v + v \otimes u) \otimes \frac{w}{1-v} \otimes \frac{u}{1-u} \\
&+ 2\, v \otimes \frac{w}{1-v} \otimes u \otimes \frac{u}{1-u} \\
&+ u \otimes (1-u) \otimes y_u y_v y_w \otimes y_u y_v y_w \\
&- 2\, u \otimes v \otimes y_w \otimes y_u y_v y_w \\
&+ 5 \text{ permutations of } (u,v,w)
\end{aligned}
$$

First entry

- Always drawn from $\{u, v, w\}$ GMSV, 1102.0062
- This is because first entry controls branch-cut location
- Only massless particles

→ all cuts start at origin in $s_{i,i+1}$, $s_{i,i+1,i+2}$

→ Branch cuts all start from 0 or ∞ in

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}^2 s_{45}^2}{s_{123}^2 s_{345}^2}$$

Final entry

- Always drawn from $\left\{\frac{u}{1-u}, \frac{v}{1-v}, \frac{w}{1-w}, y_u, y_v, y_w\right\}$

- Have seen this in the structure of various Feynman integrals [e.g. from Arkani-Hamed et al., 1108.2958] related to amplitudes Drummond, Henn, Trnka 1010.3679; LD, Drummond, Henn, 1104.2787, V. Del Duca et al., 1105.2011
- Same condition also arrived at via recent approach to supersymmetric Wilson loops Caron-Huot, 1105.5606

- We also did the analysis with the full 9 final entries

Ansatz for $S[\, R_6^{(3)}(u,v,w)\,]$

$$
\begin{matrix} \\ \\ \\ u \\ v \\ w \\ \\ \\ \\ \end{matrix}
\otimes
\begin{matrix} u \\ v \\ w \\ 1-u \\ 1-v \\ 1-w \\ y_u \\ y_v \\ y_w \end{matrix}
\otimes
\begin{matrix} u \\ v \\ w \\ 1-u \\ 1-v \\ 1-w \\ y_u \\ y_v \\ y_w \end{matrix}
\otimes
\begin{matrix} u \\ v \\ w \\ 1-u \\ 1-v \\ 1-w \\ y_u \\ y_v \\ y_w \end{matrix}
\otimes
\begin{matrix} u \\ v \\ w \\ 1-u \\ 1-v \\ 1-w \\ y_u \\ y_v \\ y_w \end{matrix}
\otimes
\begin{matrix} \frac{u}{1-u} \\ \frac{v}{1-v} \\ \frac{w}{1-w} \\ y_u \\ y_v \\ y_w \end{matrix}
$$

$3 \times 9^4 \times 6 = 118098$ parameters before imposing any constraints

Generic Constraints

- **Integrability** (immediately forbids y_u, y_v, y_w from second entry)
- S_3 permutation **symmetry** in $\{u, v, w\}$
- Even under "**parity**": every term must have an even number of y_i – 0, 2 or 4

$$i\sqrt{\Delta} \leftrightarrow -i\sqrt{\Delta}$$
$$z_+ \leftrightarrow z_-$$
$$y_i \leftrightarrow 1/y_i$$

- Vanishing in **collinear** limit $v \to 0$

$$y_u \to \frac{u}{1-w} \qquad y_v \to \frac{v(1-u)(1-w)}{(1-u-w)^2} \qquad y_w \to \frac{w}{1-u}$$

followed by $w \to 1-u$

- These 4 constraints reduce 118,098 → 35 free parameters

OPE Constraints

Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788; GMSV, 1010.5009; 1102.0062

