High-energy amplitudes in the next-to-leading order in N=4 SYM

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Outline

- Regge limit in a conformal theory.
- High-energy scattering and operator expansion in Wilson lines.
- Evolution equation for color dipoles.
- Leading order: BK equation.
- $\mathcal{N} = 4$: study of 2-dim conformal invariance at high energies.
- Conformal composite dipole operator.
- NLO BK kernel in $\mathcal{N} = 4$.
- NLO amplitude in $\mathcal{N} = 4$ SYM.
- Conclusions
- Outlook

Conformal four-point amplitude

$$A(x,y,x',y') = (x-y)^4(x'-y')^4 N_c^2 \langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle$$

$\mathcal{O} = \mathrm{Tr}\{Z^2\}$ ($Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$) - chiral primary operator
In a conformal theory the amplitude is a function of two conformal ratios

$$A = F(R,R')$$
$$R = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2}, \qquad R' = \frac{(x-y)^2(x'-y')^2}{(x-y')^2(x'-y)^2}$$

At large N_c

$$A(x,y,x',y') = A(g^2N_c) \qquad g^2N_c = \lambda \ - \ \text{‘t Hooft coupling}$$

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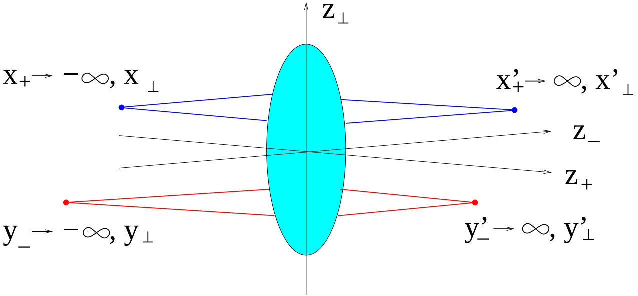
AdS/CFT gives predictions at large $\lambda \to \infty$.

Our goal is perturbative expansion and resummation of $(\lambda \ln s)^n$ at large energies in the next-to-leading approximation

$$(\lambda \ln s)^n (c_n^{\mathrm{LO}} + c_n^{\mathrm{NLO}}\lambda)$$

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+,\ y_- \to \rho' y_-,\ y'_- \to \rho' y-_-$ $\quad \rho, \rho' \to \infty$



Full 4-dim conformal group: $A = F(R, r)$

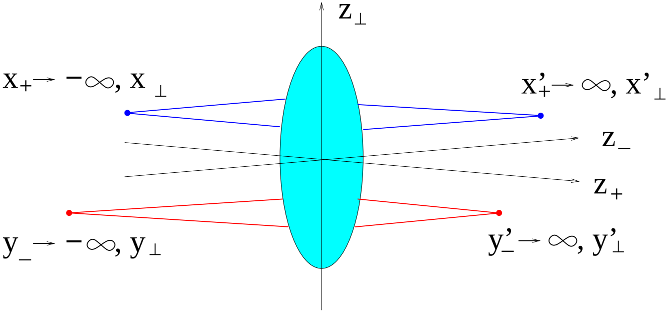
$$R = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2} \to \frac{\rho^2\rho'^2 x_+ x'_+ y_- y'_-}{(x-x')^2_\perp (y-y')^2_\perp} \to \infty$$

$$r = \frac{[(x-y)^2(x'-y')^2 - (x'-y)^2(x-y')^2]^2}{(x-x')^2(y-y')^2(x-y)^2(x'-y')^2}$$

$$\to \frac{[(x'-y')^2_\perp x_+ y_- + x'_+ y'_-(x-y)^2_\perp + x_+ y'_-(x'-y)^2_\perp + x'_+ y_-(x-y')^2_\perp]^2}{(x-x')^2_\perp (y-y')^2_\perp x_+ x'_+ y_- y'_-}$$

4-dim conformal group versus $SL(2,C)$

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+, y_- \to \rho' y_-,\ y'_- \to \rho' y-_-$ $\rho, \rho' \to \infty$



Regge limit symmetry: 2-dim conformal group $SL(2,C)$ formed from P_1, P_2, M^{12}, D, K_1 and K_2 which leave the plane $(0,0,z_\perp)$ invariant.

Pomeron in a conformal theory

$$A(x,y;x',y') \stackrel{s\to\infty}{=} \frac{i}{2}\int d\nu\, f_+(\omega(\lambda,\nu))F(\lambda,\nu)\Omega(r,\nu)R^{\omega(\lambda,\nu)/2}$$

L. Cornalba (2007)

$f_+(\omega) = \frac{e^{i\pi\omega}-1}{\sin\pi\omega}$ - signature factor

$\Omega(r,\nu)$ - solution of the eqn $(\Box_{H_3}+\nu^2+1)\Omega(r,\nu)=0$.
Explicit form:

$$\Omega(r,\nu) = \frac{\nu^2}{\pi^3}\int d^2z\Big(\frac{\kappa^2}{(2\kappa\cdot\zeta)^2}\Big)^{\frac12+i\nu}\Big(\frac{\kappa'^2}{(2\kappa'\cdot\zeta)^2}\Big)^{\frac12-i\nu} = \frac{\sin\nu\rho}{\sinh\rho}, \qquad \cosh\rho = \frac{\sqrt{r}}{2}$$

$$\zeta = p_1 + \frac{z_\perp^2}{s}p_2 + z_\perp, \qquad p_1^2 = p_2^2 = 0,\ \ 2(p_1,p_2) = s$$

$$\kappa = \frac{1}{2x_+}(p_1 - \frac{x^2}{s}p_2 + x_\perp) - \frac{1}{2y_+}(p_1 - \frac{y^2}{s}p_2 + y_\perp), \qquad \kappa^2\kappa'^2 = \frac{1}{R}$$

$$\kappa' = \frac{1}{2x'_-}(p_1 - \frac{x'^2}{s}p_2 + x'_\perp) - \frac{1}{2y'_-}(p_1 - \frac{y'^2}{s}p_2 + y'_\perp, \qquad 4(\kappa\cdot\kappa')^2 = \frac{r}{R}$$

The dynamics is described by $\omega(\lambda,\nu)$ and $F(\lambda,\nu)$.

