Aspects of the Quantum Hall Effect

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Slightly edited version of the introductory chapter from *Quixotic Order and Broken Symmetry in the Quantum Hall Effect and its Analogs*, S.A. Parameswaran, Princeton University Ph.D. Thesis, 2011.

Contents

I.   A BRIEF HISTORY, 1879-1984

In 1879, Edwin Hall, a twenty-four-year-old graduate student at Johns Hopkins University, was confounded by two dramatically different points of view on the behavior of a fixed, current-carrying wire placed in a magnetic field. The first, espoused by no less authority than Maxwell [63], was that the electromagnetic forces acted not on currents but on the conductor itself, so that if the latter were immobile there would be no effect whatsoever once transients died down. The second [19] held that the forces acted on moving charges, and so there should be measurable consequences on transport through the wire even if it were held fixed. Understandably confused, Hall consulted his doctoral advisor, Henry Rowland, and with his help designed an experiment in favor of the latter view [36], to wit: *"If the current of electricity in a fixed conductor is itself attracted by a magnet, the current should be drawn to one side of the wire, and therefore the resistance experienced should be increased."* With this succinct observation – and the experimental *tour de force* that followed – Hall became the first to study his eponymous effect. As the modern theory of metals was developed in

the mid-twentieth century, Hall effect measurements were applied to a variety of problems: they served not only as a means to measure the sign of charge carriers in different materials, but also in constructing magnetometers and sensors for various uses.

Beginning in the 1930s, a series of experiments began to probe quantum mechanical phenomena in the transport of electrons. The Shubnikov – de Haas and de Haas – van Alphen effects were the first in a series of 'quantum oscillations' in various quantities – resistivity and magnetization respectively in the initial examples, but eventually many others – observed as an applied external magnetic field was varied. Seminal work by Landau [54] on the quantization of cyclotron orbits of quadratically dispersing electrons in magnetic fields, and semiclassical extensions to more complicated situations allowed a unified explanation of the different measurements. This work also led to an appreciation of the fact that quantum oscillations provide an extremely precise technique for measuring the shapes of Fermi surfaces [76]. Experiments progressed rapidly[1], and with each successive refinement increasingly baroque Fermi surfaces were mapped out, enhancing greatly the understanding of various metallic phenomena.

Roughly in parallel with these developments, the technological applications of solid state physics developed, at a pace that multiplied tremendously following the invention of the transistor. Increasingly elaborate semiconductor devices were engineered; originally these were intended solely for industrial applications, but gradually it was recognized that there was interesting and fundamental physics to be mined, for quantum-mechanical phenomena become visible in such devices, particularly if they confine electrons in extremely clean structures of reduced dimensionality. As a harbinger of things to come, in 1966 Shubnikov–de Haas oscillations were observed in a two-dimensional electron gas (2DEG) in a silicon metal-oxide-semiconductor field-effect transistor (MOSFET) [22].

Just over a century after Hall's experiments, von Klitzing, Pepper and Dorda made careful measurements of the Hall effect, in a silicon MOSFET [111]. At magnetic fields sufficiently high that that the characteristic energies of Landau quantization were larger than the ambient temperature scale – the 'extreme quantum limit' – they observed that the Hall resistance was quantized in integer multiples of the fundamental resistance quantum[2]

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[1] For an entertaining account of the historical development of the field, see [95].

[2] Subsequently renamed the von Klitzing constant; perhaps more so than in any other branch of condensed matter physics, eponyms flourish in quantum Hall physics.

h/e^2: rather than show a smooth linear rise with changing field, the Hall resistance trace described a series of plateaus. Within each plateau, the longitudinal resistance was nearly zero, but had sharp peaks at each step between plateaus. That step-like features were seen in experimental observations was not particularly surprising, given Landau's work: the centers of the plateaus occured when the number of electrons was an integer multiple of the number of available eigenstates at a given energy, with this integer – known as the 'filling factor' ν – setting the Hall resistance. However, it rapidly became apparent that the existing theory of transport in metals was unable to account for the fantastic accuracy with which the quantization occurred, especially as samples were tuned away from the 'magic' commensurate points. Such universality, independent of microscopic details, hinted strongly that some deeper principle was at work, 'protecting' the Hall conductance from correction by such experimental complications as sample imperfections, field inhomogeneities and electron density differences.

In 1981, Laughlin gave a beautiful explanation of the universality of the experimental observations in terms of *adiabatic cycles* in the space of Hamiltonians [55]. Subsequently refined by Halperin [37], his argument rests on a simple fact: if we thread a quantized flux through the hole of a non-simply connected sample – for concreteness, say in the shape of an annulus – the Hamiltonian (and hence the spectrum) returns to itself. In physical terms, the only net result of this adiabatic cycle could be that an integer charge was transferred from one edge to the other, thereby making it possible to do work against a potential gradient in a direction transverse to the electric current induced by the changing flux. This gives rise to a Hall conductance quantized at integer values. These arguments hold quite generally – immune to details of the disorder, inhomogeneities and so on – as long as the chemical potential is in a mobility gap, i.e. if the electronic states at the Fermi level are all localized[3]. Eventually, it was realized [4, 8, 75, 105] that the Laughlin argument could be reformulated in a manner which made it clear that the Hall conductance was a topological invariant, further explaining its universal nature.

Almost simultaneously with this understanding of the importance of gauge invariance in explaining the quantization, Tsui, Störmer and Gossard performed similar experiments

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[3] Note however that for a nonzero Hall conductance it is essential that at least one electronic eigenstate below the Fermi level is extended [37].

as von Klitzing's group, in extremely clean gallium arsenide (GaAs) heterostructures [109]. They found, in addition to the integer plateaus, additional steps in the Hall resistance at *fractional* values of the filling factor, at $\nu = \frac{1}{3}, \frac{1}{5}$ and so on. The theoretical obstacle to explaining these features was stark and immediate: when $\nu < 1$, there are more available degenerate electronic states than electrons in the system, so that perturbation theory is useless to treat this problem, which quickly became known as the fractional quantum Hall effect to distinguish it from its integer predecessor.

It was Laughlin who once again came to the rescue, by proposing a truly remarkable trial wavefunction to describe the correlated electronic state at the heart of the fractional effect[4] [56]. He was able to show by numerically solving few-body examples that his *ansatz*, besides the obvious feature of being commensurate, had extremely high overlap with the true ground state[5] at $\nu = 1/3$. He was even able to construct exact wavefunctions for excited states – 'quasiholes' and 'quasielectrons' – and compute their energies. Finally, and most strikingly, he pointed out – by mapping the problem to a classical Coulomb plasma – that his excited state described particles with fractional electric charge, $e/3$, and argued that such 'fractionalized' quasiparticles were the natural excitations of the two-dimensional electron gas near $\nu = 1/3$. These ideas were soon extended by Haldane [34] and Halperin [38] to explain the 'hierarchy' of other fractional quantum Hall phases descending from the Laughlin states, and by Arovas, Schrieffer and Wilczek [6] to show that that the excitations had not only fractional charge, but also fractional statistics. The fractionalization of quantum numbers was later recognized by Wen [119] as a characteristic of what he termed topological order [112, 114], which has since been the subject of much investigation.

Since the early 1980s, when much of the groundwork for the present was laid, there has been a steady improvement in our understanding of the fractional quantum Hall effect. Powerful techniques from conformal field theory and the ever-growing power of modern computers have been brought to bear on the problem of constructing and studying increasingly elaborate trial wavefunctions for the current zoo of observed quantum Hall fractions. Besides the states with quantized Hall conductance, the global phase diagram of the quantum

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[4] A wonderfully candid discussion of the toy computations, intuitive leaps and occasional missteps that led to his insight is in Laughlin's Nobel autobiography [57].

[5] Laughlin proposed states for fillings $\nu = 1/m$, m odd; we explicitly discuss the example of the $\nu = 1/3$ state here for convenience.

FIG. 1: Transport data in the quantum Hall regime

Longitudinal (ρ_{xx}) and Hall (ρ_{xy}) resistance traces in the quantum Hall effect; region (a) of the left panel is shown in magnified form in the right panel. Notable features include integer and odd-denominator incompressible fractions, even-denominator compressible states $\left(\nu = \frac{1}{2}, \frac{3}{2}\right)$ as well as an incompressible state at $\nu = \frac{5}{2}$. Figure not shown due to copyright restrictions; reproduced from from R. L. Willett, *et. al.*, Phys. Rev. Lett. **59**, 1776 (1987) [121], figures 1-2. Copyright © 1987 by the American Physical Society.

Hall effect includes Fermi liquid-like states and modulated stripe and bubble phases. The dramatic improvement in sample mobilities in the past thirty years has meant that much of the progress has been experimentally driven: several of these phases were first seen in transport studies before they were understood theoretically. The quantum Hall effect has been shown to have a remarkable analogy with the theory of superconductivity [29, 86, 126]; it underlies perhaps the best understood itinerant ferromagnet [99]; it provided the first example[6] of the fractionalization of quantum numbers in dimensions higher than one. It has also played a central role in the modern understanding of topological phases, with analogs in magnetic systems [112], band insulators [26, 69, 92], superfluids [88] and superconductors [41]. This review is devoted to exploring just a few of the many exciting and fascinating possibilities that owe their ultimate inspiration to the strange behavior of electrons in high magnetic fields.

II.   SAMPLES AND PROBES

The quantum Hall effect is is a transport phenomenon, observed when two-dimensional electron gases (2DEGs) are placed in quantizing magnetic fields transverse to the plane in which electrons are free to move (for a sketch of the geometry, see Fig. 2.) The 2DEGs

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[6] More precisely the first example understood to be such. As pointed out to a greater or lesser degree by various authors [41, 52, 114] the oldest known fractionalized phase is the venerable superconductor!
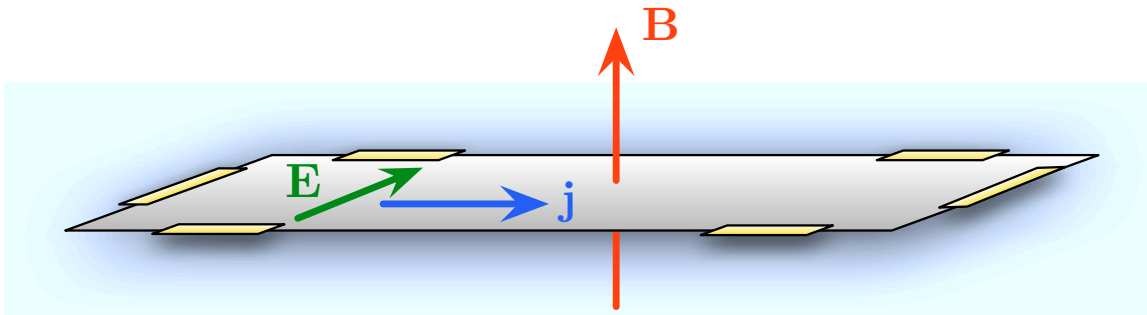
FIG. 2: Sample geometry for the quantum Hall effect.

