

Schwinger Keldysh

Consider Computing in HO:

$$G^>(t, \bar{t}) = \langle \hat{x}(t) \hat{x}(\bar{t}) \rangle \quad \nearrow$$

not $\langle T[\hat{x}\hat{x}] \rangle$

↑
not useful for TFC

Ok

$$G^>(t, \bar{t}) = \text{Tr} \rho \hat{x}(t) \hat{x}(\bar{t})$$

$$G^>(t, \bar{t}) = \int dx_1^0 \int dx_2^0 \rho(x_2^0, x_1^0) \langle x_2^0 | \hat{x}(t) x(\bar{t}) | x_1^0 \rangle$$

So

$$\int dx_1^0 \underbrace{\langle x_2^0 | \hat{x}(t) | x_f \rangle}_{\text{Conj-amp}} \underbrace{\langle x_f | \hat{x}(\bar{t}) | x_1^0 \rangle}_{\text{Amplitude with insertion at } \bar{t}}$$

Conj-amp

(ω) insertion

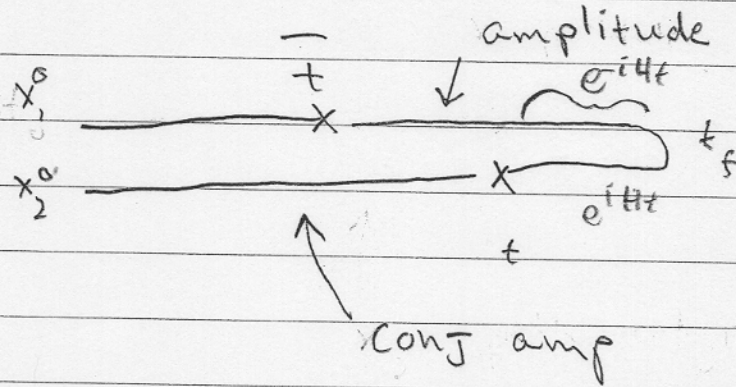
Amplitude with insertion
at \bar{t}

$$\text{Amp} = \int_{x_1^0}^{x_f} Dx_1 e^{iS_1} x_1(\bar{t})$$

$$\text{Conj amp} = \left[\langle x_f | \hat{x}(t) | x_1^0 \rangle \right]^* = \int_{x_2^0}^{x_f} Dx_2 e^{-iS_2} x_2(t)$$

So

$$G^>(t, \bar{t}) = \int dx_1^0 dx_2^0 \rho(x_1^0, x_2^0) \int_{x_1^0, x_2^0}^{X_1, X_2} DX_1 DX_2 e^{iS_1 - iS_2} X_2(t) X_1(\bar{t})$$



$$X_1 = X_2 \Big|_{t=t_f}$$

Now

$$G_{21} \equiv G^>(t, \bar{t}) = \langle X_2(t) X_1(\bar{t}) \rangle$$

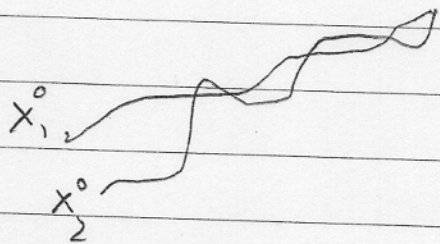
$$G_{12} \equiv G^<(t, \bar{t}) = \langle X_1(t) X_2(\bar{t}) \rangle$$

Notes useful for TFC

$$\langle T[\hat{x} \hat{x}] \rangle = \langle x_1 x_1 \rangle$$

$$\langle \tilde{T}[\hat{x} \hat{x}] \rangle = \langle x_2 x_2 \rangle$$

Now think about semi-classical limit



Amplitude is close to conjugate amp

$$x_r = \frac{x_1 + x_2}{2}$$

$$x_a = x_1 - x_2 \Rightarrow \text{small}$$

$$x_1 = x_r + x_a/2$$

$$x_a \Big|_{t_f} = 0$$

Now, this is useful

$$Z = \int Dx_r Dx_a e^{iS[x_1] - iS[x_2]} = \int Dx_r$$

$$iS[x_1] - iS[x_2] = \overset{\text{small}}{\downarrow} iS[x_r + x_a/2] - iS[x_r - x_a/2]$$

$$\approx \int_t^+ i x_a^{(+)} \frac{\delta S}{\delta x} \Big|_{x_r}$$

$$Z = \int Dx_r Dx_a e^{i \int_t^+ x_a^{(+)} \frac{\delta S}{\delta x} \Big|_{x_r}} = \int Dx_r \delta \left[\frac{\delta S}{\delta x_r} \right]$$

at linear order in x_a get classical EOM

Now lets compute for the HO:

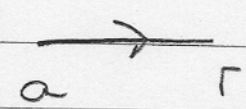
$$iS_1 = i \int dt \left[\frac{1}{2} \dot{x}_1^2 - \frac{1}{2} \omega_0^2 x_1 + F_1 x_1 \right]$$

So

$$iS_1 - iS_2 = i \int dt \left[\dot{x}_r \dot{x}_a - \omega_0^2 x_r x_a + F_r x_a + F_a x_r \right]$$

Response Fcns:

$$iG_{ra}(t, \bar{t}) = \int D x_r D x_a e^{iS_1 - iS_2} x_r(t) x_a(\bar{t})$$

①  $iG_{ra} \equiv iG_{ra}(t, \bar{t}) = \Theta(t - \bar{t}) \langle [\hat{x}_r(t), \hat{x}_a(\bar{t})] \rangle$

Note:

$$-\left(\frac{d^2}{dt^2} + \omega_0^2 \right) G_{ra}(t, \bar{t}) = \delta(t - \bar{t})$$

And Boundary condition $\langle x_r(t) x_a(\bar{t}) \rangle =$
 vanishes at whenever $\bar{t} > t$

$$G_R = \frac{1}{\omega^2 - \omega_0^2 + i\epsilon\omega}$$

Fluctuations

$$G_{rr}(t, \bar{E}) = \langle x_r(t) x_r(\bar{E}) \rangle = \frac{1}{2} \langle \{ \hat{x}(t) \hat{x}(\bar{E}) \} \rangle$$

$$G_{rr}(t, \bar{E}) = \int \rho(x_1^0, x_2^0) \int_{x_1^0, x_2^0} D x_r D x_a e^{i \int dt \dot{x}_r \dot{x}_a - \omega_0^2 x_r x_a} x_r(t) x_r(\bar{E})$$

Now to compute:

$$\textcircled{1} \quad W(x, p) = \int dx_a \rho(x_r + \frac{x_a}{2}, x_r - \frac{x_a}{2}) e^{i p x_a}$$

$$\textcircled{2} \quad x_r^{cl}(t) = m \left[x^0 \partial_{t_0} G_R(t, t_0) - \partial_{t_0} x^0 G_R(t, t_0) \right] \\ = m \left[x^0 \partial_{t_0} G_R(t, t_0) \right]$$

Then

$$G_{rr} = \int \frac{dx dp}{2\pi} W(x_r^0, p) x_r^{cl}(t) x_r^{cl}(\bar{E})$$

$$p = \partial_t x^0$$

So

$$G_{rr} = \begin{array}{c} \leftarrow \text{||} \rightarrow \\ \text{||} \rightarrow \end{array} \quad \text{or} \quad \begin{array}{c} \text{||} \rightarrow \text{||} \\ \text{||} \end{array}$$

For a thermal state

$$\rho = e^{-\beta E_n} |n\rangle\langle n|$$

Find

$$G_{rr}(\omega) = \left(\frac{1}{2} + n(\omega) \right) \left[\frac{2\pi}{2E_p} \delta(\omega - \omega_0) - \frac{2\pi}{2E_p} \delta(\omega - \omega_0) \right]$$

Fluctuation Dissipation Theorem $\rho(\omega)$

$$G_{rr}(\omega) = \left(\frac{1}{2} + n(\omega) \right) \rho(\omega) \leftarrow \text{modes are thermally occupied}$$

or

$$G^>(\omega) = G^<(\omega) e^{\beta\omega}$$

Why FDT is important:

Consider photon emission and absorption from SQG

$$2E_p \left[\partial_t + v_p \frac{\partial}{\partial x} \right] f_p = - \underbrace{f_p \Pi^>(p)}_{\text{Loss}} + \underbrace{\Pi^<(p)(1+f_p)}_{\text{gain}}$$

$$\Pi^>(p) = \text{FT}_{\text{trans}} \langle J(t) J(0) \rangle$$

$$\Pi^<(p) = \text{FT}_{\text{trans}} \langle J(0) J(t) \rangle$$

Using: $\rho(p) = \Pi^> - \Pi^<$

$$\Pi^>(p) = e^{p\hbar/T} \Pi^<$$

Find: Do it!

$$2E_p \left(\partial_t + v_p \frac{\partial}{\partial x} \right) f_p = - \left(f_p - \frac{1}{e^{p\hbar/T} - 1} \right) \overset{\text{positive def}}{\rho(p)}$$

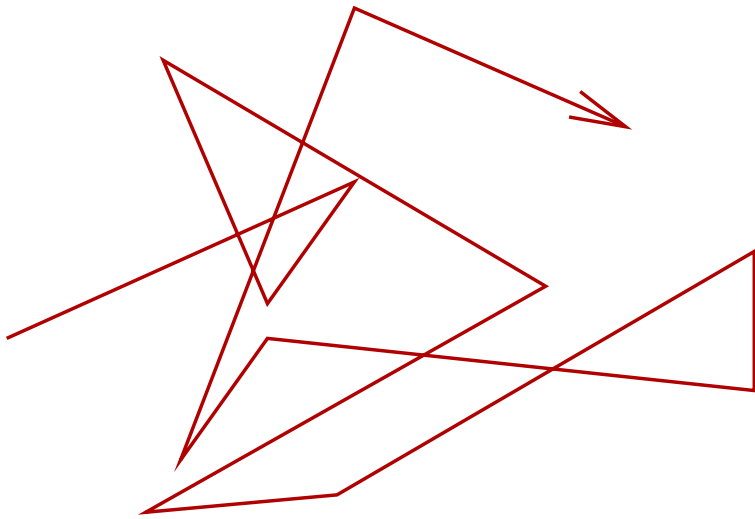
So f_p will evolve until it reaches equilibrium