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Pomeron intercept $\omega(\nu,\lambda)$ is known in two limits:

$$1. \qquad \lambda \to 0: \qquad \omega(\nu,\lambda) = \frac{\lambda}{\pi}\chi(\nu) + \lambda^2\omega_1(\nu) + ...$$

$\chi(\nu) = 2\psi(1) - \psi(\frac{1}{2}+i\nu) - \psi(\frac{1}{2}-i\nu)$ - BFKL intercept,

$\omega_1(\nu)$ - NLO BFKL intercept Lipatov, Kotikov (2000)

$$2. \qquad \lambda \to \infty: \qquad AdS/CFT \quad \Rightarrow \quad \omega(\nu,\lambda) = 2 - \frac{\nu^2+4}{2\sqrt{\lambda}} + ...$$

2 = gravition spin , next term - Brower, Polchinski, Strassler, Tan (2006)

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The function $F(\nu,\lambda)$ in two limits:

1. $\lambda \to 0$: $\quad F(\nu,\lambda) = \lambda^2 F_0(\nu) + \lambda^3 F_1(\nu) + ...$

$F_0(\nu) = \frac{\pi \sinh \pi\nu}{4\nu \cosh^3 \pi\nu}$ Cornalba, Costa, Penedones (2007)

$F_1(\nu)$ = see below G. Chirilli and I.B. (2009)

2. $\lambda \to \infty$: $\quad AdS/CFT \quad \Rightarrow \quad \omega(\nu,\lambda) = \pi^3\nu^2 \frac{1+\nu^2}{\sinh^2 \pi\nu} + ...$

L.Cornalba (2007)

Pomeron in the conformal theory

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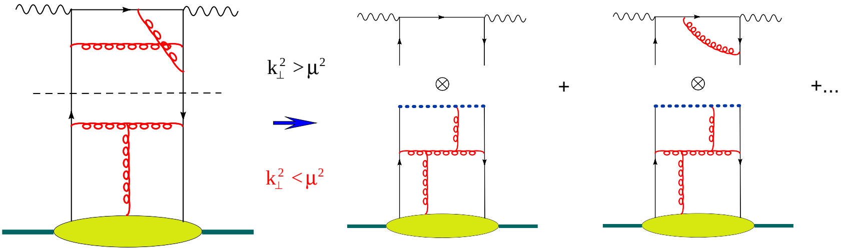
We calculate $F_1(\nu)$ (and confirm $\omega_1(\nu)$) using the expansion of high-energy amplitudes in Wilson lines (color dipoles)

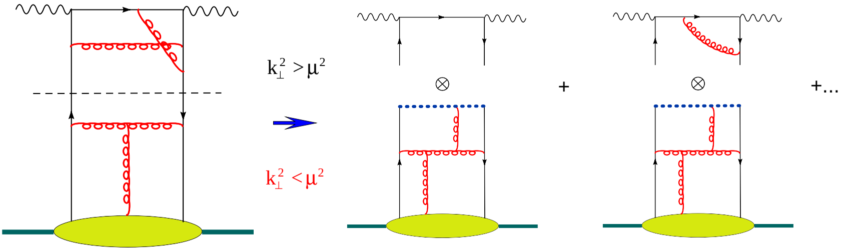


μ^2 - factorization scale (normalization point)

$k_\perp^2 \; > \; \mu^2$ - coefficient functions

$k_\perp^2 \; < \; \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)

Light-cone expansion and DGLAP evolution in the NLO



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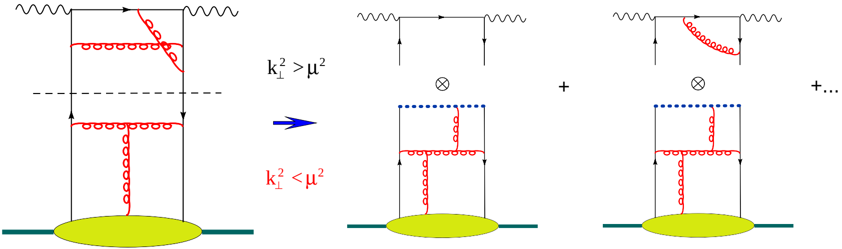
OPE in light-ray operators $(x-y)^2 \to 0$

$$T\{j_\mu(x)j_\nu(y)\} \ = \ \frac{x_\xi}{2\pi^2 x^4}\Big[1 + \frac{\alpha_s}{\pi}(\ln x^2\mu^2 + C)\Big]\bar\psi(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,y]\psi(y) + O(\frac{1}{x^2})$$

$$[x,y] \ \equiv \ Pe^{ig\int_0^1 du \ (x-y)^\mu A_\mu(ux+(1-u)y)}$$ - gauge link

Light-cone expansion and DGLAP evolution in the NLO



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Renorm-group equation for light-ray operators $\Rightarrow$ DGLAP evolution of parton densities $(x-y)^2 = 0$

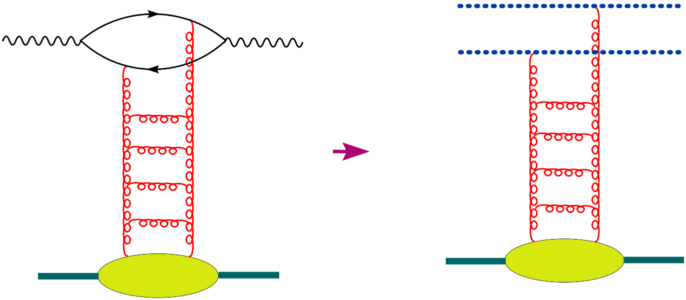
$$\mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x,y]\psi(y) \;=\; K_{\rm LO}\bar{\psi}(x)[x,y]\psi(y) + \alpha_s K_{\rm NLO}\bar{\psi}(x)[x,y]\psi(y)$$

Four steps of an OPE

- To factorize an amplitude into a product of coefficient functions and matrix elements of relevant operators.
- To find the evolution equations of the operators with respect to factorization scale.
- To solve these evolution equations.
- To convolute the solution with the initial conditions for the evolution and get the amplitude

DIS at high energy: relevant operators

- At high energies, particles move along straight lines $\Rightarrow$
  the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \mathrm{Tr}\{U(k_\perp) U^\dagger(-k_\perp)\} | B \rangle$$

$$U(x_\perp) = \mathrm{Pexp}\Big[ig \int_{-\infty}^{\infty} du\, n^\mu A_\mu(un + x_\perp) \Big] \qquad \text{Wilson line}$$