The electron gas is confined in a two-dimensional plane, and a perpendicular magnetic field **B** is applied. By means of contacts, current is passed through the sample and Hall and longitudinal voltage measurements can be made. In the figure we show the direction of the current **j** and the Hall electric field **E**.

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are typically realized in semiconductor heterostructures, where by suitable epitaxial growth techniques it is possible to build quantum wells that confine the transverse motion of electrons and thereby render them effectively two-dimensional. The thickness of the confining well controls both the spread of the electronic wavefunctions in the transverse direction as well as the spacing between the different 'subbands' for motion in this direction. While in some experiments the well thickness is increased in order to modify electron-electron interactions or make it favorable to fill multiple subbands and enhance the tunability of the system, for the purposes of this review we focus on the single subband case and in the remainder shall assume that the electrons are purely two-dimensional.

The density of electrons in the 2DEG is controlled primarily by doping with donor impurities, situated in a layer set back a fixed distance from the plane of the 2DEG; some samples may also permit some degree of tunability of density using electrostatic gates. The impurities serve as the major source of disorder: when screened by the electrons they give rise to a smooth random potential. There may also be some amount of short-range impurity scattering from imperfections at or near the plane of the 2DEG.

More recently, quantum Hall plateaus – both integer and fractional – have been observed in high-mobility two-dimensional semimetals, graphene and bilayer graphene, either in sus-

pended structures, or over a variety of substrates. Owing to the robustness of the quantum Hall effect in these materials, and their accessibility to surface probes, they are likely to play a significant role in future experimental studies.

The electrons in the 2DEG may have additional internal degrees of freedom, such as their spin, valley pseudospin in the case of degenerate conduction band minima or layer index in double quantum wells. There are additional symmetries associated with these degrees of freedom, and the complex of phenomena that accompany the breaking of these is briefly discussed in Section IX. In the remainder of this introduction, we shall generally assume that the internal degrees of freedom are 'polarized' –effectively, absent – and discuss spinless electrons unless explicitly stated otherwise.

The primary experimental probe of the quantum Hall effect is transport; measurements of currents and voltages in the Hall bar can be made by means of contacts (usually gold) on the edges of the sample. The most striking observation is the quantization of the Hall resistance into a series of plateaus, and the near-vanishing of longitudinal resistance except at a series of sharp peaks when the Hall resistance is between two different quantized values. In addition, transport through quantum point contacts [16, 18] and double point contacts [13, 123]) serve as probes of quasiparticle charge (via shot noise) and statistics (via interference measurements) respectively.

In certain specific cases, the Hall effect lends itself to study by other means besides transport. Nuclear Magnetic Resonance (NMR) measurements are particularly useful in studying the spin textures associated with quantum Hall ferromagnets [9]; surface-acoustic-wave (SAW) absorption is an important experimental signature of the Fermi-liquid like state at $\nu = \frac{1}{2}$ [122]; measurements of the charging spectra of disorder-induced compressible puddles may permit the determination of fractional quasiparticle charge in various Hall plateaus [110]; optical absorption experiments may be useful probes of spin-polarization of the $\nu = 5/2$ state [102]; and tunneling spectroscopy, both in the bulk [100] and at the edge [15] can reveal various correlation effects; to name a few.

III.   THE INTEGER EFFECT

A natural place to begin our discussion is with the integer quantum Hall effect. We first introduce the problem of noninteracting electrons in high magnetic fields, and explain

the reorganization of the spectrum into Landau levels. We then show how to explain the experimental features of the integer effect via a semiclassical percolation picture.

A.   Single-particle physics: Landau levels

An electron with effective mass m^* and charge $-e$, moving in the xy-plane under the influence of a magnetic field $\mathbf{B} = B\hat{z}$ is described by the Hamiltonian

$$H = \frac{1}{2m^*}\left(\mathbf{p} + \frac{e\mathbf{A}}{c}\right)^2 \tag{1}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$. Choosing Landau gauge, we have $A_x = 0, A_y = Bx$, so that we preserve translational invariance in the y-direction. The eigenstates of H can then be classified by their y momentum and a Landau level index n, since the eigenvalue problem reduces to that of a one-dimensional simple harmonic oscillator. We have, explicitly

$$\psi_{n,k_y}(x,y) = \frac{1}{\sqrt{L_y}}e^{ik_y y}\varphi_n\left(x + k_y\ell_B^2\right) \tag{2}$$

with $E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c$, where $\ell_B = \left(\frac{\hbar c}{eB}\right)^{1/2}$ is the magnetic length, and $\omega_c = \frac{eB}{m^*c}$ is the cyclotron frequency. The wavefunctions are written in terms of $\varphi_n(x) = \frac{1}{\sqrt{2^n n!\pi^{1/2}\ell_B}}H_n\left(\frac{x}{\ell_B}\right)e^{-x^2/2\ell_B^2}$ with H_n a Hermite polynomial, which follows from solving the one-dimensional oscillator. The wavefunction is localized around a *guiding center* coordinate $X_{k_y} = -k_y\ell_B^2$.

It is immediately apparent that there is a large degeneracy, since the energy depends only on the index n and not on k_y. To determine the degeneracy of a given energy level – henceforth referred to as a Landau level – we consider a finite rectangular strip of size $L_x \times L_y$. Keeping periodic boundary conditions in y retains k_y as a good quantum number, but it takes on discrete values, $k_y = \frac{2\pi m}{L_y}, m = 0, \pm 1, \pm 2, \ldots$. These correspond to eigenfunctions centered around $x = 0, \pm 2\pi\ell_B^2/L_y, \pm 4\pi\ell_B^2/L_y, \ldots$; the finite extent in the x direction restricts the allowed k_y values, and it then is a simple matter to count states to find a degeneracy equal N_Φ, the number of flux quantum threading the sample[7].

The knowledge of the degeneracy of a Landau level allows us to determine the filling factor, ν, which is simply the number of filled Landau levels, $\nu = \frac{N}{N_\Phi}$ where N is the total

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[7] The degeneracy is $\frac{L_x L_y}{2\pi\ell_B^2} = \frac{BL_x L_y}{hc/e} = N_\Phi$, since the numerator is the total flux and the denominator is Φ_0, the flux quantum.

number of electrons. As a result of this relation, the density of the two-dimensional electron gas can be expressed entirely in terms of the filling factor and the magnetic length, $n = \frac{\nu}{2\pi\ell_B^2}$.

It is a simple exercise to show that when exactly ν Landau levels are filled, the Hall conductance $\sigma_{xy} = \nu\frac{e^2}{h}$. However, it is not clear why it should remain tied to this value for a a finite range of filling factors about commensuration; indeed, in the absence of interactions and with translational invariance, we can show[8] that the Hall conductance cannot deviate from its classical value of B/nec. If we wish to explain the integer effect without interactions, it is necessary to consider the quantum Hall problem in a disordered system, in which translational invariance is lost.

Heuristically, it is easy to see why this is so: in the absence of disorder, the density of states of noninteracting electrons consists of a series of δ-function peaks at the energies of the Landau levels. Were we able to fix the chemical potential precisely in a gap between Landau levels, and maintain it there for a finite range of fillings about commensuration, then a finite-width plateau in the Hall conductance would automatically follow. However, it is impossible to keep the chemical potential in a gap while changing the electron density.

When disorder is present, the Landau level spectrum broadens, as electronic states become localized in the random potential. A band of extended states remains near the center of each level, but is now flanked on either side by localized states separated from it by mobility edges (see Fig. 3). As long as the chemical potential remains in the resulting *mobility* gap while the density is varied, the Hall conductance is unchanged since the electronic states being filled are all localized, and do not contribute to transport. In the following section, we sketch an argument for how this structure arises in the Landau level spectrum, using a semiclassical model of electron dynamics in a smooth random potential.

B. Semiclassical Percolation

A simple picture of the integer quantum Hall effect in the presence of a smooth disorder potential can be obtained in terms of a semiclassical percolation problem [106]. Let us recall the basic features of semiclassical dynamics of electrons in an external potential, in

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[8] The argument rests on the ability, in a translationally invariant system, to boost to a frame comoving with the current, in which the electrons are stationary but see an electric field $\mathbf{E} = \frac{B}{nec}\mathbf{j} \times \mathbf{B}$ where $\mathbf{j}$ is the current in the lab frame. The classical value of the Hall conductance follows immediately [27].
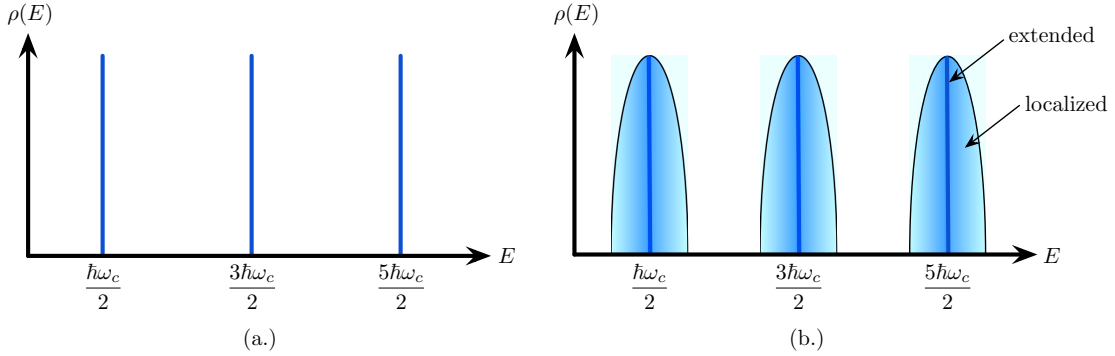
FIG. 3: Density of states of 2DEGs in high fields

(a.) Without disorder we have a series of δ-function peaks at the Landau level energies. (b.) When (bounded) disorder is included, the δ-functions broaden on a scale set by the disorder potential and all states except for those in a narrow band centered on the Landau level energy are localized.

the presence of a strong magnetic field [7]. First, the electronic motion can be separated into two parts: a slow drift along equipotential lines of the external potential, and fast cyclotron motion about these orbits. Each mode of the fast degree of freedom corresponds to a different Landau level, while the slow drift is the motion of the guiding center coordinate of the previous section. Second, these statements become increasingly precise in the limit of $B \to \infty$, since the potential becomes increasingly smooth on the length scale ℓ_B of the cyclotron orbits, which tends to zero in this limit, and it is the ratio of these scales that determines the validity of the semiclassical approximation.