Thermalization of fluctuations in strongly coupled plasmas

- Dam T. Son, DT; JHEP. arXiv:0901.2338
- Simon Caron-Huot, DT, Paul Chesler; PRD in press, arXiv:1102.1073

Heavy Quarks in equilibrium Quantum Field Theory

$$M \frac{d^2 \mathbf{x}}{dt^2} = \underbrace{-\eta \dot{\mathbf{x}}}_{\text{Drag}} + \underbrace{\xi}_{\text{Noise}}$$



“Artist’s” conception
of Brownian Motion

1. In equilibrium the drag and noise are balanced

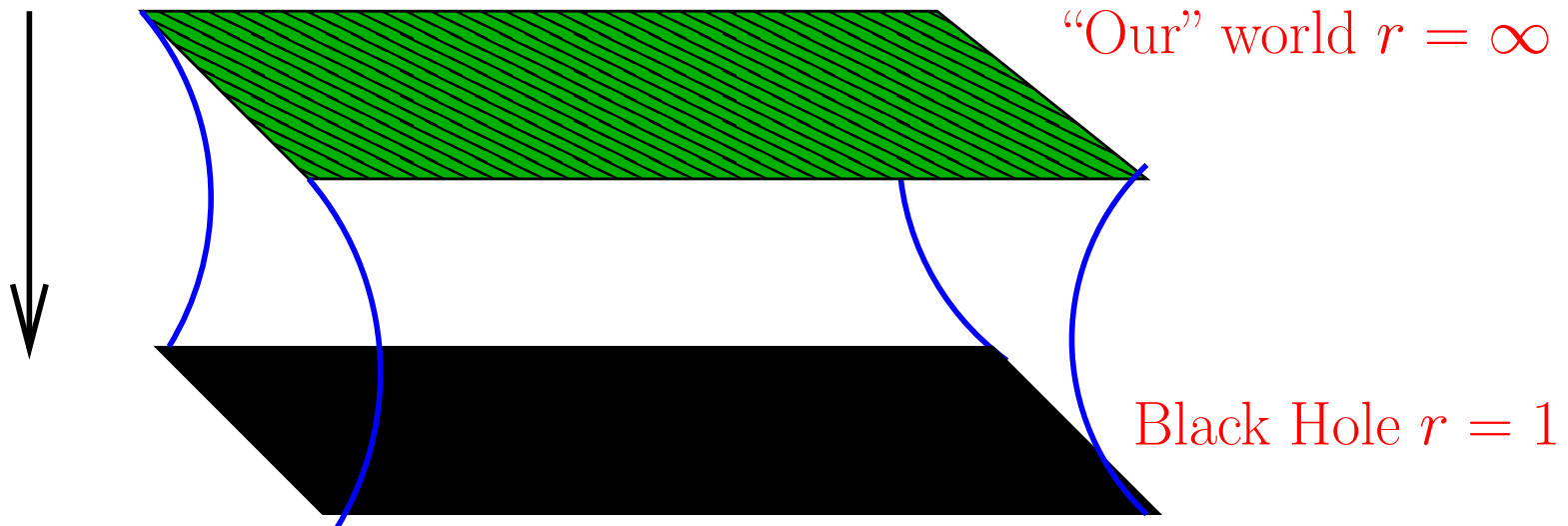
$$\langle \xi(t) \xi(t') \rangle = 2T\eta \delta(t - t') \Leftrightarrow \text{Fluctuation Dissipation Theorem}$$

AdS/CFT

- Classical solutions in curved spacetime = CFT for nonzero temperature

$$ds^2 = (\pi T)^2 r^2 \left[-f(r) dt^2 + dx^2 \right] + \frac{dr^2}{r^2 f(r)} \quad f(r) = 1 - \frac{1}{r^4}$$

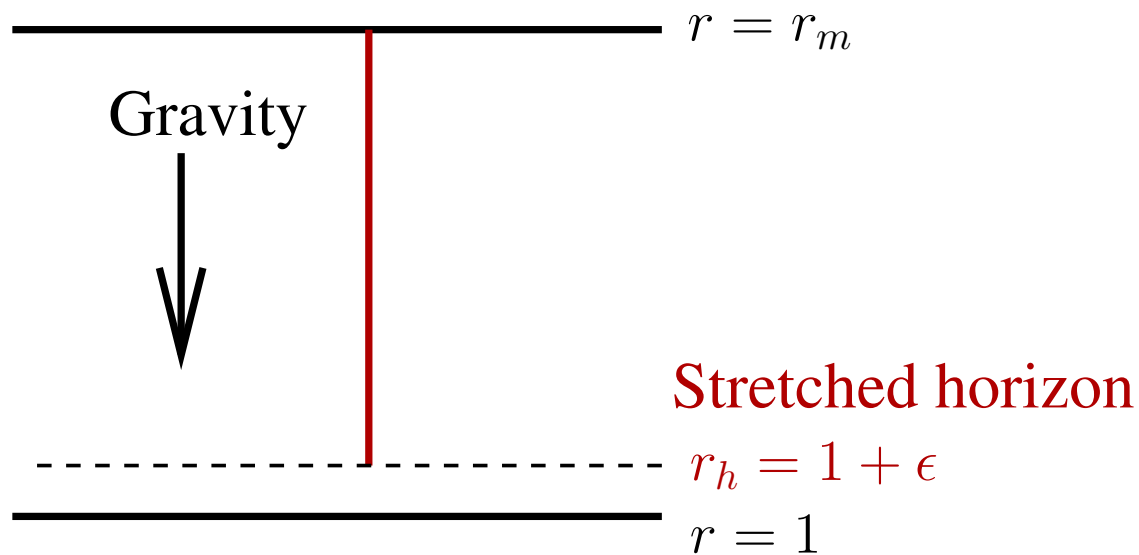
Gravity



How can a static metric be dual to equilibrium=constant fluctuations ?

Heavy Quarks in equilibrium AdS

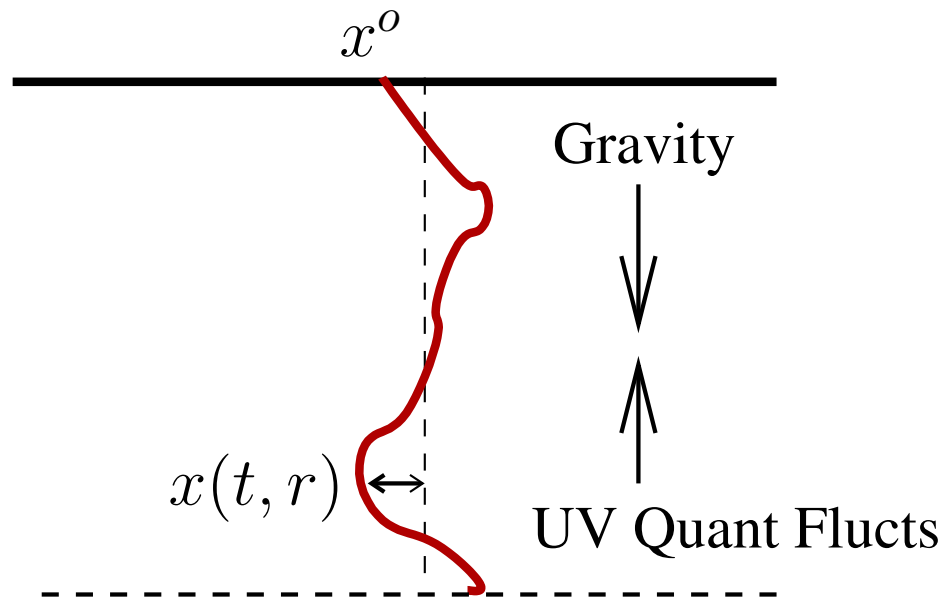
- Heavy quarks are classical strings in the 5d equilibrium AdS black hole geometry
- Solve classical string EOM and find:



Not the dual of an equilibrated quark!

Detailed Balance and Hawking Radiation:

$$M \frac{d^2 x^o}{dt^2} = \underbrace{-\eta \dot{x}^o}_{\text{Drag}} + \underbrace{\xi}_{\text{Noise}}$$



Evolves to Classical

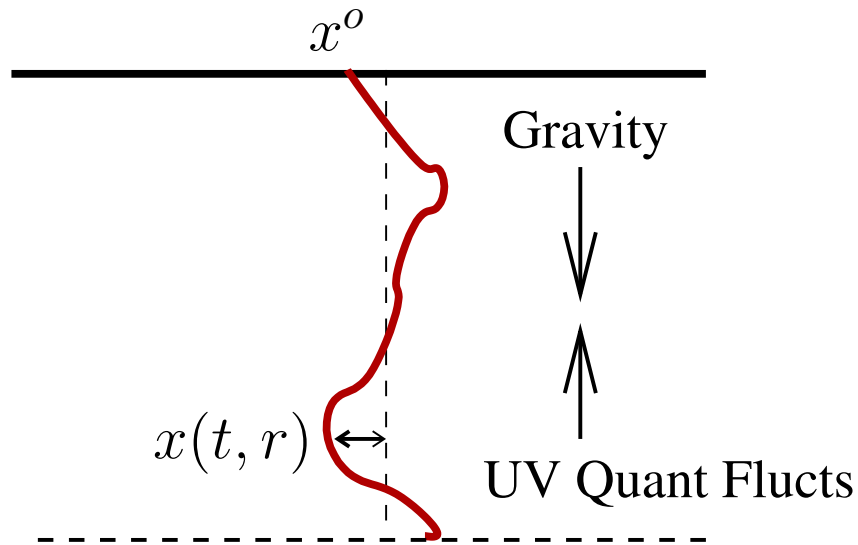
Prob Dist: (Son,DT;lancu)

$$P[x, \pi_x] \propto e^{-\beta H[x, \pi_x]}$$

Goals:

1. Will show that Hawking Radiation is balanced by gravity
2. Generalize to non-equilibrium

Detailed Balance and Hawking Radiation (Technical Discussion)



1. Fluctuations:

$$G_{rr} \equiv \frac{1}{2} \langle \{ \hat{x}(t_1, r_1), \hat{x}(t_2, r_2) \} \rangle ,$$

2. Dissipation (Spectral Density)

$$\rho_{ra-ar} \equiv \langle [\hat{x}(t_1, r_1), \hat{x}(t_2, r_2)] \rangle .$$

• Equilibrium \equiv Fluctuation Dissipation Theorem

$$G_{rr}(\omega, r_1, r_2) = \left(\frac{1}{2} + n_B(\omega) \right) \rho_{ra-ar}(\omega, r_1, r_2) \quad n(\omega) \equiv \frac{1}{e^{\omega/T} - 1}$$

Formulas

- Action for string fluctuations, $h^{\mu\nu}$ = string metric

$$S_1 - S_2 = \frac{\sqrt{\lambda}}{2\pi} \int dt dr g_{xx} \left[-\sqrt{h} h^{\mu\nu} \partial_\mu x_r \partial_\nu x_a \right] ,$$

- $h^{\mu\nu}$ is the string metric

$$h_{\mu\nu} d\sigma^\mu d\sigma^\nu = -(\pi T)^2 r^2 f(r) dt^2 + \frac{dr^2}{f(r)r^2} ,$$

- Retarded Green Function

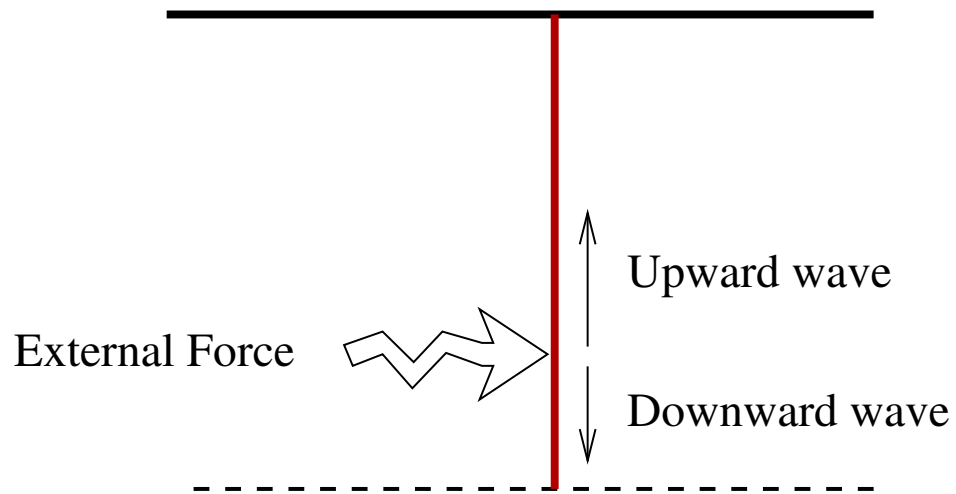
$$iG_{ra}(t_1 r_1 | t_2 r_2) \equiv \theta(t - t') \langle [\hat{x}(t_1, r_1), \hat{x}(t_2, r_2)] \rangle ,$$

$G_{ra}(t_1 r_1 | t_2 r_2)$ is the classical response to a force at $t_2 r_2$

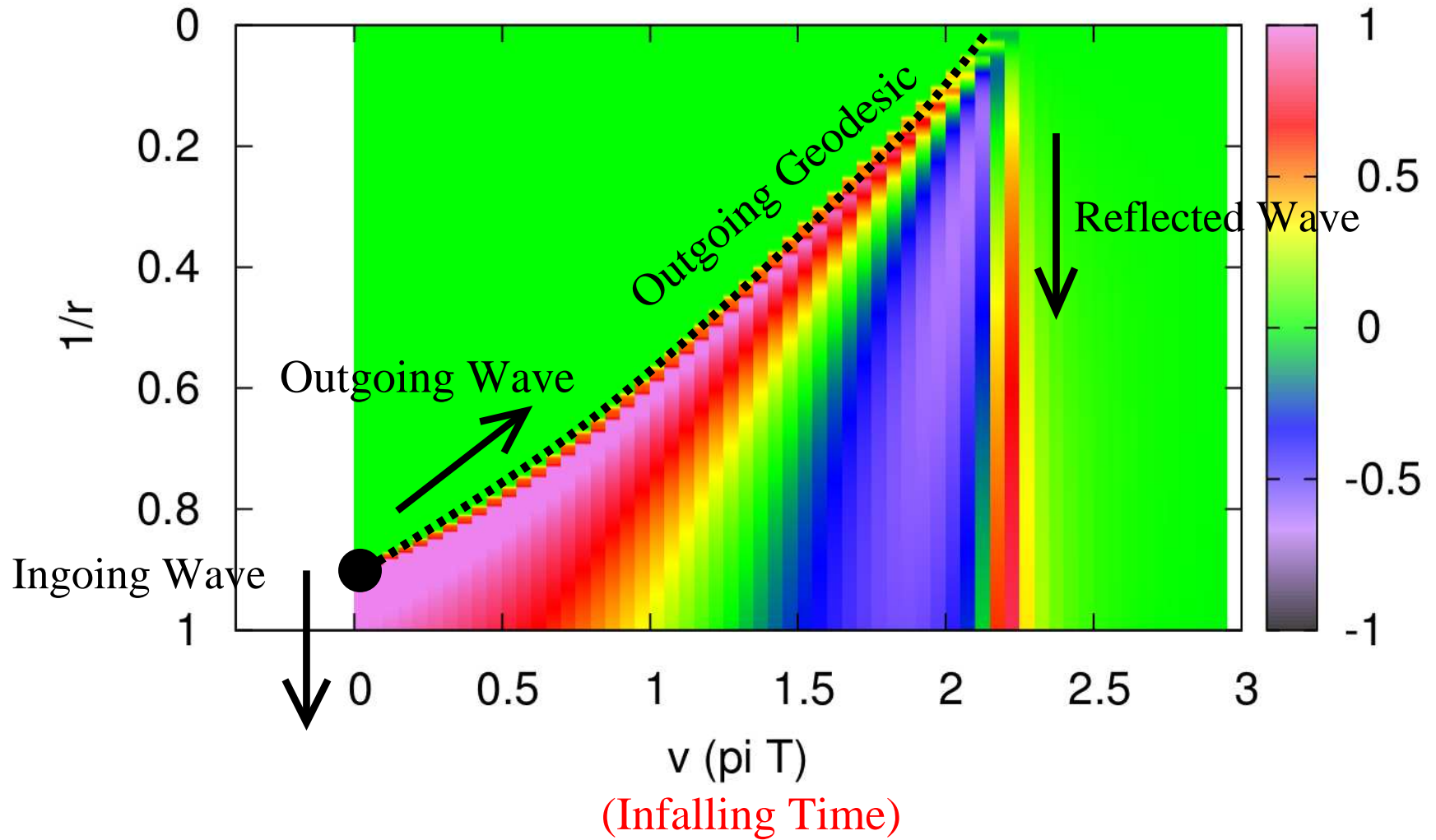
$$\frac{\sqrt{\lambda}}{2\pi} \left[\partial_\mu g_{xx} \sqrt{h} h^{\mu\nu} \partial_\nu \right] G_{ra}(t_1 r_1 | t_2 r_2) = \delta(t_1 - t_2) \delta(r_1 - r_2) ,$$

The classical Green Function or response to a force:

$$\frac{\sqrt{\lambda}}{2\pi} \left[\partial_\mu g_{xx} \sqrt{\hbar} h^{\mu\nu} \partial_\nu \right] G = \mathcal{F} \delta(t_1 - t_2) \delta(r_1 - r_2),$$

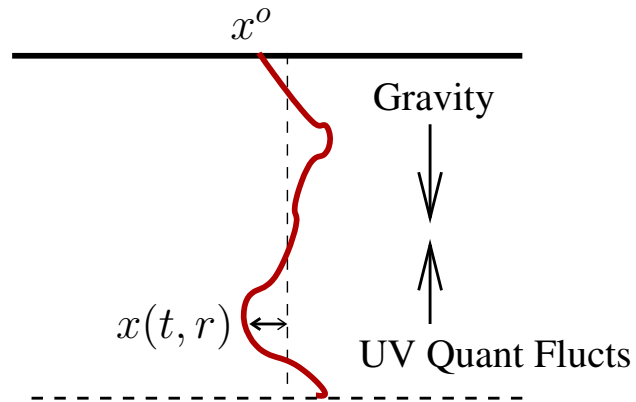


Retarded Response function



$$v = t - \frac{1}{2\pi T} \left[\tan^{-1}(r) + \tanh^{-1}(r) \right] \quad v = \text{Eddington time}$$