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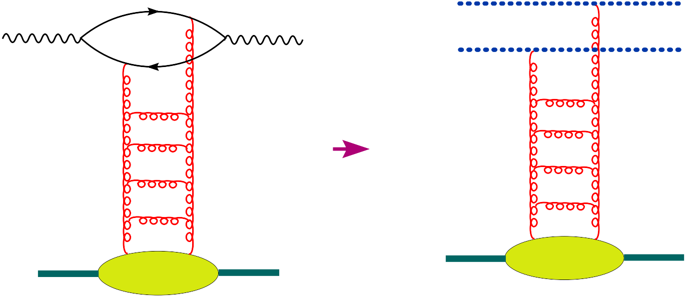

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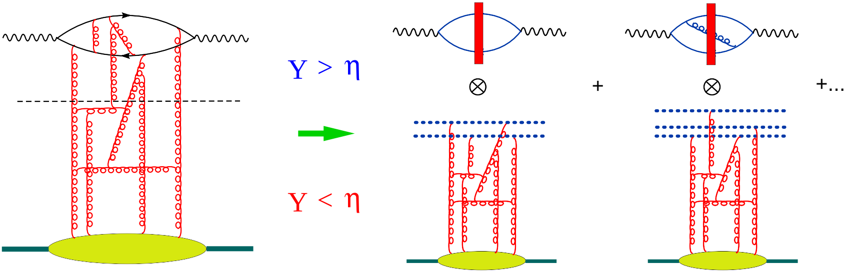
Formally, ➤ means the operator expansion in Wilson lines



η - rapidity factorization scale

Rapidity Y > η - coefficient function (“impact factor”)

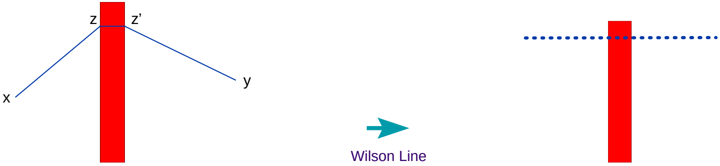
Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \text{Pexp}\left[ig\int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp)\right]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

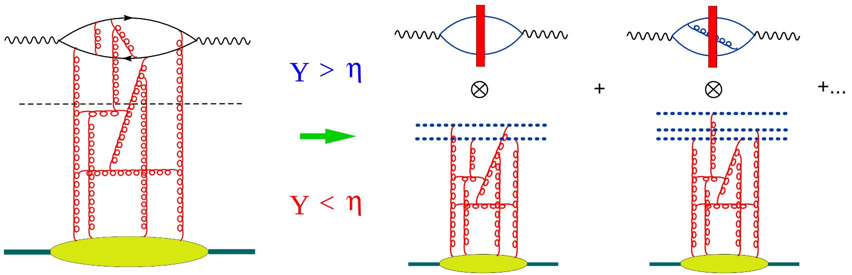
Spectator frame: propagation in the shock-wave background.

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.



[$x \to z$: free propagation]$\times$
[$U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave]$\times$
[$z \to y$: free propagation]

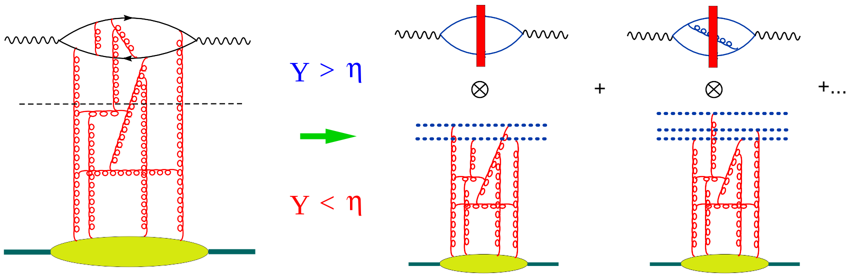
High-energy expansion in color dipoles



The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+\ \text{NLO contribution}$$

High-energy expansion in color dipoles



η - rapidity factorization scale

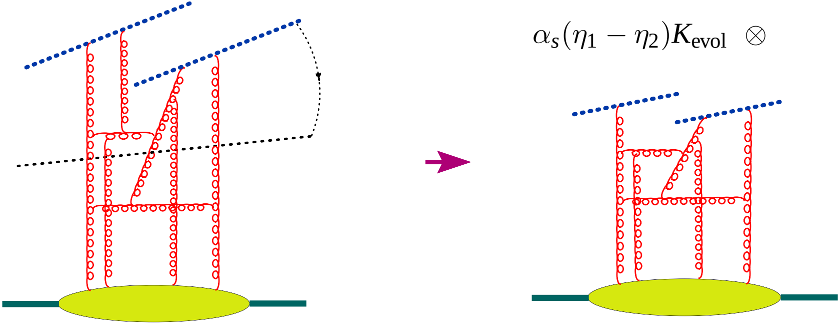
Evolution equation for color dipoles

$$\frac{d}{d\eta}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} = \frac{\alpha_s}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2}[\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} - N_c\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}] + \alpha_s K_{\mathrm{NLO}}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} + O(\alpha_s^2)$$

(Linear part of K_{NLO} = $K_{\mathrm{NLO\ BFKL}}$)

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).

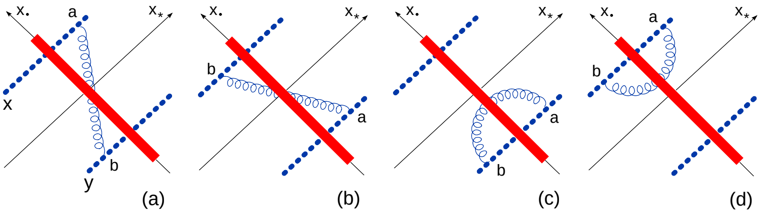
$$\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \otimes$$



Evolution equation in the leading order

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$



$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

Non linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\ (x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1,\ \alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

Why NLO correction?