For concreteness consider a sample finite in the x direction, but infinite (or at least very long) in the y direction. As the chemical potential is swept through the Landau level, different equal-energy surfaces are traced out. When it is at the bottom of the disorder potential, most of the semiclassical orbits surround low-energy regions and are therefore localized; if an electric field in the x-direction is turned on, the orbits are perturbed only weakly, and processes that involve charge transfer across the sample in the x-direction would require a conspiracy of hopping processes between the semiclassical orbits, and are hence strongly suppressed. At high chemical potential, most of the orbits surround high-energy

regions, and are hence also localized, and a similar argument goes through: there is no current in the x-direction. However, the set of semiclassical orbits now includes the orbits that lie along the edge of the sample that are extended in the y-direction. Applying a field in the x direction leads to a chemical potential difference between current carrying states in the $\pm y$ directions, and therefore there is a finite current transverse to the potential gradient, i.e. a nonzero σ_{xy}. For intermediate chemical potential, there is a point at which the semiclassical orbits percolate through the sample; orbits that carry edge currents in the $\pm y$ directions approach arbitrarily close to each other. An infinitesimal electric field can now couple the two edge states, leading to finite longitudinal resistance, accounting for the jump in R_{xx} at the plateau transition; once the percolation point is passed, a new set of edge currents has been added and σ_{xy} will therefore show a jump. This leads to the sequence of integer quantum Hall plateaus: once the chemical potential reaches the top of the disorder potential within one Landau level, we commence filling low-energy states in the next Landau level (corresponding to the next eigenenergy of the 'fast' cyclotron motion). It can be shown in the high-field limit and for particle-hole symmetric disorder that the nth plateau transition occurs when the chemical potential crosses the energy of the nth Landau level $\mu_n^* = (n + 1/2)\hbar\omega_c$.

As the statistical mechanics of percolation are well known, they can be used to make estimates for different critical exponents for the integer quantum Hall plateau transition. However, in general these are modified by quantum tunneling between different semiclassical orbits near saddle points of the disorder potential; augmenting the semiclassical treatment with suitable corrections to account for this leads to the network model proposed by Chalker and Coddington [14].

IV. WHY IS THE QUANTIZATION ROBUST?

A. Laughlin's Argument

We have so far provided a reasonable description of the features of the integer plateaus, at least in the limit of smooth disorder and high field. However, this still does not explain why the Hall conductance is so precisely and robustly quantized. We now explain this through a slightly modified version of Laughlin's argument more or less identical to that
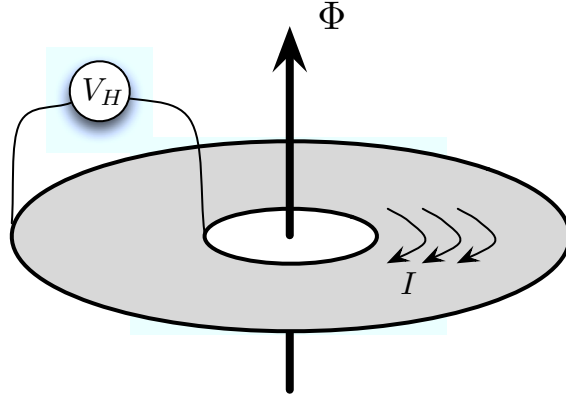
FIG. 4: Laughlin's argument in Corbino geometry

The quantum Hall liquid is confined to an annulus with a voltmeter connected between the edges. A flux Φ is adiabatically inserted through the hole; when this is equal to a multiple of the flux quantum Φ_0, the result is that an integer number of charges are transferred from the inner to the outer edge.

presented in [47], which is valid both at $T = 0$ and at finite temperature, and which is readily extended (with a few caveats) to the fractional case. Consider a quantum Hall liquid confined in an annulus, with a voltmeter connected between the inner and outer edges, that measures a voltage V_H. This is commonly referred to as the Corbino geometry. Now, imagine adiabatically inserting a flux Φ through the annulus, without letting it penetrate the sample itself. Recall that for a system interacting with an external vector potential $\mathbf{A}$ and described by a Hamiltonian H_A, the current operator is given by $\mathbf{j}(\mathbf{r}) = c\frac{\delta H_A}{\delta \mathbf{A}(\mathbf{r})}$. In an instantaneous eigenstate, $H_A|\psi_A\rangle = E_A|\psi_A\rangle$, using the fact that $|\psi_A\rangle$ is normalized it follows that $\langle \mathbf{j} \rangle_A = c\langle \psi_A|\frac{\delta H_A}{\delta \mathbf{A}}|\psi_A\rangle = c\frac{\delta E_A}{\delta \mathbf{A}}$. Taking a thermal average over eigenstates,

$$\langle \mathbf{j} \rangle = c\sum_\alpha \frac{\delta E_{\alpha,A}}{\delta \mathbf{A}} e^{-E_\alpha/kT} \equiv c\frac{\delta \langle H_A \rangle}{\delta \mathbf{A}(\mathbf{r})} \tag{3}$$

which defines the *adiabatic* derivative in which we keep the Boltzmann weight of the state α fixed while changing H_A. This is equivalent to the thermodynamic requirement of constant entropy, hence the name. Specializing to our example, and working in polar coordinates

centered on the annulus, we may write $\mathbf{A} = \hat{\theta}\frac{\Phi}{2\pi r}$, so that

$$c\frac{\partial \langle H_A \rangle}{\partial \Phi} = c\int_\Omega d^2r \frac{\delta \langle H_A \rangle}{\delta \mathbf{A}(\mathbf{r})} \cdot \frac{\partial \mathbf{A}}{\partial \Phi} = \int_\Omega d^2r \frac{\langle j_\theta \rangle}{2\pi r} = I \qquad (4)$$

where I is the azimuthal current flowing in the annulus. We ultimately wish to relate this to the voltage drop across the annulus; so far all we have shown is that the current is the adiabatic derivative of the energy with respect to the flux. Within the annulus the flux is pure gauge, and so localized states are unaffected. However, states extending around the hole see an Aharonov-Bohm phase of $2\pi\Phi/\Phi_0$ due to the inserted flux, but as long as the latter is an integer multiple of Φ_0, the Hamiltonian (and hence the spectrum) is the same, upto a gauge transformation, as at $\Phi = 0$. The only possible change induced by the adiabatic insertion of a quantized flux is to carry the system from one eigenstate to another. This process of changing a variable adiabatically so that the spectrum returns to itself is termed an *adiabatic cycle*. A simple example is furnished by the noninteracting problem, without disorder. In symmetric gauge, the eigenstates are labeled by angular momentum m, and are localized at successively higher radii with increasing m. In this basis, it is easily verified that the flux insertion procedure shifts each[9] single particle eigenfunction from m to $m+1$. The net result of a cycle is that in each filled Landau level a single electron is adiabatically transported from one edge of the annulus to the other. For p filled Landau levels, the total energy required is $\Delta E = peV_H$ where the Hall voltage is by definition the energy to move a unit test charge between edges through a weakly coupled external circuit. If we now approximate $\frac{\partial \langle H_A \rangle}{\partial \Phi}$ by $\frac{\Delta E}{\Delta \Phi}$ and set $\Delta \Phi = \Phi_0$, which is the minimal flux for an adiabatic cycle, the current follows as $I = p\frac{e^2}{h}V_H$. The Hall conductance is therefore $\sigma_{xy} = \frac{pe^2}{h}$. This can be extended to disordered systems by postulating that if the Fermi energy lies in a mobility gap, the only states that can be excited by the adiabatic insertion involve charge transfer from one edge to the other. Since weak perturbations cannot easily move the Fermi energy out of the mobility gap, the Hall conductance remains tied to its quantized value.

A further extension to the fractional effect is possible, if we postulate that in the fractional case adiabatic cycles require the insertion of an integer number $q > 1$ flux quanta[10] to transfer

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[9] In this case, *all* the single-particle states extend around the hole and therefore see the Aharonov-Bohm phase

[10] This requirement is intimately connected to the fact that fractional quantum Hall states are topologically ordered and hence have a ground state degeneracy on a topologically nontrivial manifold, such as an annulus.

p electrons between edges. The net result is a Hall conductance $\sigma_{xy} = \frac{p}{q}\frac{e^2}{h}$.

B. Hall Conductance as a Topological Invariant

Shortly after Laughlin's elucidation of the role played by gauge invariance in the universality of the Hall conductance, several workers [4, 8, 75, 105] extended his ideas to rewrite the Hall conductance as a topological invariant. The original treatment by Thouless, Kohmoto, den Nijs and Nightingale [105] considered noninteracting electrons in a periodic potential; the approach we shall follow is more or less identical to that of Niu and Thouless [75] which extends the result to interacting systems without periodicity.