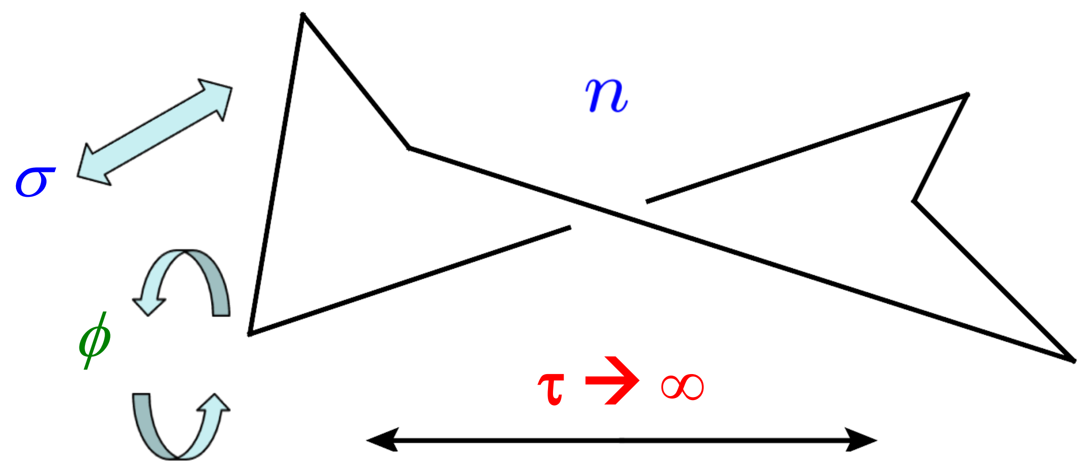
- Although $R_6^{(L)}(u,v,w)$ vanishes in the collinear limit,

$v = 1/\cosh^2\tau \to 0 \qquad\qquad \tau \to \infty$

in the **near**-collinear limit, its behavior is described by an Operator Product Expansion, with generic form

$$R_6^{(L)}(u,v,w) = R_6^{(L)}(\tau,\sigma,\phi) \sim \int dn\ C_n(g)\ \exp[-E_n(g)\tau]$$

$$\begin{aligned} u &= \frac{e^{\sigma}\sinh\tau\tanh\tau}{2(\cosh\sigma\ \cosh\tau + \cos\phi)} \\ v &= \frac{1}{\cosh^2\tau} \\ w &= u\, e^{-2\sigma} \end{aligned}$$



OPE Constraints (cont.)

- Using conformal invariance, send one long line to ∞, put other one along x^-
- Dilatations, boosts, azimuthal rotations preserve this configuration.
- σ,ϕ parametrize isometries, so classify conformal primaries by conjugate variables (twist p, spin m)
- Also expand anomalous dimensions in coupling g^2:

$$E_n(g) = E_n^{(0)} + g^2\, E_n^{(1)} + g^4\, E_n^{(2)} + \ldots$$

$$\exp[-E_n(g)\tau]$$
$$= \exp[-E_n^{(0)}\tau] \times \left[1 - g^2\,\tau\, E_n^{(1)} + g^4 (\frac{1}{2}\tau^2\, [E_n^{(1)}]^2 - \tau\, E_n^{(2)}) + \ldots\right]$$

- **Leading τ^{L-1} dependence of $R_6^{(L)}$ needs only one-loop anomalous dimension $E_n^{(1)}$**

OPE Constraints (cont.)

- As $\tau \to \infty$, $v = 1/\cosh^2\tau$ $\to$ $\tau^{L\text{-}1} \sim [\ln v]^{L\text{-}1}$
- Extract this piece from the symbol by only keeping terms with L-1 leading v entries

$$\underbrace{v \otimes \ldots \otimes v}_{\text{clip } L-1 \text{ entries}} \otimes \underbrace{\ldots}_{\text{keep } L+1 \text{ entries}}$$

$$\Delta_v^{L-1} R_6^{(L)} \propto \int dp e^{-ip\sigma} \Big[\sum_{m=1}^{\infty} \frac{[\gamma_{m+2}(p)]^{L-1} \cos(m\phi)}{p^2+m^2} + \sum_{m=2}^{\infty} \frac{[\gamma_{m-2}(p)]^{L-1} \cos((m-2)\phi)}{p^2+(m-2)^2} \Big] \times \mathcal{C}_m(p) \mathcal{F}_{m/2,p}(\tau)$$

where $\gamma_m(p) = \psi(\frac{m+ip}{2}) + \psi(\frac{m-ip}{2}) - 2\,\psi(1)$

Basso 1010.5237

First OPE Constraint

- Although $\Delta_v^2 R_6^{(3)}$ itself is rather complicated, we can easily generalize some analysis of $\Delta_v R_6^{(2)}$ in GMSV, 1102.0062 , which involves acting with various differential operators – easily applied to our symbol-level ansatz.
- We imposed 2 conditions.