- To check that high-energy OPE works at the NLO level.
- To determine the argument of the coupling constant.
- To get the region of application of the leading order evolution equation.
- To check conformal invariance (in $\mathcal{N}$=4 SYM)

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;=\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+\, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

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$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;\to\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2} \, A_+\Big(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2}\Big)\right\} \;=\; [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

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$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp / x_\perp^2$ and $x^+ \to x^+/x_\perp^2 \Rightarrow$

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \; \to \; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2} \, A_+(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2})\right\} \; = \; [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

$\Rightarrow$The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp / x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2 z}{\not{z}^4} \frac{(x-y)^2 \; \not{z}^4}{(x-z)^2 (z-y)^2}[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}]$$

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

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$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z,\bar{z})$

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

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Conformal invariance of the evolution kernel

$$\frac{d}{d\eta}[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2}\int dz\, K(x,y,z)[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}\mathrm{Tr}\{U_x U_y^\dagger\}]$$

$$\Rightarrow \left[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}\right]K(x,y,z) = 0$$

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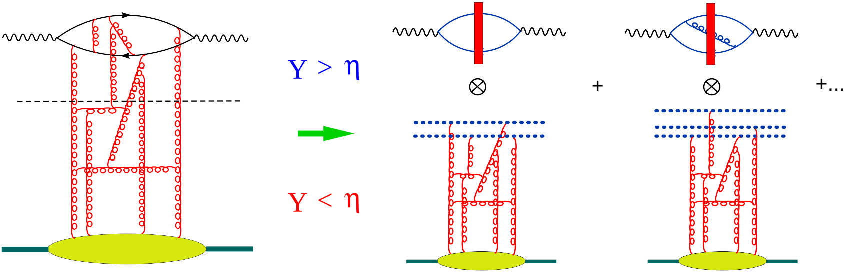
In the leading order - OK. In the NLO - ?

Expansion of the amplitude in color dipoles in the NLO



The high-energy operator expansion is

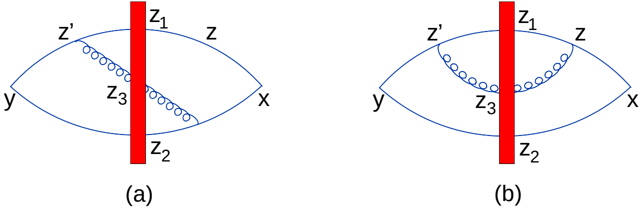
$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}(z_1,z_2){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}(z_1,z_2,z_3)[\frac{1}{N_c}{\rm Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

In the leading order - conf. invariant impact factor

$$I_{\rm LO} = \frac{x_+^{-2}y_+^{-2}}{\pi^2\mathcal{Z}_1^2\mathcal{Z}_2^2}, \qquad \mathcal{Z}_i \equiv \frac{(x-z_i)^2_\perp}{x_+} - \frac{(y-z_i)^2_\perp}{y_+} \qquad CCP, 2007$$

NLO impact factor



$$I^{\rm NLO}(x,y;z_1,z_2,z_3;\eta) \; = \; -\, I^{\rm LO} \times \frac{\lambda}{\pi^2}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\Big[\ln\frac{\sigma s}{4}\mathcal{Z}_3 - \frac{i\pi}{2} + C\Big]$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff η is not invariant

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$[{\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\rm conf} \; = \; {\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+ \frac{\lambda}{2\pi^2}\int d^2 z_3 \; \frac{z_{12}^2}{z_{13}^2 z_{23}^2}[{\rm Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c{\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2 z_{23}^2} \; + \; O(\lambda^2)$$

the impact factor becomes conformal in the NLO.

Operator expansion in conformal dipoles

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}(z_1,z_2){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$$
$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}(z_1,z_2,z_3)[\frac{1}{N_c}{\rm Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO} = -I^{\rm LO}\frac{\lambda}{2\pi^2}\int dz_3\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[\ln\frac{z_{12}^2e^{2\eta}as^2}{z_{13}^2z_{23}^2}\mathcal{Z}_3^2 - i\pi + 2C\Big]$$

The new NLO impact factor is conformally invariant
$\Rightarrow$ ${\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$ is Möbius invariant

We think that one can construct the composite conformal dipole operator order by order in perturbation theory.

Analogy: when the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator in must be corrected by finite counterterms order by order in perturbaton theory.

Non-linear evolution equation in the NLO

$$\frac{d}{d\eta}Tr\{U_x U_y^\dagger\} =$$

$$\int \frac{d^2z}{2\pi^2}\left(\alpha_s \frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z)\right)[Tr\{U_x U_z^\dagger\}Tr\{U_z U_y^\dagger\} - N_c Tr\{U_x U_y^\dagger\}] +$$

$$\alpha_s^2 \int d^2z d^2z' \left(K_4(x,y,z,z')\{U_x, U_{z'}^\dagger, U_z, U_y^\dagger\} + K_6(x,y,z,z')\{U_x, U_{z'}^\dagger, U_{z'}, U_z, U_z^\dagger, U_y^\dagger\}\right)$$

K_{NLO} is the next-to-leading order correction to the dipole kernel and K_4 and K_6 are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

Definition of the NLO kernel

In general

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

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$$\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} - \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + O(\alpha_s^3)$$

Definition of the NLO kernel

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We calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle = \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle - \langle\alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle + O(\alpha_s^3)$$

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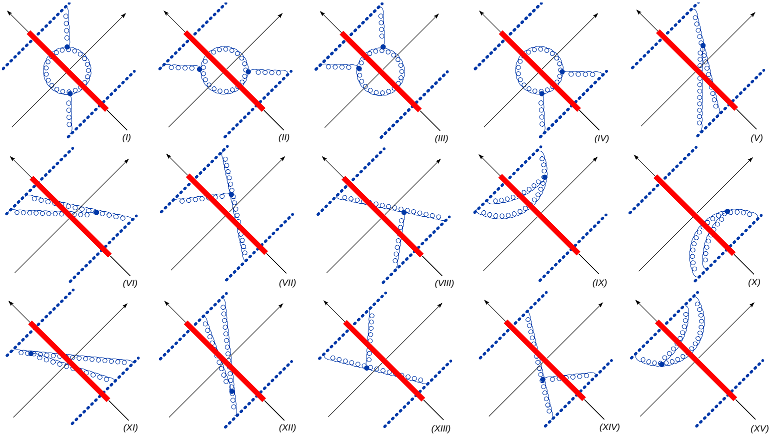
Subtraction of the (LO) contribution (with the rigid rapidity cutoff)