Let us return to our 2DEG of the previous section, now with interactions as well as an electric field $E\hat{x}$ in the plane. The linear-response Hall conductivity follows from a Kubo formula,

$$\sigma_{xy} = i\frac{e^2\hbar}{L_x L_y}\sum_{n\neq 0}\frac{\langle\Psi_0|\hat{v}_x|\Psi_n\rangle\langle\Psi_n|\hat{v}_y|\Psi_0\rangle - \langle\Psi_0|\hat{v}_y|\Psi_n\rangle\langle\Psi_n|\hat{v}_x|\Psi_0\rangle}{(E_0 - E_n)^2} \tag{5}$$

where 0 and n label the ground and excited many-body eigenstates. The velocity operators appearing in the Kubo formula are, in the same gauge as used earlier, given by

$$\hat{v}_x = \sum_{i=1}^{N}\frac{1}{m_i}\left(-i\hbar\frac{\partial}{\partial x_i}\right), \hat{v}_y = \sum_{i=1}^{N}\frac{1}{m_i}\left(-i\hbar\frac{\partial}{\partial y_i} + eBx_i\right) \tag{6}$$

Before we can use the Kubo formula, we require appropriate boundary conditions under which to solve the eigenvalue problem. Real samples have edges, and thus periodic boundary conditions are appropriate only in the y direction. However, since we are interested in the bulk contribution[11] we may make the system periodic in x direction as well with the appropiate y-dependent phase factor necessitated by translation in the magnetic field. The boundary conditions are then relaxed to

$$\Psi\left(\{x_i + L_x\}\right) = e^{i\alpha L_x}e^{-i(eB/\hbar)y_i L_x}\Psi\left(\{x_i\}\right)$$
$$\Psi\left(\{y_i + L_y\}\right) = e^{i\beta L_y}\Psi\left(\{y_i\}\right) \tag{7}$$

Note that we work explicitly in Landau gauge. It is possible to reformulate the entire problem explicitly in gauge-covariant form, but we shall continue to work in a fixed

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[11] For a discussion of possible subtleties, see [75].

gauge for clarity of presentation. In any event, our final result for σ_{xy} will be in manifestly gauge-invariant form, as appropriate to a physical observable. If we now make a unitary transformation on the many-body eigenstates $\tilde{\Psi}_n = e^{-i\alpha \sum_{i=1}^{N} x_i} e^{-i\beta \sum_{i=1}^{N} y_i} \Psi_n$, (5) becomes

$$\sigma_{xy} = i\frac{e^2}{\hbar L_x L_y} \sum_{n \neq 0} \frac{\left\langle \tilde{\Psi}_0 \left| \frac{\partial \tilde{H}}{\partial \alpha} \right| \tilde{\Psi}_n \right\rangle \left\langle \tilde{\Psi}_n \left| \frac{\partial \tilde{H}}{\partial \beta} \right| \tilde{\Psi}_0 \right\rangle - \left\langle \tilde{\Psi}_0 \left| \frac{\partial \tilde{H}}{\partial \beta} \right| \tilde{\Psi}_n \right\rangle \left\langle \tilde{\Psi}_n \left| \frac{\partial \tilde{H}}{\partial \alpha} \right| \tilde{\Psi}_0 \right\rangle}{(E_0 - E_n)^2} \quad (8)$$

where $\tilde{H}$ is the transformed Hamiltonian. It is then straightforward to reexpress the Hall conductance purely in terms of the transformed many-body ground state wavefunction[12]

$$\sigma_{xy} = i\frac{e^2}{h} \left[\left\langle \frac{\partial \tilde{\Psi}}{\partial \theta} \middle| \frac{\partial \tilde{\Psi}}{\partial \varphi} \right\rangle - \left\langle \frac{\partial \tilde{\Psi}}{\partial \varphi} \middle| \frac{\partial \tilde{\Psi}}{\partial \theta} \right\rangle \right] \quad (9)$$

where $\theta = \alpha L_x, \varphi = \beta L_y$, and each takes values on $[0, 2\pi)$.

At this point, we have simply rewritten the Hall conductance as a response of the ground state wavefunction to changes in boundary conditions; we have as yet given no reason for its quantization. We now make a crucial assumption: *that there is always a finite energy gap between the ground state and the excitations under any given boundary conditions of the form* (7). Note that it is reasonable to assume that the Kubo conductance is insensitive to boundary conditions, as long as there no long-range correlations in the ground state, which is true for the case of an incompressible liquid. As a result, we may equate the Hall conductance to its average over boundary conditions,

$$\sigma_{xy} = \overline{\sigma_{xy}} = \frac{e^2}{h} \int_0^{2\pi} \int_0^{2\pi} d\theta d\varphi \frac{1}{2\pi i} \left[\left\langle \frac{\partial \tilde{\Psi}}{\partial \varphi} \middle| \frac{\partial \tilde{\Psi}}{\partial \theta} \right\rangle - \left\langle \frac{\partial \tilde{\Psi}}{\partial \theta} \middle| \frac{\partial \tilde{\Psi}}{\partial \varphi} \right\rangle \right] = \frac{e^2}{h} C \quad (10)$$

The above expression shows that the dimensionless Hall conductance, $\sigma_{xy}/(e^2/h)$ is a topological invariant, known as the Chern number (C) [71], of the family of ground state wavefunctions. This explains why the quantization is robust: as it is a discrete topological index, the Hall conductance cannot be changed by small perturbations. Adding disorder leads to a Hall plateau at the quantized values of σ_{xy} as before.

The Chern number is an integer, and therefore the foregoing discussion satisfactorily explains the integer quantum Hall effect. The assumption that forced integer quantization was that the ground state was nondegenerate: this enabled us to rewrite σ_{xy} as a property

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[12] Assuming it is nondegenerate; this is not true for the fractional effect.

solely of the ground state wavefunction. For fractional quantization, we must require that the ground state be degenerate. The generalization of (10) to the fractional case is

$$
\begin{aligned}
\sigma_{xy} &= \overline{\sigma_{xy}} \\
&= \frac{e^2}{hd} \sum_{K=1}^{d} \int_0^{2\pi} \int_0^{2\pi} d\theta d\varphi \frac{1}{2\pi i} \left[\left\langle \frac{\partial \tilde{\Psi}_K}{\partial \varphi} \middle| \frac{\partial \tilde{\Psi}_K}{\partial \theta} \right\rangle - \left\langle \frac{\partial \tilde{\Psi}_K}{\partial \theta} \middle| \frac{\partial \tilde{\Psi}_K}{\partial \varphi} \right\rangle \right].
\end{aligned}
\tag{11}
$$

where d is the degree of degeneracy, and the Ψ_K are orthogonal and span the ground state subspace.

Unlike in the integer case, the integrals in (11) are *not* topological invariants, since a cycle in θ or φ need not return each of the degenerate states to itself. However, it is possible to show that the summation over the integrals, and hence σ_{xy}, is a topological invariant. The fractional Hall conductance is then simply a fractional multiple of a Chern number, with the fraction related to the ground-state degeneracy on the torus. For instance, for the Laughlin states with $\nu = 1/m$ which are m-fold degenerate on the torus, the argument gives $\sigma_{xy} = e^2/mh$ [75].

We close by relating the formulation above to the gauge argument. This follows immediately if we compute the current induced by the adiabatic flux insertion, and use the Kubo formula for the Hall conductance to show that the total charge transported in an adiabatic cycle is simply related to the averaged σ_{xy} [75].

V. THE FRACTIONAL EFFECT

In the previous section, we argued that we can relate the Hall conductance to a topological invariant as long as there is a gap to bulk particle-hole excitations; adding disorder then provides a mobility gap so that the Fermi energy can vary through a band of localized states while keeping the Hall conductance unchanged, leading to a plateau in σ_{xy}. For the integer Hall effect, the first step is logically straightforward, since a filled Landau level is automatically gapped to particle-hole pairs.

For the fractional effect, this is no longer the case. In the absence of interactions, we have a highly degenerate set of states, and there is no obvious reason to privilege a commensurate filling over any other[13]. Indeed, the low-energy Hilbert space of the problem, consisting of

---

[13] In fairness, a commensurate charge-density-wave (CDW) was originally proposed as a ground state, but

states exclusively belonging to the $n = 0$ energy level – commonly referred to as the lowest Landau level approximation – is completely degenerate, and in the absence of interactions there cannot be a fractional Hall conductance. Worse, as a result of this degeneracy, there is no good parameter in which one can construct a perturbation expansion to systematically include interactions. Given this all-or-nothing feature, it is a formidable task to construct a gapped many body ground state at each fractional filling. The solution – as is generally the case – is inspired guesswork, to which we now turn.

A.   Trial Wavefunctions

Our focus in this section is the trial wavefunction approach to the the quantum Hall problem. In its essence, the method rests on making a more or less physical guess for the form of the many-body wavefunction at a given filling. In some, but by no means all, cases this is an exact groundstate of a special, typically short-range, model Hamiltonian which may involve 3- and higher-body interactions. The form of the ground-state wavefunctions often suggests natural choices for excited-state wavefunctions corresponding to quasielectrons and quasiholes.

There are many different ways in which trial wavefunctions can be motivated. Laughlin's original guess blended a study of few-body examples with an intuitive leap to the N-body problem. More systematic approaches include the Jain construction, which builds fractional quantum Hall trial wavefunctions from filled pseudo-Landau levels of composite fermions; guessing trial states from conformal blocks of conformal field theories; and the Haldane-Halperin 'hierarchy' construction at various fillings, which rests on forming quantum Hall states from the quasiparticles of a parent quantum Hall liquid. Many of the model wavefunctions can also be understood in a unified fashion within a recent formulation based on the properties of Jack symmetric polynomials [10]. There are also various 'parton' approaches [44, 115, 117]. which build in fractionalization at the outset. We shall discuss composite fermions in Section VI, and in this section we briefly summarize the hierarchy construction; the conformal block technique, the Jack polynomial approach and the parton constructions,

---

this is incompatible with the strictly linear $I - V$ curves in experiments and the cusp singularity in the ground state energy at commensuration observed in numerical studies. See [108] and references therein for a discussion.

while extremely important to our understanding of the quantum Hall effect, are somewhat peripheral to our concerns hereand will be omitted for brevity.

Once constructed, model wavefunctions can be used to determine a variational upper bound on the ground state energy; alternatively, one can obtain the exact ground state for small systems by numerically diagonalizing the many-body problem[14], and compute the overlap with the trial state. Often, the overlap is extremely high, a strong indication that the ansatz captures most of the essential many-body correlations of the fractional quantum Hall phase under investigation. When there are competing states at a given filling – for instance, $\nu = 2/5$ has variously been described as a hierarchy state [34, 38], a composite fermion state [43], or the so-called 'Gaffnian' state [97]– such numerical studies of overlaps and energetics may be able to settle the question of which alternative is more likely to be stable in real systems. On occasion, trial states with very different physics have extremely high overlap – for instance, the Gaffnian and the composite fermion states in this example. Comparing the ground state entanglement spectrum has been proposed as a resolution to such issues [90]. The entanglement spectrum can also be used to systematically show adiabatic continuity between model Hamiltonians and the more realistic Coulomb interaction case [104].

As an illustrative example, we shall consider Laughlin's wavefunction for the $\nu = \frac{1}{m}$ states. We shall be fairly concise, as the trial wavefunction approach has been the subject of several reviews, e.g. [84]; our approach shall hew closely to that of [27]. Also, we pick the pedagogically simpler example of the disk geometry, although most numerical studies seek to avoid the complication of edges by studying the problem on the sphere or the torus. Finally, we shall work in symmetric gauge, as this is ideally suited to wavefunction studies of the lowest Landau level.