1) $$\mathcal{S}[\,\mathcal{D}_+\mathcal{D}_-\Delta_v^2 R_6^{(3)}(u,v,w)\,] = 0$$

where the annihilators of the two conformal blocks are:

$$\mathcal{D}_\pm = \frac{4}{1-v}\Big[-z_\pm u\partial_u - (1-v)v\partial_v - z_\pm w\partial_w$$
$$+ (1-u)vu\partial_u u\partial_u + (1-v)^2 v\partial_v v\partial_v + (1-w)vw\partial_w w\partial_w$$
$$+ (-1+u-v+w)((1-v)u\partial_u v\partial_v - vu\partial_u w\partial_w + (1-v)v\partial_v w\partial_w)\Big]$$

Second OPE Constraint

2) $\mathcal{S}[\Box\Delta_w^2\Delta_v^2 R_6^{(3)}(u,v,w)] \;\propto\; \mathcal{S}[\Box\Delta_w\Delta_v R_6^{(2)}(u,v,w)]$

$$= \frac{w(1-u+v-w)}{(1-v)(1-w)}$$

where

$$\Box = -(\partial_\sigma^2 + \partial_\phi^2)$$
$$= \frac{4uw}{1-v}[u\partial_u + w\partial_w - (1-u)\partial_u u\partial_u - (1-w)\partial_w w\partial_w + (1-u-v-w+2uw)\partial_u\partial_w]$$

removes the $p^2 + m^2$ denominator factor in $\Delta_v^{L-1} R_6^{(L)}$

Solution to Constraints

• OPE constraints 1) and 2) are mutually consistent, and reduce the symbol ansatz to just 2 parameters:

$$\mathcal{S}[R_6^{(3)}] = \mathcal{S}[X] + \alpha_1\,\mathcal{S}[f_1] + \alpha_2\,\mathcal{S}[f_2]$$

• If we had not imposed the final-entry condition, there would have been 24 more parameters/functions.

• $f_{1,2}$ have no double-v discontinuity, so they cannot be determined from the OPE without putting in (considerably) more information than $E_n^{(1)}$

• Note that at 2 loops, $\Delta_v R_6^{(2)}$ **uniquely** determines $R_6^{(2)}$ thanks to first-entry condition and symmetry

Reconstructing functions

- $\mathcal{S}[f_1]$ is only made from $\{u, v, w, 1-u, 1-v, 1-w\}$ and is so simple we can integrate it in terms of [harmonic] polylogarithms of a single variable:

$$f_1(u,v,w) \;=\; h(u)h(v) + h(u)h(w) + h(v)h(w) + k(u) + k(v) + k(w)$$

$$h(u) = \frac{1}{3}\ln^3 u + \ln u\,\mathrm{Li}_2(1-u) - \mathrm{Li}_3(1-u) - 2\,\mathrm{Li}_3(1-1/u)$$

$$\begin{aligned} k(u) = {} & -\ln^3 u\, H_3 + \frac{3}{2}\ln^2 u\,(H_4 - H_{2,2} - 4\,H_{3,1}) \\ & - \log u\,(H_{2,3} - 6\,H_{4,1} + H_{2,1,2} + 6\,H_{2,2,1} + 18\,H_{3,1,1}) \\ & + 3\,H_{2,4} + 4\,H_{3,3} + 3\,H_{4,2} + H_{2,1,3} - H_{2,2,2} - 2\,H_{2,3,1} \\ & - 2\,H_{3,1,2} + 9\,H_{4,1,1} - 2\,H_{2,1,2,1} - 9\,H_{2,2,1,1} - 24\,H_{3,1,1,1} \end{aligned}$$

Reconstructing functions (cont.)