Statistical Fluctuations



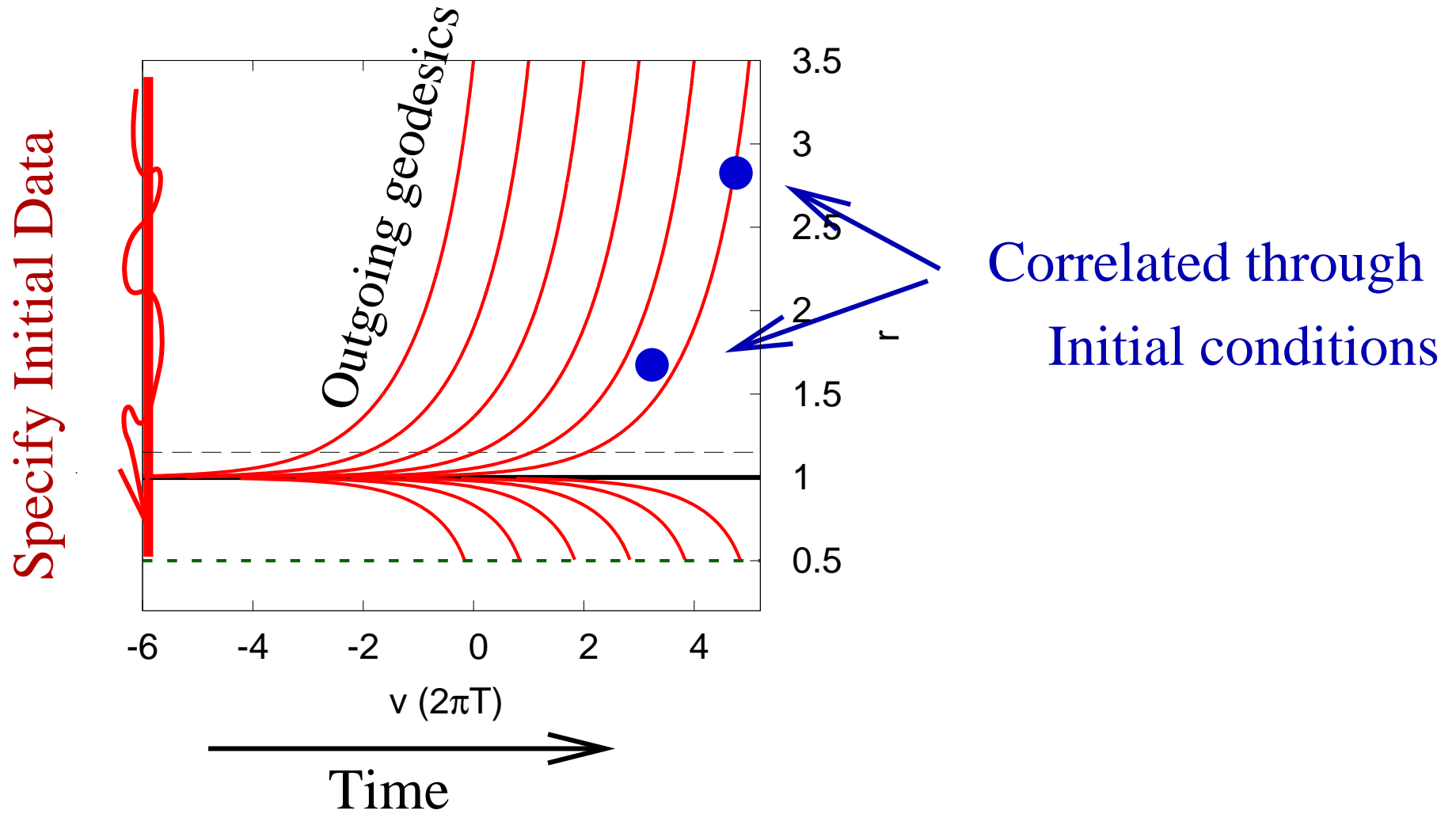
$$G_{rr} = \frac{1}{2} \langle \{x(t_1, r_1), x(t_2, r_2)\} \rangle$$

- The statistical correlator obeys the homogeneous EOM

$$\frac{\sqrt{\lambda}}{2\pi} \left[\partial_\mu g_{xx} \sqrt{\hbar} h^{\mu\nu} \partial_\nu \right] G_{rr}(t_1 r_1 | t_2 r_2) = 0$$

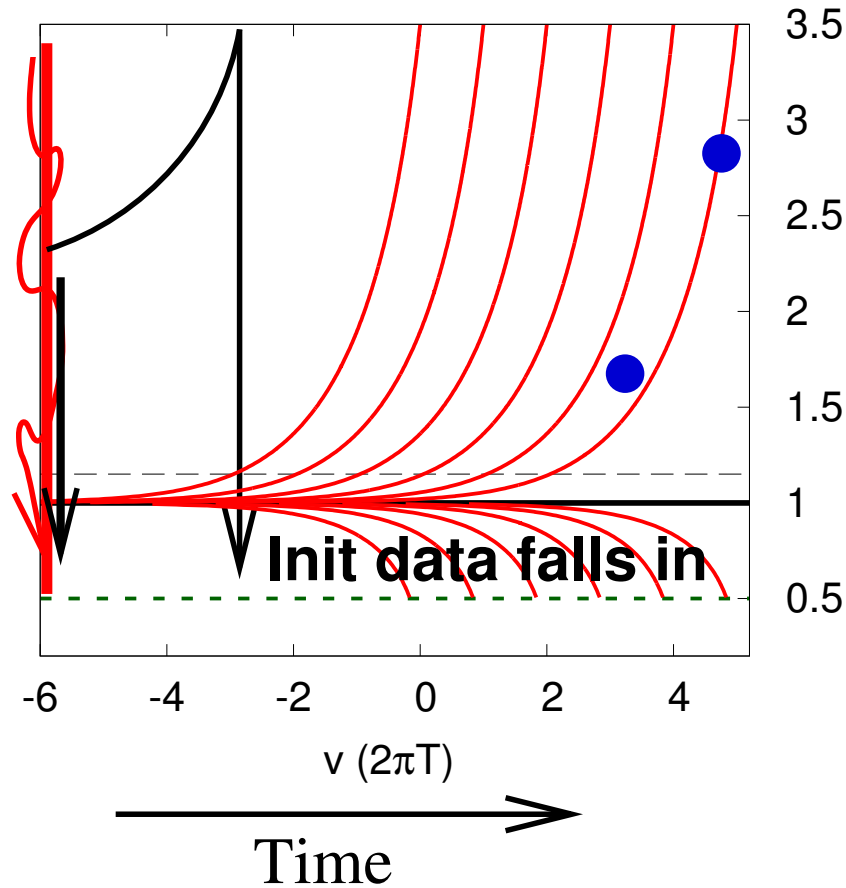
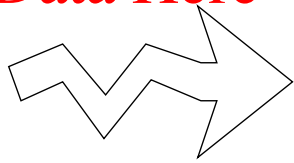
- So:
 1. Specify the correlations (or density matrix) in the past
 2. Final state fluctuations are correlated only through initial conditions

Correlations through Initial conditions



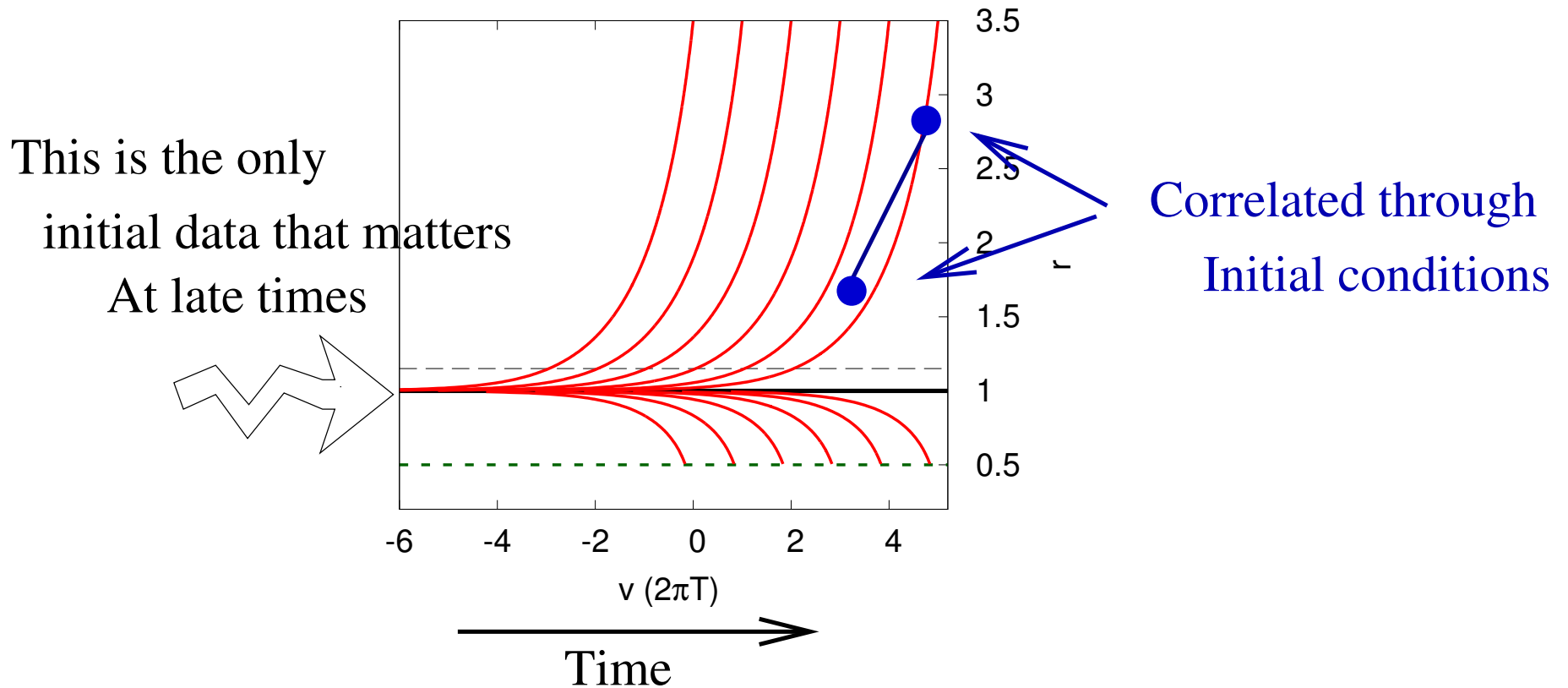
Correlations through Initial conditions

Consider Init
Data Here



Points uncorrelated
by this Init data

Correlations through Initial conditions



1. Final correlation come from correlated initial data very near horizon
 - Short Wavelength
2. Initial data is inflated by near horizon geometry