With these preliminaries, we are ready to begin our discussion. In symmetric gauge, working in two-dimensional complex coordinates $z = x + iy$ the wavefunction of an N-electron system that is confined to the lowest Landau level can always be written as the product of a function f analytic in all the electron coordinates $z_1, z_2, \ldots, z_N$ with a Gaussian factor [15],

$$\Psi(z_1, z_2, \ldots, z_N) = f_N[z] e^{-\frac{1}{4} \sum_{i=1}^{N} |z_i|^2}, \tag{12}$$

---

[14] Perhaps surprisingly, many quantum Hall systems appear to converge to the thermodynamic limit with only ~ 10 electrons.

[15] For brevity, we shall denote functions of all the electron coordinates $f(z_1, z_2, \ldots, z_N)$ as $f_N[z]$.

with the obvious requirement from the Pauli principle that that f is totally antisymmetric. Here and below we have chosen to set the magnetic length, $\ell_B = 1$. As the Gaussian factor is set by the cyclotron degree of freedom, the Hilbert space of the lowest Landau level reduces to the space of analytic functions in N complex variables. An orthonormal basis of single-particle states for the lowest Landau level is thus provided by functions of the form $\varphi_k(z) = \frac{z^k}{\sqrt{2\pi 2^k k!}} e^{-|z|^2/4}$ [28]. These states each have angular momentum k and it is easy to show that their probability density is peaked at radius $\sqrt{2k}$.

Laughlin made the following guess for the wavefunction at $\nu = \frac{1}{m}$, m odd:

$$f_N^m[z] = \prod_{i<j}^{N} (z_i - z_j)^m. \tag{13}$$

Since m is an odd integer, analyticity and antisymmetry are immediate. In addition, the $m = 1$ state is simply a Slater determinant built out of the single-particle states φ_k, i.e.

$$\Psi_N^1[z] = \begin{vmatrix} \varphi_0(z_1) & \varphi_0(z_2) & \dots & \varphi_0(z_N) \\ \varphi_1(z_1) & \varphi_1(z_2) & \dots & \varphi_1(z_N) \\ \vdots & & & \\ \varphi_{N-1}(z_1) & \varphi_{N-1}(z_2) & \dots & \varphi_{N-1}(z_N) \end{vmatrix} \tag{14}$$

which clearly corresponds to a filled lowest Landau level. For $m > 1$, we verify that Ψ_N^m has the correct filling as follows. Since the highest degree of any of the z_is in f is $m(N-1)$, it follows that this is the highest possible angular momentum in the decomposition of Ψ_N^m in the single-particle basis. The area of the droplet described by Ψ_N^m is thus $A = 2\pi m(N-1)$; from this, the filling factor is $\nu = \frac{2\pi N}{A}$. Since we have already verified $\nu = 1$ for $m = 1$, it follows that $\nu = \frac{1}{m}$ in the general case of arbitrary odd m [47]. Thus, we have produced a trial wavefunction that has the correct filling and lives entirely in the lowest Landau level.

The Laughlin wavefunction has an additional and extremely useful property: we can show that it is the *exact* ground state for a short-range model Hamiltonian. To see this we observe, following Haldane [34], that we can expand any translationally and rotationally invariant two-body interaction projected onto a single Landau level as[16] [34, 107]

$$V = \sum_{m'=0}^{\infty} \sum_{i<j} v_{m'} \mathbf{P}_{m'}(ij) \tag{15}$$

---

[16] Note that within the lowest Landau level, the kinetic energy is quenched and the Hamiltonian reduces to just the interaction term, $H = V$.

where $\mathbf{P}_m(ij)$ are operators that project onto states such that particle i and j have relative angular momentum m. The coefficients of the expansion, v_m, are known as Haldane pseudopotentials and depend on the Landau level under consideration; for repulsive interactions they are all positive. Since the projectors for different angular momenta do not commute, this rewriting does not immediately simplify the problem. However, if we consider a model Hamiltonian defined by $v_{m'} > 0$ for $m' < m$ and zero otherwise, then it is clear that the Laughlin state $\Psi_N^m[z]$ is an exact, zero-energy eigenstate for any N, since any two electrons have a relative angular momentum of at least m. It is also possible to show that the model Hamiltonian has a gap, since any excitation involves reducing the relative angular momentum of at least one pair of electrons and therefore costs positive energy.

It would be truly remarkable if every trial wavefunction had a corresponding model Hamiltonian for which it is the exact ground state. Unfortunately, this is not the case; there exist several different trial states for which no simple model Hamiltonian is known. However, an infinite family of states belonging to the so-called Read-Rezayi sequence [89] can be shown to be exact ground states of model Hamiltonians, albeit ones involving n-body interactions with $n > 2$. These include the Laughlin states as well as the Moore-Read state for even-denominator fillings.

B. Plasma Analogy

The Laughlin wavefunction exemplifies another extremely useful property in common with several different wavefunctions that share its Jastrow (pair product) form[17], namely that the ground state correlations reduce to the finite-temperature equilibrium correlations of an interacting classical system in the *same* dimension. This is accomplished by computing the ground state probability density and noting that it can be written as a classical Boltzmann weight

$$|\Psi_N^m[z]|^2 = e^{-\beta H_{\mathrm{cl}}} \tag{16}$$

where $\beta = \frac{2}{m}$, and

$$H_{\mathrm{cl}} = -m^2 \sum_{i<j} \log |z_i - z_j| + \frac{m}{4} \sum_k |z_k|^2 \tag{17}$$

---

[17] These include several other quantum Hall trial states, as well the AKLT ground states that are discussed in Section X

corresponds to the energy of a two-dimensional Coulomb plasma of particles of charge $-m$ moving in a uniform background charge density (reinstating ℓ_B for clarity) $\rho_B = -\frac{1}{2\pi\ell_B^2}$. The long-range Coulomb forces[18] enforce charge neutrality in the plasma, which requires $mn + \rho_B = 0$, from which we recover the fact that $\nu = 2\pi\ell_B^2 n = \frac{1}{m}$. Working in momentum space, we have $H_{\mathrm{cl}} = \frac{1}{2L^2}\sum_{\mathbf{q}}\frac{2\pi m^2}{q^2}\rho_{\mathbf{q}}\rho_{-\mathbf{q}}$ upto unimportant self-energy corrections, where we take $L_x = L_y = L$. From this, we get (at $q \gg \ell_B^{-1}$) that the density-density correlations in the Laughlin state are suppressed at long wavelengths,

$$\langle\rho_{\mathbf{q}}\rho_{-\mathbf{q}}\rangle = \frac{L^2}{2\pi m}q^2. \tag{18}$$

By performing Monte Carlo simulations of (16), we can verify both this result, as well as the fact that the long-range plasma forces lead to liquid-like correlations.

It is worth mentioning that while it appears that the Laughlin construction works for arbitrary odd m, in fact for $m \geq 7$ the Coulomb interaction energy is minimized, not by a quantum Hall liquid, but by a triangular electron Wigner crystal. The plasma analogy also eventually leads to a crystal state, but at much lower filling; the Laughlin state stops being a good variational state well before this.

C.   Neutral Excitations and the Single-Mode Approximation

We turn now to the neutral collective excitations of the Laughlin liquid. In the absence of additional degrees of freedom, the only neutral collective modes are phonon-like density wave excitations. In keeping with the approach we have taken thus far, we would ideally like to compute the neutral excitation spectrum variationally. To do this, we use the Single Mode Approximation (SMA), originally employed to study the collective mode spectrum of superfluid ⁴He by Feynman [20] and Bijl [11]. In its essence, the SMA approach relies on obtaining a variational upper bound for excitation energies by constructing a trial wavefunction for the excited state, that is orthogonal to the ground state at each wavevector $\mathbf{k}$. Typically, this is done by multiplying the ground state wavefunction with a density operator $\rho_{\mathbf{k}}$. When the SMA is applied to the fractional quantum Hall effect, we require an additional

---

[18] Note that these are purely fictitious; they always exist in the classical companion plasma to the Laughlin state, even when the physical Hamiltonian is short-ranged. They are simply a means of enforcing the correlations built into the Laughlin *ansatz*.

projection of the trial excited state wavefunction into the lowest Landau level, in order to ensure that we correctly restrict the Hilbert space and thereby capture the intra-Landau level excitation scale set by the Coulomb energy $e^2/\varepsilon\ell_B$. Explicitly, we have

$$\Psi_N^{m;\mathbf{q}}[z] = \bar{\rho}_{\mathbf{q}}\Psi_N^m[z] \tag{19}$$

where the bar denotes an operator projected into the lowest Landau level. While we shall not present details here, it can be shown [30] that the SMA calculation always gives a gap at all q. Near $q = 0$, for Coulomb interactions the result is

$$\Delta_{SMA}(q) \equiv \frac{\langle\Psi_N^{m;\mathbf{q}}|V|\Psi_N^{m;\mathbf{q}}\rangle}{\langle\Psi_N^{m;\mathbf{q}}|\Psi_N^{m;\mathbf{k}}\rangle} = c\frac{e^2}{\varepsilon\ell_B} + \mathcal{O}(q^2) \tag{20}$$

where c is a numerical constant depending on details of the interaction. More careful analysis over a greater range of wavevectors, as well as numerical studies, reveals that the collective mode spectrum has a minimum at $q \sim \ell_B^{-1}$, commonly referred to as the magnetoroton in analogy with the roton minimum in superfluid Helium [30].