- Terms in $\mathcal{S}[f_2]$ can contain y_i in the form

$$a_1 \otimes a_2 \otimes a_3 \otimes a_4 \otimes y_1 \otimes y_2$$

with
$$a_i \in \{u, v, w, 1-u, 1-v, 1-w\}$$
$$y_i \in \{y_u, y_v, y_w\}$$

- We think f_2 is not much more complicated than $R_6^{(2)}$ (at least one way of writing it)
- Terms in $\mathcal{S}[X]$ can have up to four y_i

$$a_1 \otimes a_2 \otimes y_1 \otimes y_2 \otimes y_3 \otimes y_4$$

X will be harder to integrate, but you are welcome to have a go (we provide the 12,504 term symbol at arXiv)

How to determine the α_i?

- We reconstructed an "ultra-pure" function f_1 obeying

$$\partial_u f_1(u,v,w) = \frac{1}{u(1-u)}[\text{pure function}]$$

the functional equivalent of the final-entry condition

$$\mathcal{S}[f_1] = \ldots \otimes \frac{u}{1-u}$$

- The collinear limit of f_1 diverges beyond symbol level:

$$\lim_{v\to 0} f_1 = \zeta_2[\ln w(\frac{1}{2}\ln u \ln^2(1-u) + \ln u \mathrm{Li}_2(u) \\ +2\ln(1-u)\mathrm{Li}_2(u) - 3\mathrm{Li}_3(u) + 3H_{2,1}(u)) + \ldots$$

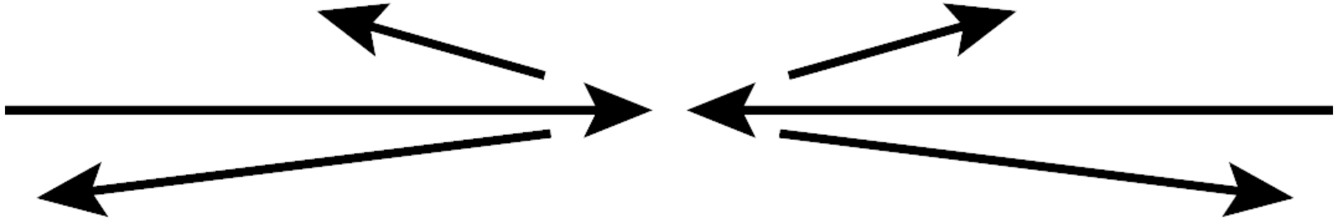
- Curiously, this behavior cannot be cured by

$$\zeta_2 \times [\text{ultra-pure degree 4 function}]$$

- Optimistically, it will only be cured by $X,\ f_2$ → **fix** $\alpha_{1,2}$

The multi-Regge limit

- One kinematic region in which we **can already** integrate the symbol is the so-called multi-Regge kinematics, with large rapidity separations between the 4 final-state gluons:

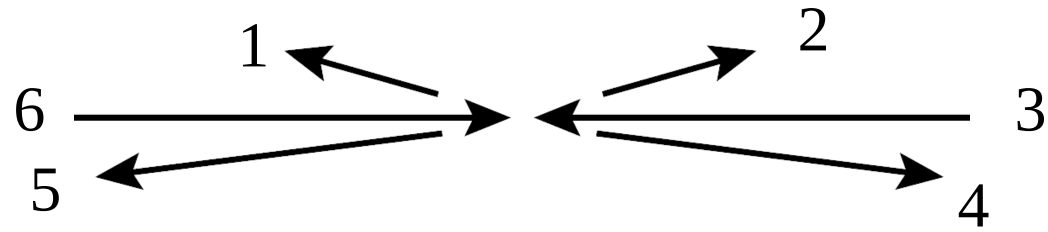


- Properties of the planar N=4 SYM amplitude in this limit have been studied extensively already:

Bartels, Lipatov, Sabio Vera, 0802.2065, 0807.0894; Lipatov, 1008.1015; Lipatov, Prygarin, 1008.1016, 1011.2673; Bartels, Lipatov, Prygarin, 1012.3178, 1104.4709.