Initial Data from Quantum Fluctuations

1. Initial data is determined at short distance = Flat Space Physics
2. Scalar Field in 1+1D vacuum flat space

$$\frac{1}{2} \langle \{ \phi(X_1), \phi(X_2) \} \rangle = -\frac{1}{4\pi K} \log |\mu \eta_{\mu\nu} \Delta X^\mu \Delta X^\nu| \quad K = \text{norm of action}$$

3. String flucfs in near horizon geometry

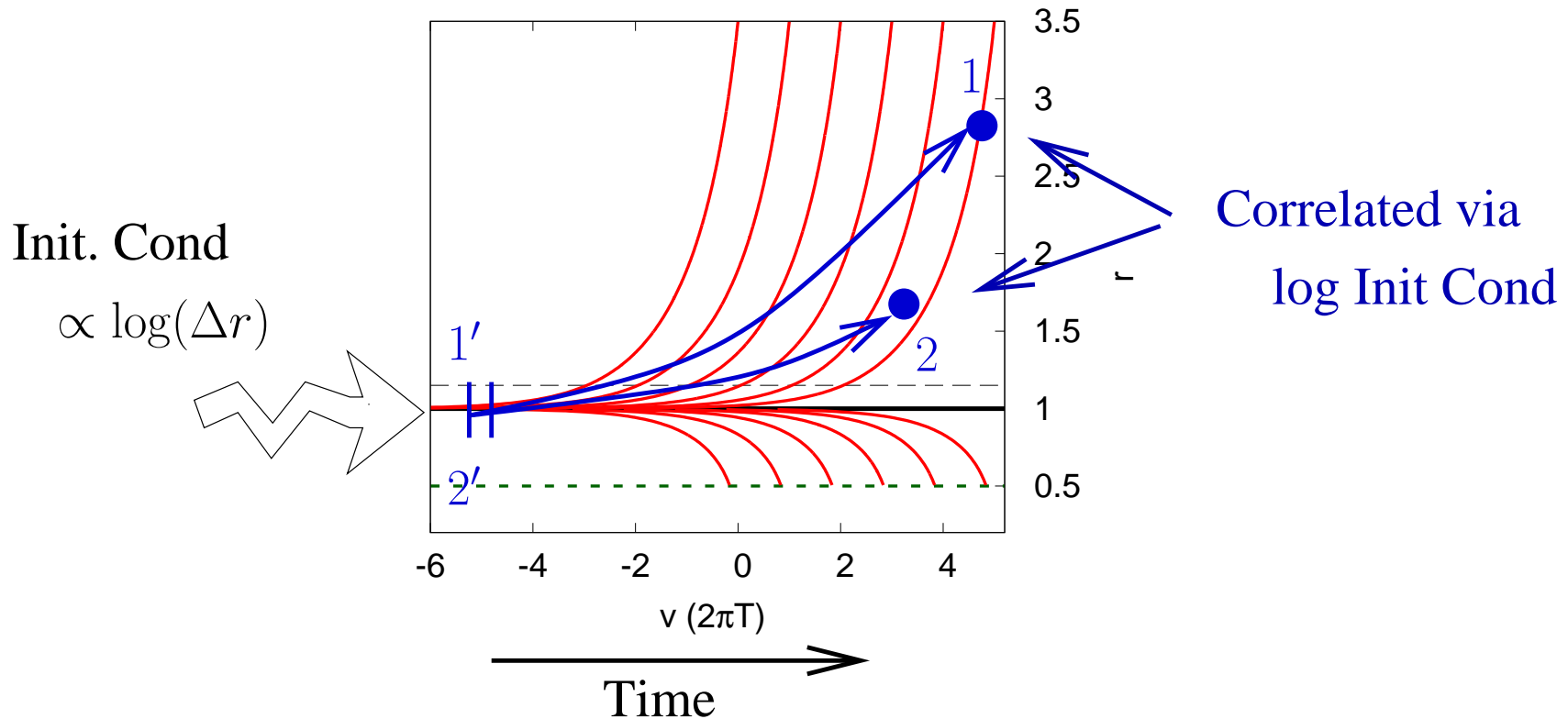
$$S^{\text{near-horizon}} = \eta \int dt dr \left[-\frac{1}{2} \sqrt{h} h^{\mu\nu} \partial_\mu x \partial_\nu x \right] \quad \eta = \text{Drag Coefficient}$$

The near horizon initial condition is:

$$G_{rr}(v_1 r_1 | v_2 r_2) \rightarrow -\frac{1}{4\pi\eta} \log \left| \mu \overbrace{2\Delta v \Delta r}^{\text{local } \Delta s^2} \right|$$

Summary: Specify IC and Solve Equations of Motion

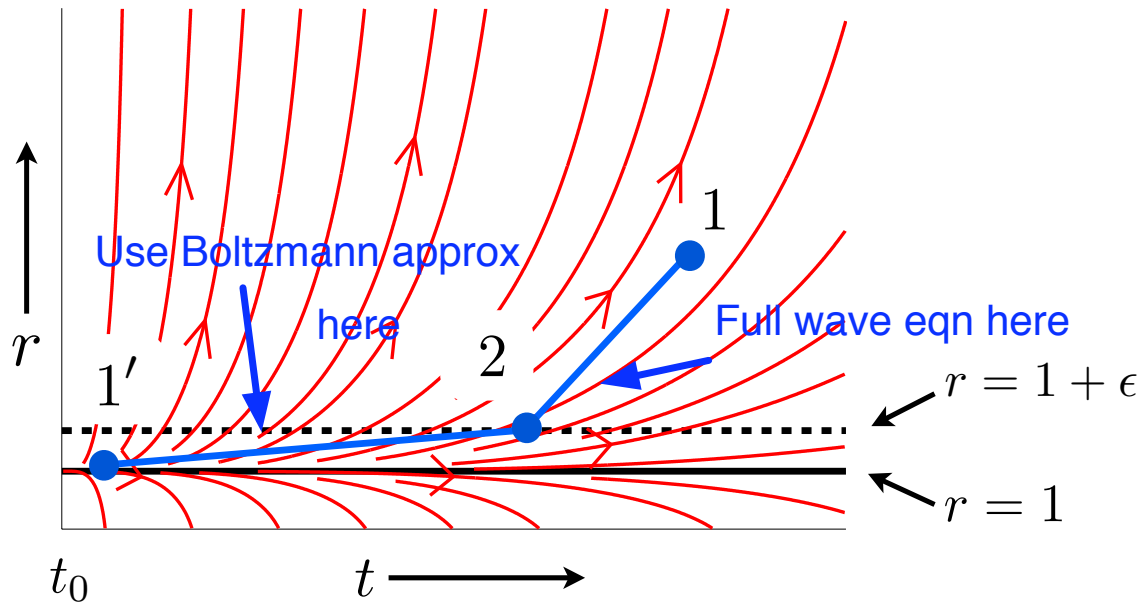
$$\frac{\sqrt{\lambda}}{2\pi} \left[\partial_\mu g_{xx} \sqrt{\hbar} h^{\mu\nu} \partial_\nu \right] G_{rr}(t_1 r_1 | t_2 r_2) = 0$$



$$G_{rr}(1|2) = \left[\frac{\sqrt{\lambda}}{2\pi} \int dr'_1 g_{xx} \sqrt{\hbar} h^{tt}(r'_1) G_{ra}(1|1') \overleftrightarrow{\partial}_{t'_1} \right] \text{Like Harmonic Oscillator}$$

$$\times \left[\frac{\sqrt{\lambda}}{2\pi} \int dr'_2 g_{xx} \sqrt{\hbar} h^{tt}(r'_2) G_{ra}(2|2') \overleftrightarrow{\partial}_{t'_2} \right] G_{rr}(1'|2'),$$

From initial data to final correlations in two steps:

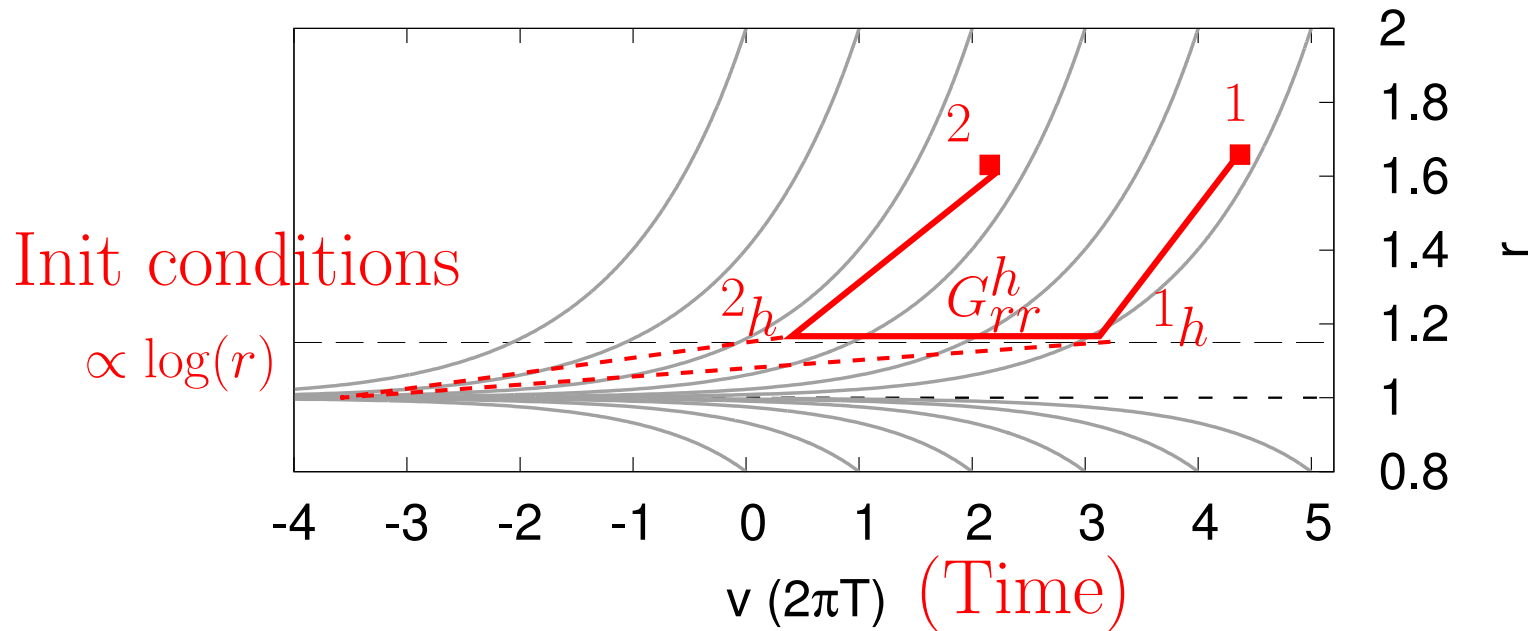


$$G_{ra}(1|1') = \int dt_2 G_{ra}(1|2) \left[\eta \sqrt{h} h^{rr}(r_2) \overleftrightarrow{\partial}_{r_2} \right]_{r_2=1+\epsilon} G_{ra}(2|1'),$$

- (a) From horizon to stretched horizon – Waves are very short wavelength
- Use collisionless Boltzmann approximation (geodesic/WKB/eikonal approx)
- (b) The stretched horizon to boundary – Waves are longer wavelength
- Use full wave equation

Fluctuations from Equations of Motion

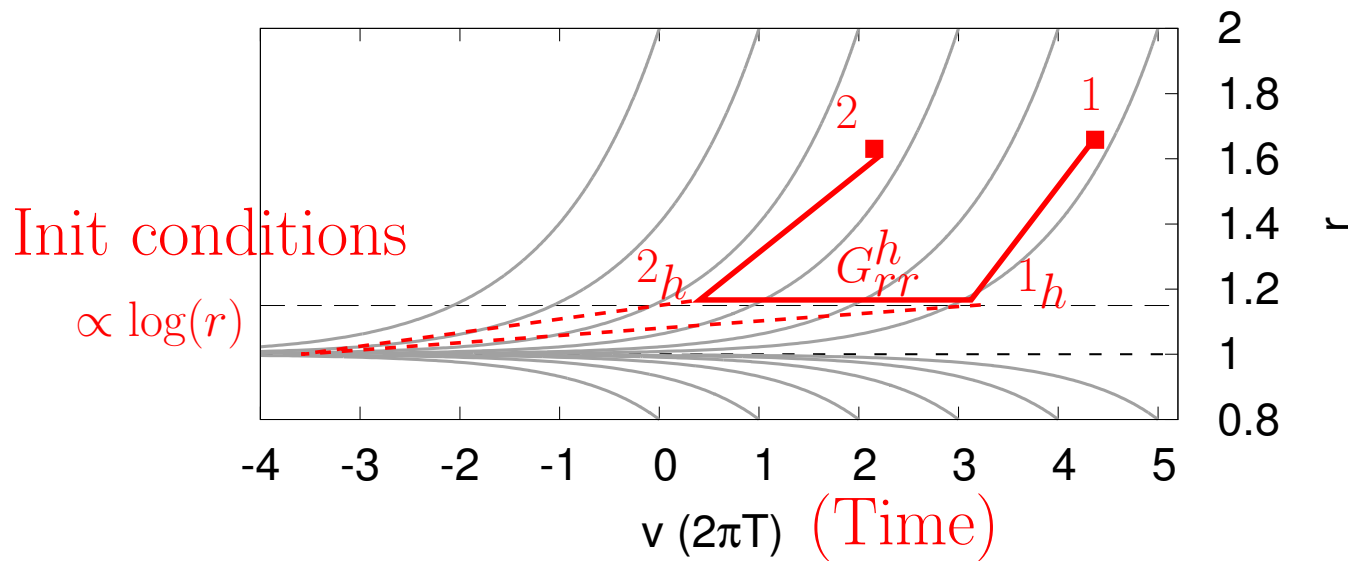
$$\underbrace{G_{rr}(1|2)}_{\text{bulk fluc}} = \int dt_{1h} dt_{2h} \underbrace{G_R(1|1_h) G_R(2|2_h)}_{\text{outgoing Green fcn}} \underbrace{G_{rr}^h(1_h|2_h)}_{\text{horizon fluc}},$$



The fluctuations on the stretched horizon are from UV vacuum fluc in past

$$\begin{aligned} G_{rr}^h(t_1|t_2) &= \text{Blow-up of initial data } \propto \log(r) \\ &= -\frac{\eta}{\pi} \partial_{t_1} \partial_{t_2} \log |1 - e^{-2\pi T(t_1 - t_2)}|. \end{aligned}$$

The horizon fluctuations and the Lyapunov exponent



1. Thermal looking:

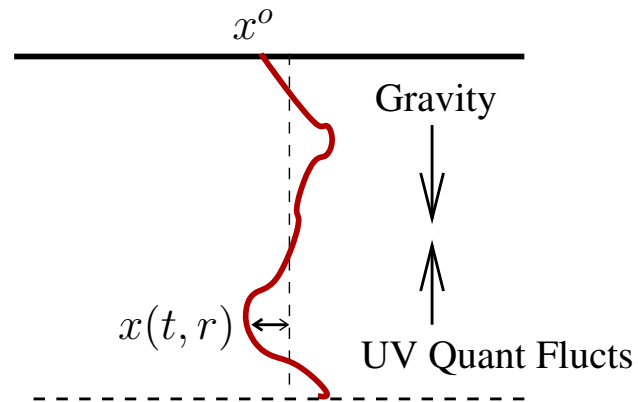
$$G_{rr}^h(\omega) = \text{Fourier-Trans of } -\frac{\eta}{\pi} \partial_{t_1} \partial_{t_2} \log |1 - e^{-2\pi T(t_1 - t_2)}|$$

$$= \left(\frac{1}{2} + n(\omega)\right) 2\omega\eta \quad n(\omega) \equiv \frac{1}{e^{\omega/T} - 1}$$

2. Temperature \propto inflation rate

$$2\pi T = \text{Lyapunov exponent of diverging geodesics}$$

Dissipation - Spectral Density



$$\rho_{ra-ar} = \langle [\hat{x}(t_1, r_1), \hat{x}(t_2, r_2)] \rangle$$

- The spectral density also obeys the EOM

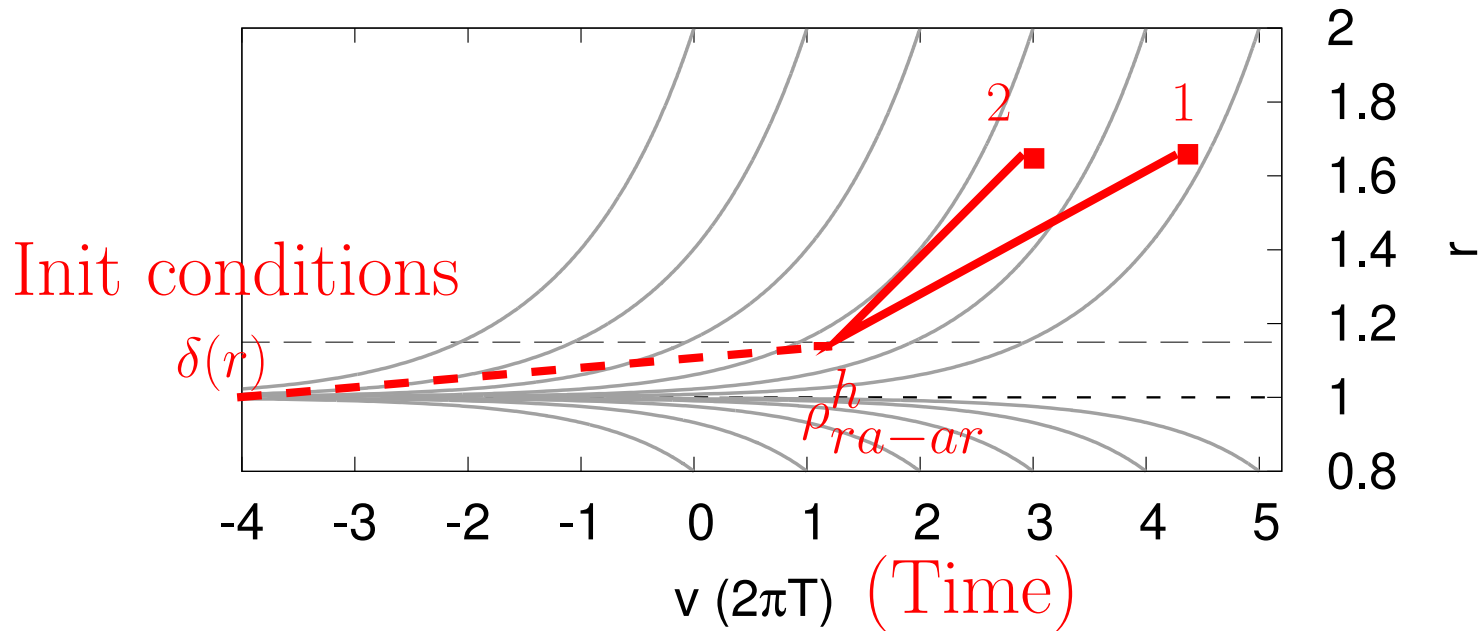
$$\frac{\sqrt{\lambda}}{2\pi} \left[\partial_\mu g_{xx} \sqrt{h} h^{\mu\nu} \partial_\nu \right] \rho_{ra-ar}(t_1 r_1 | t_2 r_2) = 0$$