D. Fractionally Charged Excitations

In addition to neutral collective modes, quantum Hall states also have charged quasiparticle excitations, conventionally referred to as quasielectrons and quasiholes[19], corresponding to negative and positive charge respectively. These are nucleated when the filling factor is altered from a commensurate value, either by varying the charge density or the magnetic field. The wavefunction for a quasihole located at Z is [56]

$$\Psi_{\mathrm{qh},N}^m[z; Z] = \prod_{i=1}^{N}(z_i - Z)\Psi_N^m[z]. \tag{21}$$

What about a quasielectron? Naively, we would guess that to obtain a quasielectron wavefunction we should simply replace the product in the quasihole wavefunction with its complex conjugate, but this leads to a wavefunction that is not restricted to the lowest Landau level. Projecting back to the restricted Hilbert space, the z_i^*s are mapped to derivatives, leading to [56]

$$\Psi_{\mathrm{qe},N}^m[z; Z] = \prod_{i=1}^{N}\left(2\frac{\partial}{\partial z_i} - Z^*\right)\Psi_N^m[z] \tag{22}$$

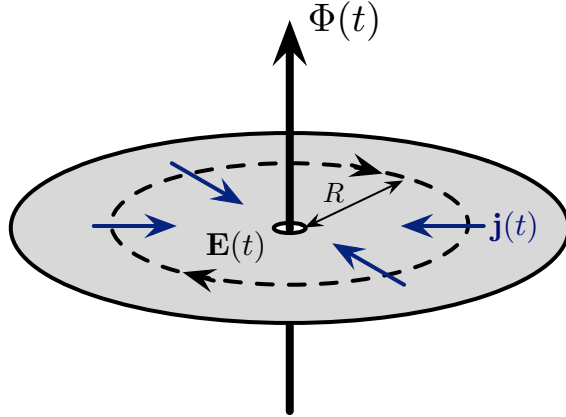[19] We shall reserve the term quasiparticle for situations when we mean 'either quasielectron or quasihole'.

FIG. 5: Flux insertion argument for fractional charge.

where the derivatives act only on the polynomial part of Ψ_N^m. Owing to the rather complicated form of the quasielectron wavefunction, we shall work primarily with quasiholes in the following[20]

The quasihole wavefunction can also be studied using the plasma mapping, which leads to a classical Boltzmann weight of the form $e^{-\beta(H_{cl}+H_i)}$. Here H_{cl} and β have the same values as in Section V B, and

$$H_i = -m \sum_{i=1}^{N} \log |z_i - Z| \tag{23}$$

is the energy of unit charge impurity at Z interacting with the mobile charge-m particles of the plasma. Since the plasma attempts to maintain charge neutrality it will screen the impurity. The resulting screening cloud has a net deficit of $1/m$ plasma particles. When translated into a physical electric charge, the quasihole represents an excitation with fractional electric charge,

$$q^* = \frac{e}{m} \tag{24}$$

An alternative way to see that the quasihole has fractional charge is to perform the following *gedanken* experiment to produce a quasihole: drill a hole in the Laughlin liquid at Z, and adiabatically insert a flux Φ_0 (see Fig. 5). Consider a loop of radius R encircling the

---

[20] As it happens, operators creating a quasihole are relatively easy to construct within conformal field theory, but the same cannot be said for the quasielectron; for a discussion of the subtleties involved, see [40].

point of flux insertion, and sufficiently far away from it $(R \gg \ell_B)$ that we can assume the quantum Hall fluid responds with the bulk σ_{xy}. By Faraday's law, the changing flux induces an azimuthal electric field $\mathbf{E}(t) = -\frac{1}{c}\frac{d\Phi}{dt}\hat{\theta}$; this leads to a radial current density $\mathbf{j} = \sigma_{xy}E\hat{\mathbf{r}}$ from the Hall response. Once the flux insertion is complete, we see that a total charge

$$q^* = \int dt \int \mathbf{j}(t) \cdot d\mathbf{s} = \frac{\sigma_{xy}}{c}\Phi_0 = \nu e \tag{25}$$

flows into the area around the quasihole. Thus, the quasihole has a charge excess whose magnitude is a fractional multiple of the electron charge, localized in a region of size ℓ_B around Z. This argument makes it clear that the fractional charge is inextricably linked with the fractional Hall conductance. For a lucid discussion of some of the subtleties involved in thinking about the fractional charge of Hall effect quasiparticles, we refer the reader to [27]. It can also be shown that the quasiparticles, in addition to fractional charge, also possess fractional statistics [6].

E.   Life on the Edge

So far, our discussion has exclusively focused on the bulk of the sample, where the incompressibility of the fractional quantum Hall droplet leads to a gap both to neutral collective modes as well as to quasiparticle excitations. The edge of a quantum Hall droplet in contrast supports gapless excitations, whose field theory is that of a chiral Luttinger liquid [113]. The subject of quantum Hall edge theories is vast and extremely technical, and is well beyond the scope of this introduction. Here, we content ourselves with showing within a hydrodynamic approach that the edge of a $\frac{1}{m}$ Laughlin state is described by a chiral Luttinger liquid, closely following the presentation of [116]. The central point [60, 103] is to note that the only low-lying excitations of an incompressible, irrotational droplet that is gapped in the bulk are surface waves along the edge of the droplet. By identifying these with the edge excitations of the quantum Hall liquid and with an appropriate quantization procedure, we can infer a 1D quantum theory of the edge. Consider a quantum Hall droplet with filling factor ν, confined in a finite region by a potential well. The electric field from the confining potential generates a persistent current along the edge,

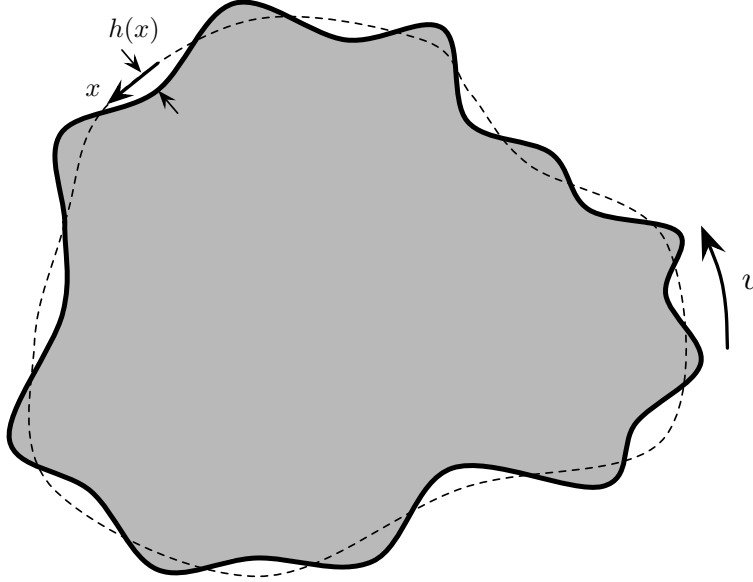$$\mathbf{j} = \sigma_{xy}\hat{\mathbf{z}} \times \mathbf{E} \tag{26}$$

FIG. 6: Edge excitations of a quantum Hall droplet.

because of the nonzero Hall conductance; this implies that near the edge electrons drift with velocity $v = cE/B$; we assert (without attempting a proof) that this must also be the velocity of the edge excitations. If we pick x as a coordinate along the edge, we may write a 1D density $\rho(x) = nh(x)$ where $n = \nu/2\pi\ell_B^2$ is the bulk density (see Fig. 6.) Since the edge waves are gapless, and propagate unidirectionally[21] with velocity v, they should be described by a chiral wave equation

$$(\partial_t - v\partial_x)\rho = 0 \tag{27}$$

The Hamiltonian is simply the energy, which we can compute classically as the work done in displacing the charge a distance h against the electric field:

$$H = \int dx \, \frac{1}{2}e\rho(x)h(x)E = \int dx\pi\frac{v}{\nu}\rho(x)^2 \tag{28}$$

We can rewrite (27) and (28) in momentum space as

$$\begin{aligned} \dot{\rho}_k &= ivk\rho_k \\ H &= 2\pi\frac{v}{\nu}\sum_{k>0}\rho_k\rho_{-k} \end{aligned} \tag{29}$$

---

[21] We choose $\mathbf{B}$ to point in the $-\hat{\mathbf{z}}$ direction so that v is in the positive sense.

where $\rho_k = \frac{1}{\sqrt{L}} \int dx e^{ikx} \rho(x)$ and L is the length of the edge. From these, we infer that if we take as generalized coordinates $q_k = \rho_k$ $(k > 0)$, then the corresponding canonical momenta are $p_k = \frac{2\pi i}{\nu k} \rho_{-k}$, and Hamilton's equations are satisified. Quantizing the theory is then a simple matter of imposing canonical commutation relations, $[q_k, p_{k'}] = i\delta_{kk'}$, whence

$$[\rho_k, \rho'_k] = \frac{\nu}{2\pi} k\delta_{k+k'}; \quad [H, \rho_k] = v\rho_k \tag{30}$$

where $k, k' = \frac{2\pi m}{L}$ with m an integer. These commutation relations form a $U(1)$ Kac-Moody algebra, and are precisely those obtained by bosonization of an interacting theory of chiral fermions; in other words, they describe a chiral Luttinger liquid [33]. Using standard techniques from bosonization, we can show that the operator that creates an electron is $\psi(x) \sim e^{i\frac{2\pi}{\nu} \int^x dx' \rho(x')}$ and that the electron propagator along the edge (for the case $\nu = \frac{1}{m}$) is:

$$G(x,t) = \langle T\psi^\dagger(x,t)\psi(0)\rangle \sim \frac{1}{(x - vt)^m} \tag{31}$$

This corresponds to a Luttinger parameter [33] m, which is the final piece of information we need to specify the edge theory.

F. Hierarchies

The Laughlin states are excellent variational ground states for filling factor $1/m$; however, it experiments show a host of other fractions which are not simple Laughlin fillings. How are we to understand these new fractions?

The solution lies in any of a number of 'hierarchy constructions' that in effect reduce the problem non-Laughlin fillings to an already solved problem. One approach – pioneered by Haldane [34] and Halperin [38] – is to arrange affairs so that the quasiparticles of an existing quantum Hall state themselves form a Laughlin liquid. Another idea, due to Jain [43], is to consider the fractional quantum Hall effect as the *integer* effect of a new 'composite' fermion. Finally, there is a third construction due to Wen and Zee [120] which we shall not consider here. For a succinct comparison of the advantages and shortcomings of the Jain and Haldane-Halperin approaches, we defer to [47]. We note that while they are perfectly good variational states, not all members of a given hierarchy may be realized since other phases, such as quasiparticle Wigner crystals, may have a lower energy.

1. Haldane-Halperin Hierarchy

We start with the Haldane-Halperin hierarchy construction, which proceeds iteratively at as follows: fractional quantum Hall states at a given level of the hierarchy are obtained by forming Laughlin states of the quasielectrons or quasiholes of the previous level; the uppermost level of the hierarchy consists of the Laughlin states.