Multi-Regge kinematics

$$u = \frac{s_{12}^2 s_{45}^2}{s_{123}^2 s_{345}^2} \to 1$$



$$\frac{v}{1-u} \to x$$

$$\frac{w}{1-u} \to y$$

And a very nice change of variables [LP, 1011.2673] is to (w, w^*) :

$$x = \frac{1}{(1+w)(1+w^*)}$$

$$y = \frac{ww^*}{(1+w)(1+w^*)}$$

$$y_u \to 1$$

$$y_v \to \frac{1+w^*}{1+w}$$

$$y_w \to \frac{(1+w)w^*}{w(1+w^*)}$$

2 symmetries: conjugation $w \leftrightarrow w^*$
and inversion $w \leftrightarrow 1/w, \;\; w^* \leftrightarrow 1/w^*$

Physical 2→4 multi-Regge limit

- If the multi-Regge limit is approached from the Euclidean side, the remainder function vanishes

Brower et al., 0801.3891; Del Duca, Duhr, Glover, 0809.1822

- To get a nonzero result, for the physical region, one must first let $u \to u\, e^{-2\pi i}$, by clipping either one or two u entries (for L <4) from front of symbol, replacing them by $-2\pi i$

$$R_6^{(L)} \to (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u)\, [g_r^{(L)}(w,w^*) + 2\pi i\, h_r^{(L)}(w,w^*)]$$

Three-loop results

- All classical polylogarithms in this limit

$$g_2^{(3)}(w,w^*) = \frac{1}{8}\, g_0^{(2)}(w,w^*) - \frac{1}{32} \log|1+w|^2 \log\frac{|1+w|^2}{|w|^2} \log\frac{|1+w|^4}{|w|^2}$$

LLA, agrees with LP, 1011.2673

$$\begin{aligned} g_1^{(3)}(w,w^*) \;=\; & \frac{1}{8}\Bigg\{ \log|w|^2 \left[\mathrm{Li}_3\left(\frac{w}{1+w}\right) + \mathrm{Li}_3\left(\frac{w^*}{1+w^*}\right)\right] \\ & + (5\log|1+w|^2 - 2\log|w|^2)\left[\mathrm{Li}_3(-w) + \mathrm{Li}_3(-w^*)\right] \\ & - \frac{3}{2}\log|w|^2 \log\frac{|1+w|^4}{|w|^2}\left[\mathrm{Li}_2(-w) + \mathrm{Li}_2(-w^*)\right] \\ & - \frac{1}{12}\log^2|1+w|^2\left[\log|w|^2(\log|w|^2 + 2\log|1+w|^2) - 10\log^2\frac{|1+w|^2}{|w|^2}\right] \\ & + \frac{1}{2}\log|w|^2 \log\frac{|1+w|^2}{|w|^2}\log(1+w)\log(1+w^*) - 2\zeta_3\log|1+w|^2\Bigg\} \\ & + \left(\frac{5}{2} + \gamma'\right)\zeta_2\, g_1^{(2)}(w,w^*) + \ldots \end{aligned}$$

NLLA, new

beyond-the-symbol ambiguity

vanishes when final-entry condition imposed

Three-loop results (cont.)

- Degree 5 NNLLA

- We also get the real parts $h_r^{(L)}$. Together they satisfy (for $c=0$!) an all-orders relation, based on a dispersion relation for 3 → 3 multi-Regge scattering