- But initial conditions are given by the canonical commutation relations

$$\eta \sqrt{h} h^{tt}(r_1) \lim_{t_2 \rightarrow t_1} \partial_{t_1} \rho_{ra-ar}(t_1 r_1 | t_2 r_2) = i\delta(r_1 - r_2).$$

Spectral Density

$$\underbrace{\rho_{ra-ar}(1|2)}_{\text{bulk spectral fcn}} = \int dt_{1h} dt_{2h} \underbrace{G_R(1|1_h) G_R(2|2_h)}_{\text{outgoing Green fcn}} \underbrace{\rho_{ra-ar}^h(1_h|2_h)}_{\text{horizon spectral fcn}},$$

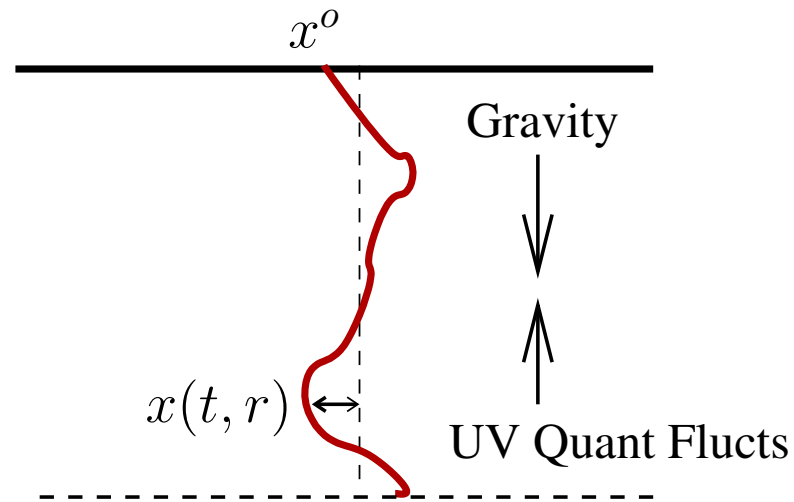


Where the horizon spectral density

$$\begin{aligned} \rho_{ra-ar}^h(t_1, t_2) &= \text{local due to canonical commutation relations} \\ &= 2\eta \left[-i\delta'(t_1 - t_2) \right] \quad (2\omega\eta \text{ in Fourier space}) \end{aligned}$$

Detailed Balance

$$G_{rr}(\omega, r_1, r_2) = \left(\frac{1}{2} + n(\omega)\right) \rho(\omega, r_1, r_2)$$



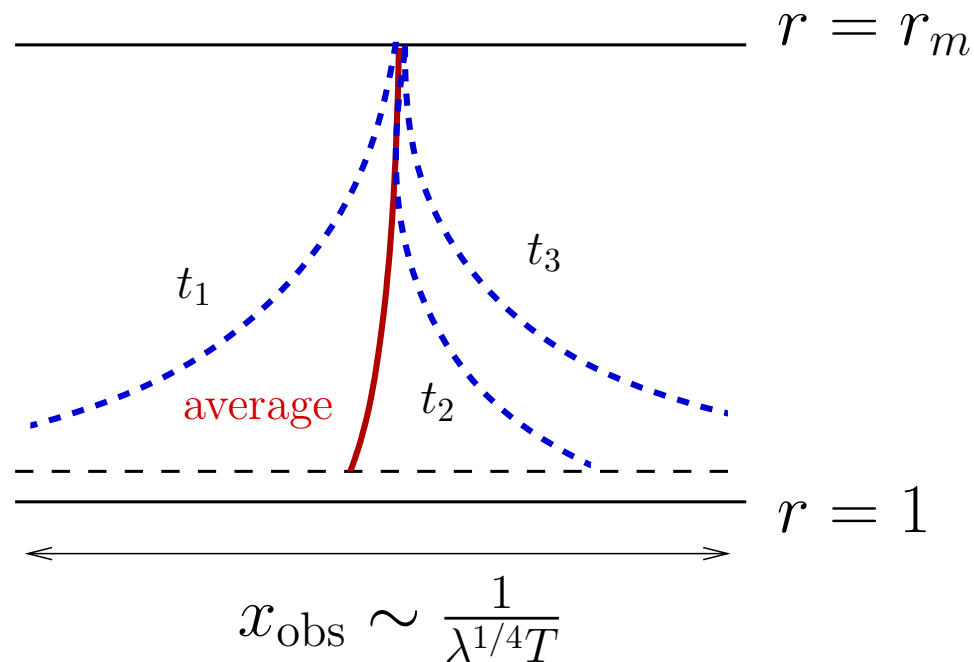
1. Fluctuations (Anti-commutator)

$$\underbrace{G_{rr}(\omega, r_1, r_2)}_{\text{bulk fluc}} = \underbrace{G_R(\omega, r_1|r_h) G_R(\omega, r_2|r_h)}_{\text{outgoing Green fcn}} \underbrace{\left(\frac{1}{2} + n(\omega)\right) 2\omega\eta}_{\text{Horizon-fluc}}$$

2. Dissipation: (Commutator)

$$\underbrace{\rho_{ra-ar}(\omega, r_1, r_2)}_{\text{bulk spec dense}} = \underbrace{G_R(\omega, r_1|r_h) G_R(\omega, r_2|r_h)}_{\text{outgoing Green fcn}} \underbrace{2\omega\eta}_{\text{Horizon spec dense}}$$

Fluctuation dissipation and stochastic dynamics



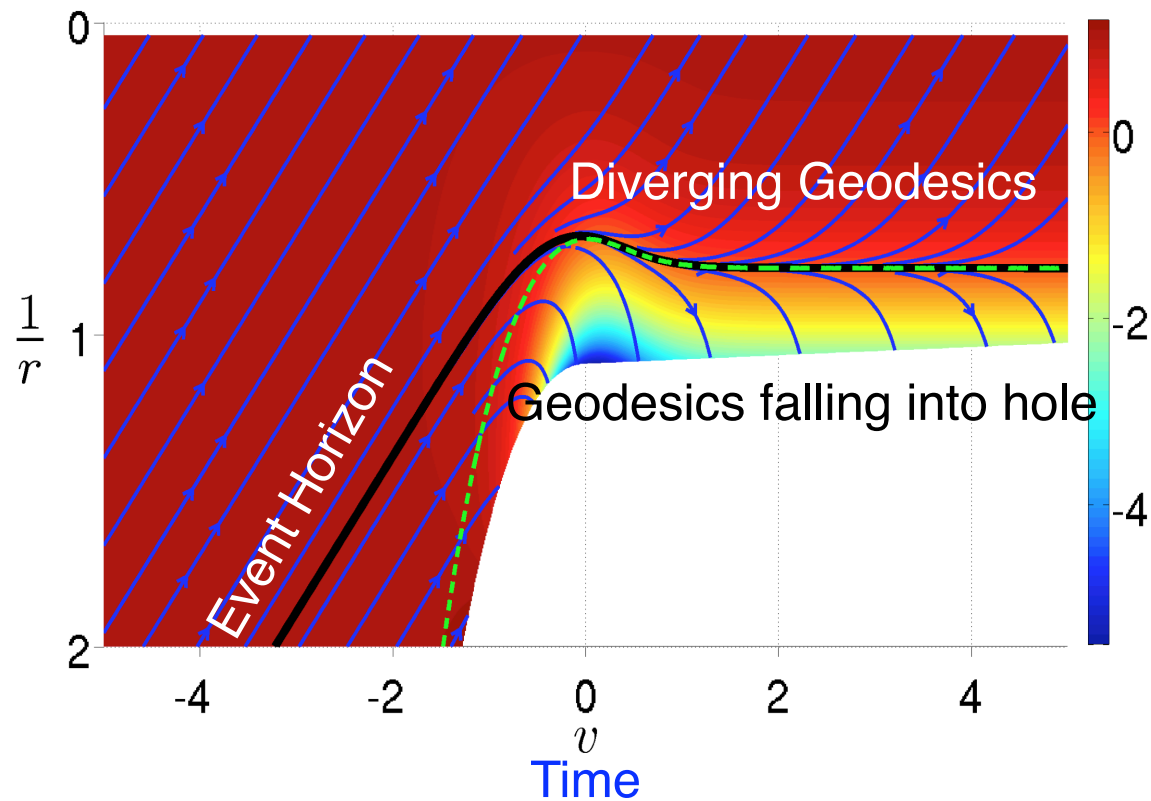
1. Every step t_1, t_2, t_3 fluctuates to a new trailing string – \rightarrow random force
2. The average of the trailing strings gives the drag – average string \rightarrow drag