Let us begin with a parent Laughlin state with $\nu = \frac{1}{m}$, and discuss how to construct the next level of the hierarchy[22]. As we have shown previously, on the disk the flux and filling factor at commensuration are related by $N_\Phi = m(N-1)$. If we in addition add N_{qp} quasiparticles, this is replaced by

$$N_\Phi = m(N-1) + \alpha N_{\mathrm{qp}} \tag{32}$$

with $\alpha = -1(+1)$ for quasielectrons (quasiholes). If we now require that the added quasiparticles are themselves in a Laughlin $\frac{1}{p}$ state, we must have a similar condition

$$N = p(N_{\mathrm{qp}} - 1) \tag{33}$$

with p even. This requires some explanation. The evenness of p is because the quasiparticles are nominally bosonic, and so they can only form even-denominator Laughlin states. The replacement of N_Φ by N is because the number of independent single-*quasiparticle* states is N rather than N_Φ, which is the number of available electronic states. From (32) and (33), it follows that in the thermodynamic limit the first-level hierarchy state has filling

$$\nu = \frac{N}{N_\Phi} = \frac{p}{mp + \alpha} \tag{34}$$

To determine the charge carried by quasiparticle of the hierarchy state, consider adding a single electron to the system, so that we take $N \to N^0 + 1$. We then have

$$
\begin{aligned}
N_\Phi &= m(N^0 + 1 - 1) + \alpha N_{\mathrm{qp}}^0 \\
&= m(N^0 - 1) + \alpha(N_{\mathrm{qp}}^0 + \alpha m) \\
&\equiv m(N^0 - 1) + \alpha N_{\mathrm{qp}}
\end{aligned}
\tag{35}
$$

---

[22] Our discussion closely parallels that of Haldane in [84].

where we have defined a modified quasiparticle number $N_{\text{qp}} = N_{\text{qp}}^0 + \alpha m$, which follows from the fact that an electron is equivalent to αm Laughlin quasiparticles. We then find

$$
\begin{aligned}
N^0 + 1 &= p(N_{\text{qp}}^0 - 1) + 1 \\
&= p(N_{\text{qp}}^0 + \alpha m - 1) - \alpha(mp - \alpha) \\
&= p(N_{\text{qp}} - 1) - \alpha(mp - \alpha)
\end{aligned}
\tag{36}
$$

which corresponds to a state with $(mp - \alpha)$ quasiparticle excitations of the daughter incompressible fluid. It follows that the latter have charge $\pm\frac{1}{mp-\alpha}$ i.e. the reciprocal of the denominator of the filling fraction.

By iterating this procedure, one can construct an infinite hierarchy of incompressible states, at filling factors given by appropriately terminating the continued fraction

$$
\nu = \cfrac{1}{m + \cfrac{\alpha_1}{p_1 - \cfrac{\alpha_2}{p_2 - \cdots}}}.
\tag{37}
$$

If the underlying system is fermionic – as is the case in a 2DEG – this always leads to an odd denominator. Whether any of these states are stabilized in the lowest Landau level once again depends sensitively on the interactions.

2. Composite Fermions and the Jain Hierarchy

To motivate the Jain approach, it is instructive to rewrite the Laughlin wave function in the following suggestive form: with $m = 2k + 1$, we have

$$
\Psi_{N;\text{Laughlin}}^m[z] = \prod_{i<j}^N (z_i - z_j)^m e^{-\frac{1}{4}\sum_r |z_r|^2} = \prod_{i<j}^N (z_i - z_j)^{2k} \Psi_N^1[z]
\tag{38}
$$

which is a Jastrow factor multiplying a Slater determinant corresponding to a filled lowest Landau level. Jain's insight was to realize that the last term could be replaced by other Slater determinants, corresponding to p filled Landau levels, $\Psi_N^p[z, z^*]$; since the latter obviously involve terms from higher Landau levels, the resulting wavefunction must be projected into the lowest Landau level. When this is done, we obtain a trial wavefunction that describes quantum Hall states at filling $\nu = \frac{1}{2k+p^{-1}}$:

$$
\Psi_{N;\text{Jain}}^{(k,p)}[z] = \mathcal{P}\left[\prod_{i<j}^N (z_i - z_j)^{2k} \Psi_N^p[z, z^*]\right]
\tag{39}
$$

A very useful understanding of the Jain hierarchy can be given within composite fermion Chern-Simons theory (which is the subject of the next section). Here, we perform a statistical gauge transformation that attaches $2k$ flux quanta to each electron, whose formal implementation introduces a Chern-Simons term into the action. The resulting particles obey fermionic statistics, and are termed composite fermions. Since composite fermions carry flux, the magnetic field seen by any one of them is the sum of the external, fixed magnetic field and the statistical pseudomagnetic field due to all the others. Thus, when the composite fermions are at uniform density they cancel part of the external magnetic field. As charged particles in the residual magnetic field, they give rise to an auxiliary Landau problem, with its own pseudo-Landau levels, sometimes termed "Λ" levels to emphasize their fictitious nature. When an integer number p of these are filled, the result is the Jain state[23] with $\nu = \frac{p}{2pk+1}$.

VI. HALF-FILLED LANDAU LEVELS

In the previous section, we showed two different ways to construct an infinite hierarchy of incompressible states at odd-denominator fillings. While somewhat involved, they allow us to explain many of the observed quantized Hall plateaus. However, experiments see two distinct behaviors when a Landau level is *half* filled: at $\nu = \frac{1}{2}$, there is much evidence in favor of a compressible, Fermi liquid-like state, whereas at $\nu = \frac{5}{2}$ – corresponding to a half-filled $n = 1$ Landau level above filled lowest Landau levels for each spin polarization – there is a clear plateau in σ_{xy}. Evidently, neither the Haldane-Halperin hierarchy nor Jain's construction can explain the latter[24]; as for the Fermi liquid-like state, this appears to be another beast entirely.

In this section, we shall show that both the compressible state at $\nu = \frac{1}{2}$ and the incompressible $\nu = \frac{5}{2}$ state may be understood in a unified fashion in terms of composite fermions, albeit in a manner that sets the incompressible state apart from Jain's hierarchy. To do so, we shall have to introduce two new pieces of physics: the composite fermion Chern-Simons

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[23] Within the field theoretic approach, a careful treatment of the fluctuations of the Chern-Simons field may be necessary to recover the wavefunction. For an example at $\nu = \frac{1}{3}$, see [62].

[24] The attentive reader might worry that the possibility left unmentioned might solve the problem; she can rest assured that the Wen-Zee approach is also left wanting at even denominators.

theory and paired quantum Hall states of composite fermions.

A. Composite Fermions and the Halperin-Lee-Read Theory

We shall begin by discussing the Halperin-Lee-Read (HLR) Chern-Simons theory of the half-filled Landau level. Our treatment shall be necessarily, even criminally, brief, and we refer the reader interested in further details to both HLR's original work [39] and the excellent pedagogical review by Simon [96].

There are many different ways of motivating the Chern-Simons approach. Perhaps the conceptually most straightforward route is to begin in the first-quantized Hamiltonian formalism, and observe that the unitary transformation,

$$\Psi \rightarrow \tilde{\Psi} \equiv e^{i2k \sum_{i<j} \text{Im} \log(z_i - z_j)} \Psi \tag{40}$$

on the many-body wavefunction maintains the antisymmetry of the wavefunction, and hence the statistics of the underlying particles are unchanged. We further observe that the Hamiltonian is changed by this transformation: the vector potential $\mathbf{A}(\mathbf{r}_j) \rightarrow \mathbf{A}(\mathbf{r}_j) + \mathbf{a}(\mathbf{r}_j)$, where the 'statistical' gauge field $\mathbf{a}$ is transverse, $\nabla_j \cdot \mathbf{a}(\mathbf{r}_j) = 0$ and satisfies

$$\nabla \times \mathbf{a}(\mathbf{r}_j) \equiv b(\mathbf{r}_j) = -2k\Phi_0 \sum_{i \neq j} \delta(\mathbf{r}_i - \mathbf{r}_j). \tag{41}$$

The physical content of this statement is that each composite fermion – as the gauge-transformed electrons are known – sees a $2k$-flux tube at the coordinates of each of the others. In other words, we have attached a pair of fluxes to each electron in order to convert it to a composite fermion. The composite fermions thus move in a gauge field that is the sum of the statistical and background (i.e., external) contributions[25].

With these preliminaries, we are ready to make the leap to a field theory. Consider a system of fermions, interacting with an external magnetic field $\mathbf{A}$, and a statistical Chern-Simons gauge field. This may be described by the path integral, $\mathcal{Z} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}a_\mu e^{-S}$,

---

[25] Our argument, with the minimal modification that $2k$ is replaced by $2k + 1$, applies equally well to the composite boson approach; indeed part of our discussion is adopted from [47] which focuses on the latter.

where

$$S = \int_0^\beta d\tau d^2r \left[\psi^\dagger (i\hbar c\partial_0 - ea_0 - \mu)\psi + \frac{\hbar^2}{2m^*} \left| \left(-i\partial_i - \frac{e}{\hbar c}(a_i + A_i) \right) \psi \right|^2 \right.$$
$$\left. - \frac{1}{2k\Phi_0} a_0 \varepsilon^{ij} \partial_i a_j + \frac{1}{2} \int d^2r' \psi^\dagger(\mathbf{r})\psi(\mathbf{r})V(\mathbf{r}-\mathbf{r}')\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}') \right] \quad (42)$$

The above action is written in Coulomb gauge, where we impose the transversality condition[26] $\nabla \cdot \mathbf{a} = 0$. Here, V is be the electron-electron interaction and μ is the chemical potential. We have chosen not to impose an external A_0. As usual, $\Phi_0 = hc/e$ is the quantum of flux.

The equations of motion can be obtained by varying the action with respect to ψ and a_μ. For ψ we obtain the standard equations for a nonrelativistic fermion moving in the combined field $a_\mu + A_\mu$, with a density-density interaction V. The equations of motion for the Chern-Simons field are given by

$$b \equiv \nabla \times \mathbf{a} = -2k\Phi_0\rho(\mathbf{r})$$
$$\varepsilon^{ij}e_j \equiv \varepsilon^{ij}(\partial_0 a_j - \partial_j a_0) = -\frac{2k\Phi_0}{c}\mathbf{j}(\mathbf{r}) \quad (43)$$

where

$$\rho(\mathbf{r}) = \psi^\dagger(\mathbf{r})\psi(\mathbf{r})$$
$$\text{and} \quad \mathbf{j}(\mathbf{r}) = \frac{\hbar}{2m^*i}\left[\psi^\dagger\nabla\psi - \psi\nabla\psi^\dagger\right]_{\mathbf{r}} - \frac{e}{m^*c}(\mathbf{a} + \mathbf{A})\psi^\dagger(\mathbf{r})\psi(\mathbf{r}) \quad (44)$$

are the density and current density of the composite fermions, and we have defined the 'statistical electric field' $\mathbf{e}$. Note that varying the action with respect to $\mathbf{a}$ does *not*, strictly speaking, lead to an equation for e_j as written, since by our gauge fixing $\varepsilon^{ij}\partial_0 a_j = 0$ so that $\mathbf{e} = -\nabla a_0$. Nevertheless we give the definition of $\mathbf{e}$ that is gauge-invariant which we can always obtain from a gauge-invariant form of S where we undo the gauge fixing in the path integral.