BLP, 1012.3178

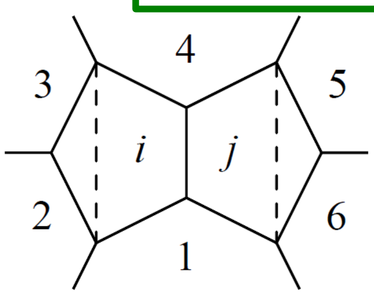
$$
\begin{aligned}
g_0^{(3)}(w,w^*) \;=\; -\frac{1}{32}\Bigg\{ & -60\Big[2\Big(\mathrm{Li}_5(-w)+\mathrm{Li}_5(-w^*)\Big) - \log|w|^2\Big(\mathrm{Li}_4(-w)+\mathrm{Li}_4(-w^*)\Big)\Big] \\
& + 12\Bigg[2\Bigg(\mathrm{Li}_5\Big(\frac{w}{1+w}\Big)+\mathrm{Li}_5\Big(\frac{1}{1+w}\Big)+\frac{1}{24}\log w\,\log^4(1+w) \\
& \qquad + \mathrm{Li}_5\Big(\frac{w^*}{1+w^*}\Big)+\mathrm{Li}_5\Big(\frac{1}{1+w^*}\Big)+\frac{1}{24}\log w^*\,\log^4(1+w^*)\Bigg) \\
& \qquad + \log\frac{|1+w|^2}{|w|^2}\Bigg(\mathrm{Li}_4\Big(\frac{w}{1+w}\Big)+\mathrm{Li}_4\Big(\frac{w^*}{1+w^*}\Big)\Bigg) \\
& \qquad + \log|1+w|^2\Bigg(\mathrm{Li}_4\Big(\frac{1}{1+w}\Big)-\frac{1}{6}\log w\,\log^3(1+w) \\
& \qquad\qquad + \mathrm{Li}_4\Big(\frac{1}{1+w^*}\Big)-\frac{1}{6}\log w^*\,\log^3(1+w^*)\Bigg)\Bigg] \\
& - 2\Big(5\,(\log^2|w|^2-\log^2|1+w|^2)+6\,\log|w|^2\,\log|1+w|^2\Big)\Big(\mathrm{Li}_3(-w)+\mathrm{Li}_3(-w^*)\Big) \\
& - 2\,\log|w|^2\,\log\frac{|1+w|^4}{|w|^2}\Bigg(\mathrm{Li}_3\Big(\frac{w}{1+w}\Big)+\mathrm{Li}_3\Big(\frac{w^*}{1+w^*}\Big)\Bigg) \\
& - 6\,\log|w|^2\,\log|1+w|^2\,\log\frac{|1+w|^2}{|w|^2}\Big(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\Big) \\
& + \frac{5}{3}\log^5|1+w|^2-\frac{5}{2}\log|w|^2\,\log^4|1+w|^2+\frac{4}{3}\log^2|w|^2\,\log^3|1+w|^2 \\
& - \log|w|^2\,\log^2(1+w)\,\log^2(1+w^*)-2\,\log^3|1+w|^2\,\log(1+w)\,\log(1+w^*) \\
& + \zeta_2\,\log|w|^2\,\log|1+w|^2(\log|w|^2-3\,\log|1+w|^2)+4\,\zeta_3\,\log|w|^2\,\log|1+w|^2-48\,\zeta_5\Bigg\} \\
& + \zeta_3\,d_1\,g_1^{(2)}(w,w^*)+\zeta_2\,\gamma''\,g_0^{(2)}(w,w^*)+\zeta_2\,d_2\,\log|1+w|^2\,\log\frac{|1+w|^2}{|w|^2}\,\log\frac{|1+w|^4}{|w|^2}\,.
\end{aligned}
$$

beyond-the-symbol ambiguities

A new representation for $R_6^{(2)}(u,v,w)$

- In one way a slight step backwards, because classical polylogarithmic nature is no longer manifest.
- However, it makes a connection with loop integrals for scattering amplitudes. Arkani-Hamed et al, 1008.2958; 1012.6032
- Also no explicit square roots.
- Template for more complicated amplitudes?