Non-equilibrium

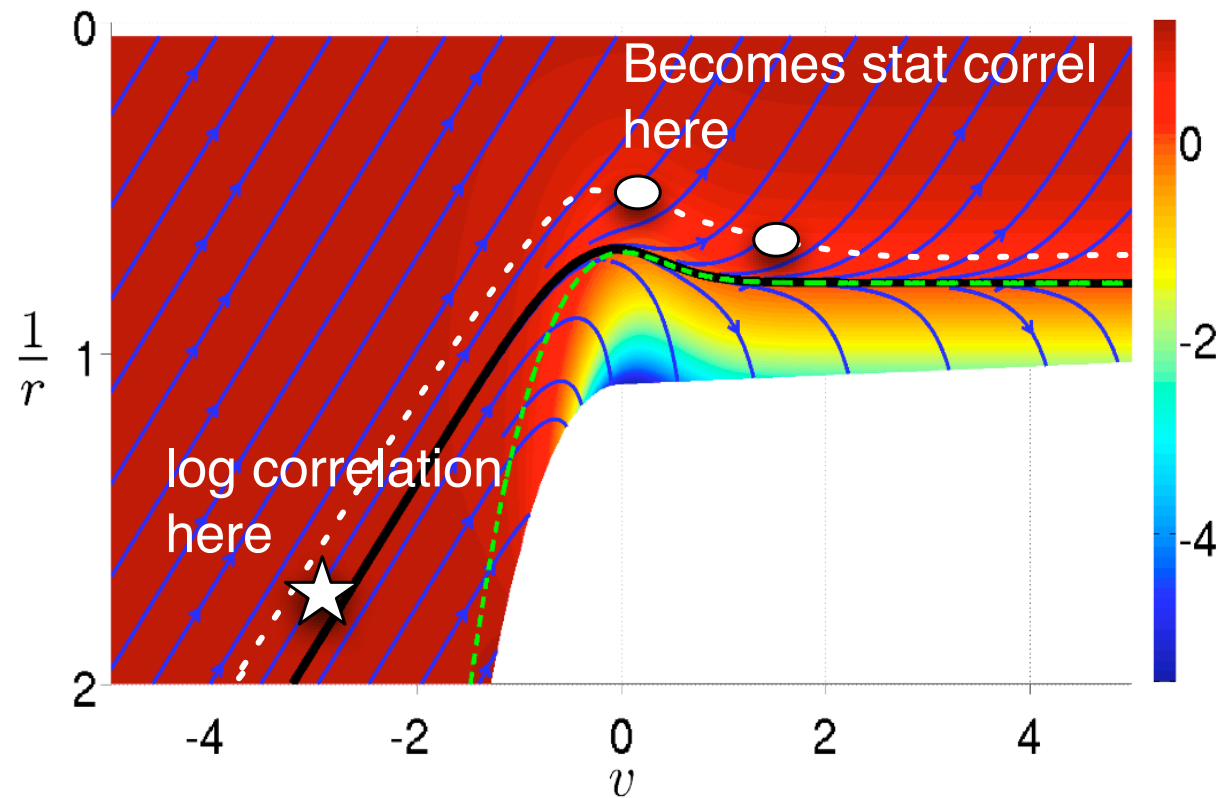
Non-equilibrium setup

Chesler-Yaffe

1. Chesler&Yaffe create QGP by turning a gravitational pulse in vacuum
2. Corresponds to non-equilibrium geometry with BH formation in AdS_5



Fluctuations in non-equilibrium



- Surface Properties – on event horizon

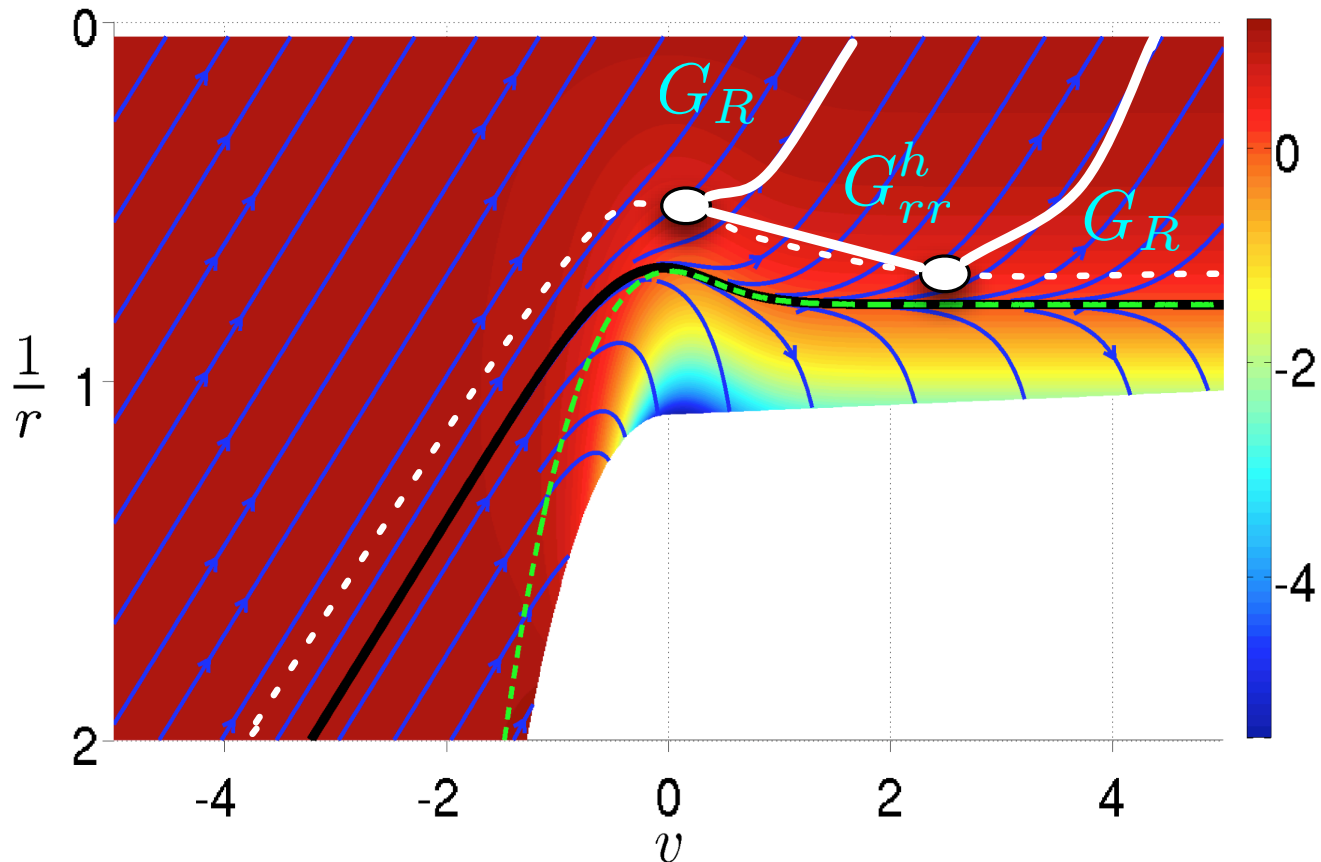
$$\underbrace{2\pi T_{\text{eff}}(v)}_{\text{Lyapunov exponent}} \equiv \left. \frac{\overbrace{\frac{1}{2} \frac{\partial A(r, v)}{\partial r}}^{\text{Metric-coeff}}}{\partial r} \right|_{r=r_h(v)} \propto \text{extrinsic curvature}$$

Result:

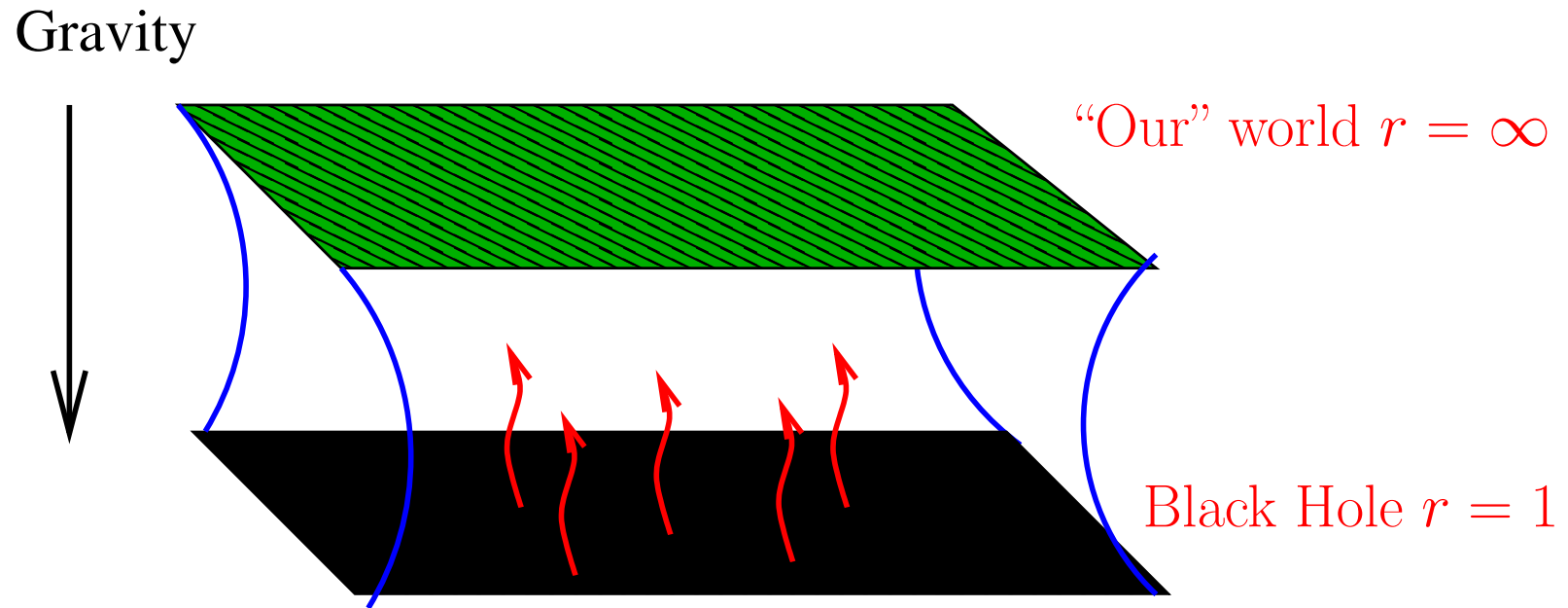
- General form of near horizon fluctuations in non-equilibrium

$$G_{rr}^h(v_1|v_2) = -\frac{\sqrt{\eta(v_1)\eta(v_2)}}{\pi} \partial_{v_1} \partial_{v_2} \log \left| 1 - e^{-\int_{v_1}^{v_2} 2\pi T_{\text{eff}}(v') dv'} \right|.$$

- Can map the near horizon fluctuations up to boundary (numerics in progress)



Not conclusions, but picture:



Gravity pulls down, but quantum fields fluctuate, reaching equilibrium