$\Rightarrow$ $\left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

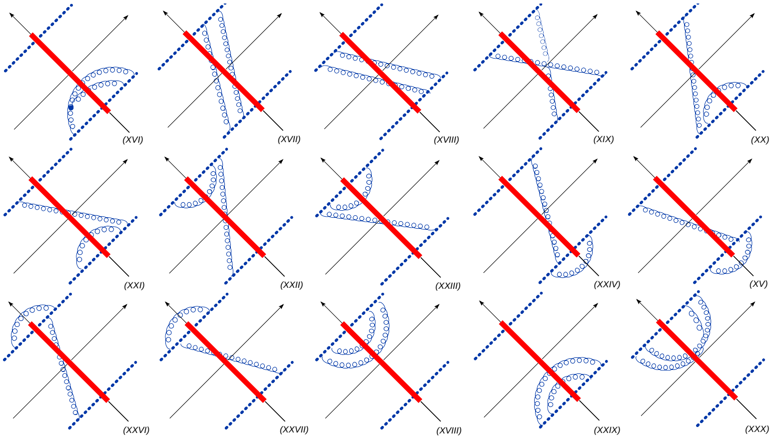
Typical integral

$$\int_0^1 dv\,\frac{1}{(k-p)_\perp^2 v + p_\perp^2(1-v)}\left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2}\ln\frac{(k-p)_\perp^2}{p_\perp^2}$$

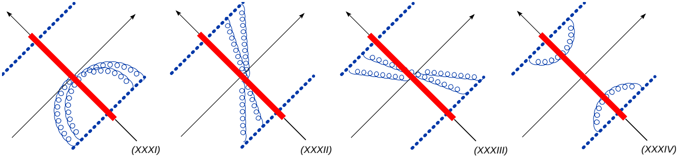
Gluon part of the NLO BK kernel: diagrams



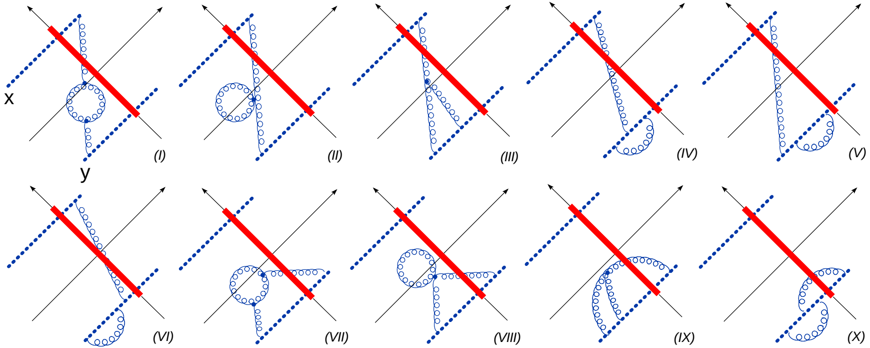
Diagrams for 1→3 dipoles transition



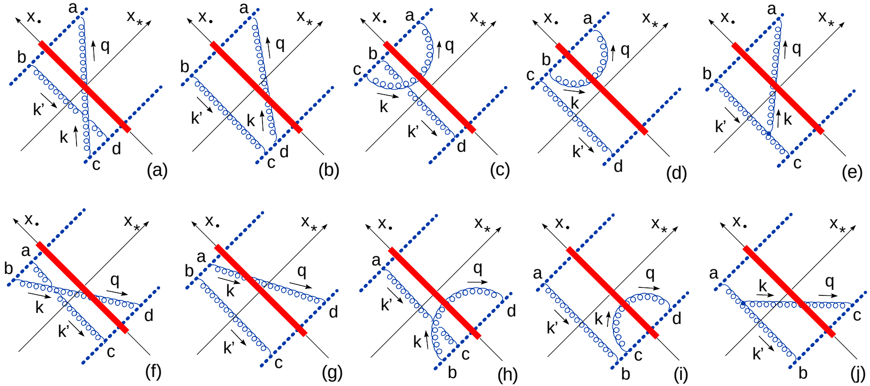
Diagrams for 1→3 dipoles transition



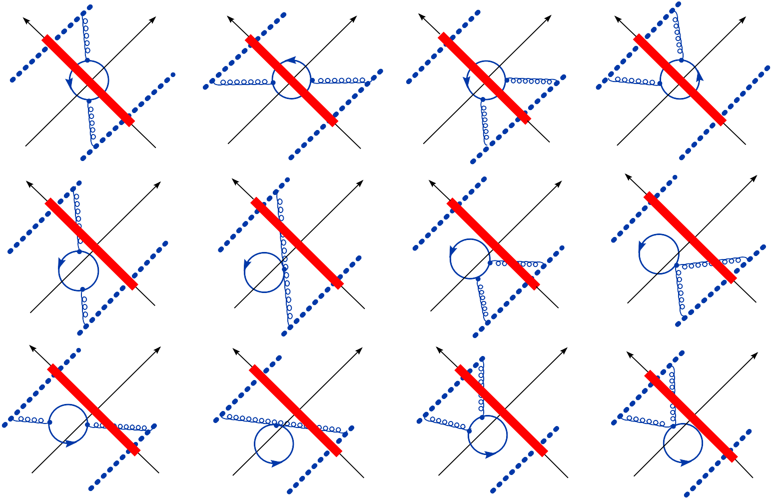
"Running coupling" diagrams



1 → 2 dipole transition diagrams



Gluino and scalar loops



Evolution equation for color dipole in $\mathcal{N} = 4$ (I.B. and G. Chirilli)

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.

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&\times\,[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
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Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant

Evolution equation for composite conformal dipoles in $\mathcal{N}=4$

$$\frac{d}{d\eta}\big[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]^{\rm conf}$$
$$= \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1-\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}\Big]\big[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]^{\rm conf}$$
$$-\frac{\alpha_s^2}{4\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}$$
$$\times\ \mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}[(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4})^{bb'}-(z_4\to z_3)]$$

Now Möbius invariant!