$$R_6^{(2)}(u,v,w) = \frac{1}{4}[\Omega^{(2)}(u,v,w) + \Omega^{(2)}(v,w,u) + \Omega^{(2)}(w,u,v)] + R_6^{(2),\mathrm{rat}}(u,v,w)$$



$$x_{1a}^2 = x_{2a}^2 = x_{3a}^2 = x_{4a}^2 = 0$$

$$\Omega^{(2)}(u_1,u_2,u_3) = -\frac{x_{35}^2 x_{26}^2 (x_{14}^2)^2}{x_{ab}^2} \int \frac{d^4x_i}{i\pi^2} \int \frac{d^4x_j}{i\pi^2} \frac{x_{ai}^2 x_{bj}^2}{x_{1i}^2 x_{2i}^2 x_{3i}^2 x_{4i}^2 x_{ij}^2 x_{4j}^2 x_{5j}^2 x_{6j}^2 x_{1j}^2}$$

$R_6^{(2),\mathrm{rat}}(u,v,w)$

$$R_6^{(2),\mathrm{rat}}(u,v,w) = -\frac{1}{2}\Big[\frac{1}{4}(\mathrm{Li}_2(1-1/u) + \mathrm{Li}_2(1-1/v) + \mathrm{Li}_2(1-1/w))^2 + r(u) + r(v) + r(w) - \zeta_4\Big]$$

$$\begin{aligned} r(u) = & -\mathrm{Li}_4(u) - \mathrm{Li}_4(1-u) + \mathrm{Li}_4(1-1/u) \\ & -\ln u\,\mathrm{Li}_3(1-1/u) - \frac{1}{6}\ln^3 u \ln(1-u) \\ & +\frac{1}{4}(\mathrm{Li}_2(1-1/u))^2 + \frac{1}{12}\ln^4 u \\ & +\zeta_2(\mathrm{Li}_2(1-u) + \ln^2 u) + \zeta_3 \ln u \end{aligned}$$

$\Omega^{(2)}(u,v,w)$

$$\partial_{y_w}\Omega^{(2)} = \frac{1 - y_u y_v}{(1-y_w)(1-y_u y_v y_w)} Q_\phi^{(1)}(y_u, y_v, y_w)$$

$\rightarrow$ $$\Omega^{(2)}(u,v,w) = -6\,\zeta_4 + \int_1^u \frac{du_t}{u_t(u_t-1)} Q_\phi^{(1)}(u_t, v_t, w_t)$$

$$v_t = \frac{(1-u)\,v\,u_t}{u\,(1-v) + (v-u)\,u_t}$$

$$w_t = 1 - \frac{(1-w)\,u_t\,(1-u_t)}{u\,(1-v) + (v-u)\,u_t}$$

$$\begin{aligned} Q_\phi^{(1)}(u,v,w) = {} & 2\left[\mathrm{Li}_3(1-w) + \mathrm{Li}_3\left(1 - \frac{1}{w}\right)\right] \\ & + \ln w \left[-\mathrm{Li}_2(1-w) + \mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \ln u \,\ln v - 2\,\zeta_2\right] - \frac{1}{3}\ln^3 w \\ & - 2\,\mathrm{Li}_3(1-u) - \mathrm{Li}_3\left(1 - \frac{1}{u}\right) - 2\,\mathrm{Li}_3(1-v) - \mathrm{Li}_3\left(1 - \frac{1}{v}\right) \\ & + \ln\left(\frac{u}{v}\right)\left[\mathrm{Li}_2(1-u) - \mathrm{Li}_2(1-v)\right] + \frac{1}{6}\ln^3 u + \frac{1}{6}\ln^3 v - \frac{1}{2}\ln u\,\ln v\,\ln(uv) \end{aligned}$$

Conclusions

- We solved, up to a few constants, for the 3-loop 6-gluon amplitude in N=4 SYM.
- A similar approach has also been used to constrain the 3-loop 8-gluon Wilson loop in special 2-dimensional kinematics Heslop, Khoze, 1109.0058
- Remarkably, we never had to determine a multi-loop integrand, or carry out directly a multi-loop Feynman integral (or Wilson loop integral).
- Key constraints came from the OPE expansion, along with an assumption about the form of the symbol.
- In multi-Regge limit, many predictions made, and final-entry assumption cross-checked.
- It's possible to handle non-MHV amplitudes in the same way, using OPEs for "super-loops" Sever, Vieira, 1108.1575

What further secrets lurk within the hexagon?