NLO BFKL equation in $\mathcal{N}=4$ SYM

To find $A(x,y;x',y')$ we need the linearized (NLO BFKL) equation. With two-gluon accuracy

$$\hat{\mathcal{U}}^\eta(x,y) \ = \ 1-\frac{1}{N_c^2-1}\mathrm{Tr}\{\hat{U}_x^\eta\hat{U}_y^{\dagger\eta}\}$$

Conformal dipole operator in the BFKL approximation

$$\hat{\mathcal{U}}^\eta_{\rm conf}(z_1,z_2) \ = \ \hat{\mathcal{U}}^\eta(z_1,z_2)+\frac{\alpha_s N_c}{4\pi^2}\int d^2z\frac{z_{12}^2}{z_{13}^2z_{23}^2}\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}[\hat{\mathcal{U}}^\eta(z_1,z_3)+\hat{\mathcal{U}}^\eta(z_2,z_3)-\hat{\mathcal{U}}^\eta(z_1,z_2)]$$

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Define

$$\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2) = \hat{\mathcal{U}}^\eta(z_1,z_2)+\frac{\alpha_s N_c}{4\pi^2}\int d^2z\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\ln\frac{a e^{2\eta} z_{12}^2}{z_{13}^2 z_{23}^2}[\hat{\mathcal{U}}^\eta(z_1,z_3)+\hat{\mathcal{U}}^\eta(z_2,z_3)-\hat{\mathcal{U}}^\eta(z_1,z_2)]+\ldots$$

such that $\frac{d}{d\eta}\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2)=0$.

$\Rightarrow$The evolution can be rewritten in terms of a

NLO BFKL

$$
\begin{aligned}
& a\frac{d}{da}\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2) \\
& = \frac{\alpha_s N_c}{2\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[1-\frac{\alpha_s N_c}{4\pi}\frac{\pi^2}{3}\Big]\big[\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_3)+\hat{\mathcal{U}}^a_{\rm conf}(z_2,z_3)-\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2)\big] \\
& + \frac{\alpha_s^2 N_c^2}{8\pi^4}\int \frac{d^2z_3 d^2z_4}{z_{34}^4}\ \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2}\Big\{2\ln\frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2}+\Big[1+\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\Big]\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}\Big\}\hat{\mathcal{U}}^a_{\rm conf}(z_3,z_4) \\
& \qquad + \frac{3\alpha_s^2 N_c^2}{2\pi^3}\zeta(3)\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2)
\end{aligned}
$$

Eigenfunctions are determined by conformal invariance

$$
E_{\nu,n}(z_{10},z_{20}) \ = \ \Big[\frac{\tilde{z}_{12}}{\tilde{z}_{10}\tilde{z}_{20}}\Big]^{\frac{1}{2}+i\nu+\frac{n}{2}}\Big[\frac{\bar{z}_{12}}{\bar{z}_{10}\bar{z}_{20}}\Big]^{\frac{1}{2}+i\nu-\frac{n}{2}}
$$

The expansion in eigenfunctions

$$
\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2) \ = \ \sum_{n=0}^{\infty}\int d^2z_0\int d\nu\ E_{\nu,n}(z_{10},z_{20})\hat{\mathcal{U}}^a_{z_0,\nu,n} \quad \Rightarrow \quad a\frac{d}{da}\hat{\mathcal{U}}^a_{z_0,\nu,n} \ = \ \omega(n,\nu)\hat{\mathcal{U}}^a_{z_0,\nu,n}
$$

$\omega(n,\nu) \equiv$ pomeron intercept = eigenvalue of the BFKL equation

Pomeron intercept

Pomeron intercept = the eigenvalue of the BFKL equation

$$\omega(n,\nu) = \frac{\alpha_s}{\pi}N_c\Big[\chi(n,\frac{1}{2}+i\nu)+\frac{\alpha_s N_c}{4\pi}\delta(n,\frac{1}{2}+i\nu)\Big],$$
$$\delta(n,\gamma) = 6\zeta(3)-\frac{\pi^2}{3}\chi(n,\gamma)-\chi"(n,\gamma)-\ 2\Phi(n,\gamma)-2\Phi(n,1-\gamma)$$

where $\gamma=\frac{1}{2}+i\nu$ and

$$\chi(n,\gamma)=2\psi(1)-\psi(\gamma+\frac{n}{2})-\psi(1-\gamma+\frac{n}{2})$$
$$\Phi(n,\gamma) = \int_0^1\frac{dt}{1+t}\,t^{\gamma-1+\frac{n}{2}}\Big\{\frac{\pi^2}{12}-\frac{1}{2}\psi'\Big(\frac{n+1}{2}\Big)-\mathrm{Li}_2(t)-\mathrm{Li}_2(-t)$$
$$-\Big(\psi(n+1)-\psi(1)+\ln(1+t)+\sum_{k=1}^{\infty}\frac{(-t)^k}{k+n}\Big)\ln t-\sum_{k=1}^{\infty}\frac{t^k}{(k+n)^2}[1-(-1)^k]\Big\}$$

Pomeron intercept

Pomeron intercept = the eigenvalue of the BFKL equation

$$\omega(n,\nu) = \frac{\alpha_s}{\pi}N_c\Big[\chi(n,\frac{1}{2}+i\nu) + \frac{\alpha_s N_c}{4\pi}\delta(n,\frac{1}{2}+i\nu)\Big],$$
$$\delta(n,\gamma) = 6\zeta(3) - \frac{\pi^2}{3}\chi(n,\gamma) - \chi"(n,\gamma) - 2\Phi(n,\gamma) - 2\Phi(n,1-\gamma)$$

where $\gamma = \frac{1}{2}+i\nu$ and

$$\chi(n,\gamma) = 2\psi(1) - \psi(\gamma+\frac{n}{2}) - \psi(1-\gamma+\frac{n}{2})$$
$$\Phi(n,\gamma) = \int_0^1 \frac{dt}{1+t}\, t^{\gamma-1+\frac{n}{2}}\Big\{\frac{\pi^2}{12} - \frac{1}{2}\psi'\Big(\frac{n+1}{2}\Big) - \mathrm{Li}_2(t) - \mathrm{Li}_2(-t)$$
$$- \Big(\psi(n+1) - \psi(1) + \ln(1+t) + \sum_{k=1}^{\infty}\frac{(-t)^k}{k+n}\Big)\ln t - \sum_{k=1}^{\infty}\frac{t^k}{(k+n)^2}[1-(-1)^k]\Big\}$$

Coincides with Lipatov & Kotikov

Agrees with $j \to 1$ asymptotics of 3-loop splitting functions

Vogt, Moch, Vermaseren,(2003)

NLO impact factor

$$(x-y)^4T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \frac{(x-y)^4}{\pi^2}\int d^2z_1d^2z_2\frac{(x_+y_+)^{-2}}{\mathcal{Z}_1^2\mathcal{Z}_2^2}\Big\{\hat{\mathcal{U}}^{\rm conf}$$
$$-\frac{\lambda}{2\pi^2}\int\frac{d^2z_3\ z_{12}^2}{z_{13}^2z_{23}^2}\Big[\ln\frac{az_{12}^2\mathcal{Z}_3^2}{z_{13}^2z_{23}^2}-i\pi\Big][\hat{\mathcal{U}}^{\rm conf}(z_1,z_3)+\hat{\mathcal{U}}^{\rm conf}(z_2,z_3)-\hat{\mathcal{U}}^{\rm conf}(z_1,z_2)]\Big\}$$

With two-gluon accuracy ($\mathcal{R} \equiv \frac{(x-y)^2z_{12}^2}{x_+y_+\mathcal{Z}_1\mathcal{Z}_2}$ - conformal ratio)

$$(x-y)^4T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \frac{(x-y)^4}{\pi^2}\int d^2z_1d^2z_2\frac{(x_+y_+)^{-2}}{\mathcal{Z}_1^2\mathcal{Z}_2^2}\Big\{1$$
$$-\frac{\lambda}{2\pi^2}\Big[4\mathrm{Li}_2(1-\mathcal{R})-\frac{2\pi^2}{3}+2\big(\ln\frac{1}{\mathcal{R}}+\frac{1}{\mathcal{R}}-2\big)\ln\frac{a\mathcal{Z}_1\mathcal{Z}_2}{z_{12}^2}\Big]\hat{\mathcal{U}}^{\rm conf}(z_1,z_2)$$

The impact factor should not scale with energy $\Rightarrow$ $a=\frac{x_+y_+}{(x-y)^2}$ (analog of $\mu^2=Q^2$ in DIS)

$$(x-y)^4T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \frac{1}{\pi^2}\int\frac{d^2z_1d^2z_2}{z_{12}^4}\ \mathcal{R}^2\Big\{1$$
$$-\frac{\lambda}{2\pi^2}\Big[4\mathrm{Li}_2(1-\mathcal{R})-\frac{2\pi^2}{3}+2\big(\ln\frac{1}{\mathcal{R}}+\frac{1}{\mathcal{R}}-2\big)\ln\frac{1}{\mathcal{R}}\Big]\Big\}\hat{\mathcal{U}}^{\rm conf}(z_1,z_2)$$

NLO impact factor in Mellin representation

The projection onto the conformal eigenfunctions $\left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^\gamma$ $(\gamma = \frac{1}{2} + i\nu)$:

$$\int dz_1 dz_2 (x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\}\left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^\gamma = \left(\frac{\kappa^2}{(2\kappa\cdot\zeta_0)^2}\right)^\gamma [I^A_{\rm LO}(\gamma) + I^A_{\rm NLO}(\gamma)]\hat{\mathcal{U}}(z_0,\gamma),$$

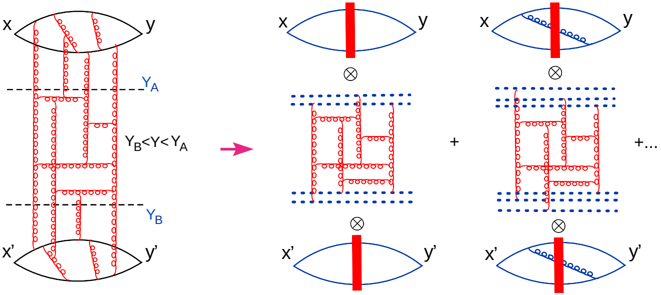
$$\hat{\mathcal{U}}(z_0,\gamma) = \int d^2z_1 d^2z_2 \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^\gamma \hat{\mathcal{U}}(z_1,z_2)$$

$$I^A_{\rm LO}(\gamma) = \frac{\Gamma^2(1-\gamma)}{\Gamma(2-2\gamma)}\Gamma(1+\gamma)\Gamma(2-\gamma)$$

$$I^A_{\rm NLO}(\gamma) = \frac{\lambda}{8\pi^2} I^A_{\rm LO}\Big[-2\frac{\pi^2}{\sinh^2\pi\gamma} + \frac{2\pi^2}{3} + \frac{\chi(\gamma)-2}{\gamma(1-\gamma)} + 2C\chi(\gamma)\Big]$$

$$\chi(\gamma) \equiv \chi(0,\gamma) = 2C - \psi(\gamma) - \psi(1-\gamma)$$

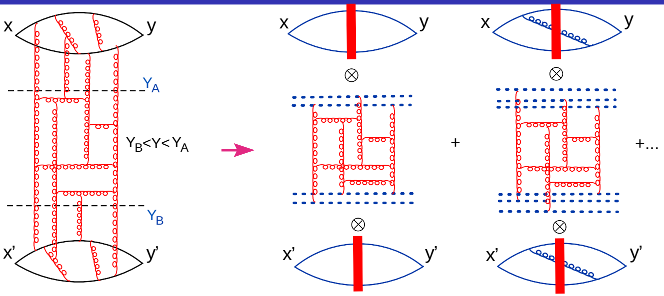
NLO Amplitude in $\mathcal{N}$=4 SYM theory: factorization in rapidity



$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

$a_0 = \frac{x_+y_+}{(x-y)^2}$, $b_0 = \frac{x'_-y'_-}{(x'-y')^2}$ $\Leftrightarrow$ impact factors do not scale with energy
$\Rightarrow$ all energy dependence is contained in $[\mathrm{DD}]^{a_0,b_0}$ ($a_0b_0 = R$)

NLO Amplitude in $\mathcal{N}$=4 SYM theory: factorization in rapidity

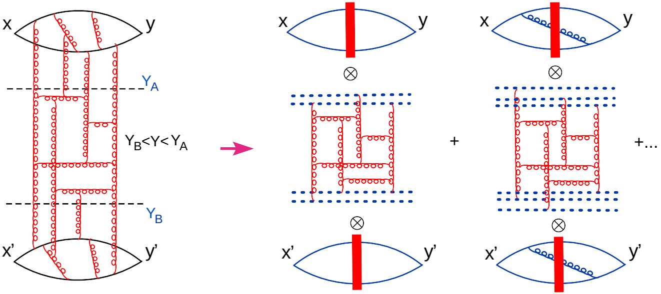


$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

Dipole-dipole scattering

$$[DD] = \int d\nu \int dz_0 \,\Big(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\Big)^{\frac{1}{2}+i\nu}\Big(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\Big)^{\frac{1}{2}-i\nu} D(\frac{1}{2}+i\nu;\lambda)R^{\omega(\nu)/2}$$

$$D(\gamma;\lambda) \;=\; \frac{\Gamma(-\gamma)\Gamma(\gamma-1)}{\Gamma(1+\gamma)\Gamma(2-\gamma)}\Big\{1-\frac{\lambda}{4\pi^2}\Big[\frac{\chi(\gamma)}{\gamma(1-\gamma)}-\frac{\pi^2}{3}\Big]+O(\lambda^2)\Big\}$$



$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$

$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

Result : (G.A. Chirilli and I.B.)

$$F(\nu) = \frac{N_c^2}{N_c^2-1}\frac{4\pi^4\alpha_s^2}{\cosh^2\pi\nu}\left\{1+\frac{\alpha_s N_c}{\pi}\left[\frac{\pi^2}{2}-\frac{2\pi^2}{\cosh^2\pi\nu}-\frac{8}{1+4\nu^2}\right]+O(\alpha_s^2)\right\}$$

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The NLO BK kernel in for the evolution of conformal composite dipoles in $\mathcal{N} = 4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel agrees with NLO BFKL equation.
- The correlation function of four Z^2 operators is calculated at the NLO order.
- NLO photon impact factor is calculated.

Outlook: relation to conformal light-ray operators

Gluon parton density $\mathcal{D}(x_B, \mu^2)$ is proportional to matrix element of the light-ray operator

$$\mathcal{O}(x_B, \mu^2) \;=\; \int d\lambda\; e^{i\lambda x_B}\; \mathrm{Tr}\{G_{+i}(\lambda e)[\lambda e, 0]G_{+i}(0)[0, \lambda e]\}^{\mu}$$

Conformal light-ray operator $\mathcal{O}_j$ (j - conformal spin in $SL(2,R)$ group)

$$\mathcal{O}_j^{\mu} \;=\; \int d\lambda\; \lambda^{1-j}\; \mathrm{Tr}\{G_{+i}(\lambda e)[\lambda n, 0]G_{+i}(0)[0, \lambda e]\}^{\mu}$$

Anomalous dimension

$$\mu\frac{d}{d\mu}\mathcal{O}_j \;=\; \gamma_j(\alpha_s)O_j$$

At $j = n$ γ_n is an anomalous dimension of the local twist-2 operator

$$G^{+i}(D^{+})^{n-2}G_i^{+}$$

Expansion of conformal dipoles in conformal light-ray operators?

In the leading order relation this expansion is trivial: $x_\perp^2$ is the normalization point of gluon light-ray operator and $x_B = e^{-\eta}$:

$$\mathrm{Tr}\{\partial_i U_x \partial^i U_0\}^\eta = \mathcal{D}^{\mu^2 = x_\perp^{-2}}_{x_B = e^{-\eta}} + O(x_\perp^2) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{dj}{2\pi i} \frac{\Gamma(j-1)}{x_B^{j-1}} (x_\perp^2 \mu^2)^{-\gamma_j} \mathcal{O}_j^{\mu^2}$$

$$= \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{d\omega}{2\pi i} \Gamma(\omega) e^{\omega\eta} (x_\perp^2 \mu^2)^{-\gamma_\omega} \mathcal{O}_\omega^{\mu^2}$$

This should be compared to LO rapidity evolution of color dipole $\omega_{\gamma=\frac{1}{2}+i\nu} = \omega(\nu)$ - pomeron intercept)

$$\mathrm{Tr}\{\partial_i U_x \partial^i U_0\}^\eta = = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\gamma}{2\pi i} e^{\omega_\gamma(\eta-\eta_0)} (x_\perp^2 \mu^2)^{-\gamma} \int d^2 z \, (z_\perp^2)^{1-\gamma} \mathcal{U}(z_\perp)^{\eta_0}$$

We see that

$$\omega = \omega(\gamma, \alpha_s) \Leftrightarrow \gamma = \gamma(\omega, \alpha_s) \simeq \sum \frac{\alpha_s^n}{\omega^n} = \frac{\alpha_s}{\omega} + \frac{\alpha_s^3}{\omega^3} + \ldots$$

Outlook: rapidity evolution of TMD's

$$\text{Weizsacker – Williams gluon TMD}: \quad D(x_B, k_\perp) \sim \int d^2k_\perp \, e^{ik_\perp \cdot z_\perp}$$

$$\times \int dudv \langle [-\infty, u]_z G_{+i}(z_\perp + up_1)[u, -\infty]_z [-\infty, u]_0 G_{+i}(vp_1)[u, -\infty]_0 e^{i(u-v)x_B \frac{s}{2}} \rangle$$

Compare to ($U_i \equiv U_i^\dagger i\partial_i U$)

$$\{U_i(z_\perp)U_i(0_\perp)\}^\eta = \int dudv [-\infty, u]_z G_{+i}(z_\perp + up_1)[u, -\infty]_z [-\infty, u]_0 G_{+i}(vp_1)[u, -\infty]_0$$

$\Rightarrow$ same operator with different rapidity cutoff.

Evolution equation (leading order)

$$\frac{d}{d\eta}(U_i^a(x)U_i^a(y))^\eta$$

$$= -\frac{\alpha_s}{\pi^2}\Big(\nabla_i^x \int dz \frac{(x-z, y-z)}{(x-z)^2(y-z)^2}(U_x^\dagger U_y + 1 - U_x^\dagger U_z - U_z^\dagger U_y) \overleftarrow{\nabla}_i^y \Big)^{aa}$$

$$-\frac{\alpha_s}{\pi^2}\Big[\int \frac{dz}{(x-z)^2}\big[f^{abc}(U_x^\dagger \partial_i U_z)^{bc} U_i^a(y) + N_c U_i^a(x) U_i^a(y) + x \leftrightarrow y\big]$$

NLO evolution equation for WW gluon TMD - ?.