The composite fermion density is obviously equal to the density of the original electrons; we observe in addition that since ψ is always coupled to the combined field $a_\mu + A_\mu$, we may write

$$\frac{ie}{c}j_\mu \equiv \frac{\delta S}{\delta A_\mu} = \frac{\delta S_0}{\delta A_\mu} = \frac{\delta S_0}{\delta a_\mu} \quad (45)$$

---

[26] In two dimensions, this means the natural second term in the gauge action, $\mathbf{a} \times \partial_0 \mathbf{a}$ vanishes; a manifestly gauge-invariant action can be obtained by undoing the gauge-fixing. See [47] for a discussion in the composite boson context.

where S_0 is the action without the Chern-Simons term, thereby relating the electron and composite fermion current densities. The first of the Chern-Simons equations implements the flux-attachment transformation and is therefore equivalent to (41); the second simply ensures that the flux attachment is preserved under time evolution, and can be seen more or less as a consequence of the continuity equation relating ρ and $\mathbf{j}$.

At half-filling ($\nu = \frac{1}{2}$), we observe that $\langle \rho \rangle = n = \frac{B}{2\Phi_0}$. The mean-field solution of the first Chern-Simons equation yields $\langle b \rangle = -2k\Phi_0 n$, so that for $k = 1$, the mean-field statistical magnetic field exactly cancels the external magnetic field, $\langle b \rangle + B = 0$, and the composite fermions see zero field on average. Barring various instabilities – which in channels other than the particle-particle channel are precluded by various considerations, for rotationally invariant systems [94] – the ground state of a system of fermions in zero field is to form a Fermi sea. This in essence is the heart of composite fermion theory: Jain's states can then be understood as arising when the Landau diamagnetism of the $\nu = \frac{1}{2k}$ composite fermion metal leads to the formation of a state with p filled pseudo-Landau levels. The Fermi liquid-like state has been verified both in numerics [91] and by surface acoustic wave absorption [122] and magnetic focusing [31] experiments.

For now, let us focus on the case of half-filling ($k = 1$), although our results apply for any even-denominator filling $\nu = \frac{1}{2k}$. Thanks to the Chern-Simons constraint, we are free to trade the quartic density-density interaction for a quadratic interaction between fluxes that modifies the propagator for the gauge field. Naïvely it seems that we can solve the theory exactly by integrating out the fermions. However, the fermionic excitations are gapless, and as a result this procedure is not controlled and can lead to various non-analytic and/or non-local terms in the resulting effective action[27]. We can go one step beyond the saddle-point solution and incorporate Gaussian fluctuations of the Chern-Simons field, as originally done by HLR. As we show below, the resulting "random-phase approximation" (RPA) response is that of a compressible phase with a Hall resistance tied to $\rho_{xy} = \frac{2e^2}{h}$, and a longitudinal resistance that arises from scattering off both charged impurities as well as the random flux configuration produced due to the rearrangement of the electron density in a disordered external potential [39].

We provide a telegraphic account of how one computes the composite fermion conduc-

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[27] An example of the law of conservation of difficulty, or the 'no free lunch' theorem.

tivity [96]. While computing the full electromagnetic response in the RPA is somewhat involved [39, 96], the determination of the conductivity tensor – which is ultimately the most significant observable in the quantum Hall context – is fairly straightforward. We begin by observing that the equation for the Chern-Simons electric field can be written as

$$\mathbf{e} = 2k\Phi_0(\hat{\mathbf{z}} \times \mathbf{j}) \equiv -\rho_{\mathrm{CS}}\mathbf{j} \tag{46}$$

where

$$\rho_{\mathrm{CS}} = \frac{2kh}{e^2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{47}$$

is the Chern-Simons resistivity tensor. On the other hand, we have already shown that the composite fermions interact with the combined gauge field $a_\mu + A_\mu$; the resulting response equation takes the form

$$\mathbf{j} = \rho_{\mathrm{CF}}^{-1}(\mathbf{e} + \mathbf{E}) \tag{48}$$

Here ρ_{CF} is the resistivity of composite fermions in the effective magnetic field $\langle b \rangle + B$. Of course, the Chern-Simons field is a purely internal quantity, and therefore cannot be measured by a physical voltmeter; the conductivity tensor measured in any experiment is the response of the current to the external field $\mathbf{E}$. To determine the measured conductivity σ, we must therefore eliminate $\mathbf{e}$ using the Chern-Simons relation, to find a resistivity addition rule: $\mathbf{j} = \sigma\mathbf{E}$, with

$$\sigma^{-1} = \rho = \rho_{\mathrm{CF}} + \rho_{\mathrm{CS}} \tag{49}$$

Using (49), we can compute the response of various composite fermion states, both compressible and incompressible.

For the case of half-filling, the effective field vanishes and therefore ρ_{CF} is purely diagonal,

$$\rho_{\mathrm{CF}} = \begin{pmatrix} \rho_{xx}^{\mathrm{CF}} & 0 \\ 0 & \rho_{yy}^{\mathrm{CF}} \end{pmatrix} \tag{50}$$

leading to the the following measured conductivity:

$$\sigma = \frac{1}{\rho_{xx}^{\mathrm{CF}}\rho_{yy}^{\mathrm{CF}} + \left(\frac{2h}{e^2}\right)^2} \begin{pmatrix} \rho_{yy}^{\mathrm{CF}} & -\frac{2h}{e^2} \\ \frac{2h}{e^2} & \rho_{xx}^{\mathrm{CF}} \end{pmatrix} \tag{51}$$

We see immediately that is *not* the response of a quantized Hall phase, which is appropriate: we do not expect a plateau from a compressible state. Note that within the composite

fermion Chern-Simons theory, the Hall *resistance* is quantized, although there is dissipation i.e. nonzero longitudinal resistance [28].

If we perform the composite fermion construction away from half-filling, the statistical and external fields no longer cancel exactly and $\langle b \rangle$ is nonzero. If, in this weakened field, the composite fermions form integer quantum Hall states with p filled pseudo-Landau levels, we have

$$\rho_{\text{CF}} = \frac{h}{pe^2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{52}$$

which leads to a measured conductivity

$$\sigma = \frac{1}{2k + \frac{1}{p}} \frac{e^2}{h} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{53}$$

as appropriate to the Jain hierarchy state at filling $\nu = \frac{p}{2pk+1}$. As promised, we have recovered the results of Jain's variational approach within a field theoretic framework.

Fluctuations of the Chern-Simons gauge field should not significantly modify the qualitative features of results away from half filling, since the states with p filled pseudo-Landau levels are all gapped. However, this is emphatically *not* the case for the gapless composite Fermi liquid at $\nu = \frac{1}{2}$. Here, the inclusion of gauge fluctuations beyond the RPA leads to various singularities. When the RPA corrections are taken into account, the propagator of the gauge field is renormalized by the particle-hole excitations of the composite fermions, and acquires both infrared and ultraviolet divergences. While the latter can be explained away as unphysical, the former lead to singularities in various physical quantities when the RPA-improved propagator is used in computations that go beyond the RPA, such as the fermion self-energy and corrections to the fermion-gauge vertex. In particular, the composite fermion spectral function vanishes logarithmically for Coulomb interactions, and even faster for short-ranged interactions between the bare electrons [39, 101]. Various careful studies [48–51, 53, 73] show that while the low-frequency, long-wavelength *response* does not deviate significantly from the RPA result, other quantities– for instance, the $2k_F$ susceptibility – may acquire divergences or nonanalyticities. Such singularities make the composite fermion

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[28] This result, which is a natural consequence of the vanishing of the composite fermion Hall conductance σ_{xy}^{CF} in the HLR approach, is somewhat controversial; in particular, particle-hole symmetry – which applies in the lowest Landau level approximation – requires that at half-filling the electron Hall *conductance*, rather than the Hall *resistance*, be quantized. See [58] for a discussion of this point.

approach a far from controlled theory. Ultimately, the most serious objection – which also applies also to the composite boson Landau-Ginzburg theories – is on very physical grounds: namely, that the Chern-Simons approach fails to properly build in the the lowest Landau level structure. Frequently quoted in this context is the "effective mass problem", which in its most obvious form is that the gaps predicted for incompressible states by the Chern-Simons-RPA method have an unphysical dependence on the band mass of the electrons, rather than being set purely by the interaction scale $e^2/\varepsilon\ell_B$ as appropriate to a fractional quantum Hall state constrained to the lowest Landau level [96].

There have been various attempts to construct a more tractable theory for $\nu = \frac{1}{2}$. These include both phenomenological Fermi liquid approaches [96, 98, 101], as well as pioneering work by Read [87] and by Pasquier and Haldane [82] for the bosonic case at $\nu = 1$ and the Hamiltonian theory of Murthy and Shankar [70]. In one way or another, each of the latter three approaches attempts to build a theory for composite fermions that lives entirely within the lowest Landau level and thus avoids the singularities inherent in the HLR approach. While they have some success, they are fairly cumbersome to work with, and so we continue to use the Chern-Simons approach with all the necessary caveats. In the end, any predictions that we make will have the flavor of phenomenological guesswork: they will have to find their vindication in numerical data or experimental fact.

B.   Paired States of Composite Fermions

We have successfully explained the compressible state at filling factor $\nu = \frac{1}{2}$ as a Fermi liquid of composite fermions. How are we to understand the state at $\nu = \frac{5}{2}$, which is incompressible and has a quantized Hall conductance?

One solution, which has the merit of hewing closely to the historical development of the subject, is to simply guess a wavefunction; this was done by Moore and Read , where they used the conformal block approach to produce a wavefunction for an even-denominator incompressible state [68]. Rather than take this route, we shall sacrifice historical accuracy for the sake of physical clarity. To motivate our approach, let us return to the computation of the response of the composite Fermi liquid. There, we observed that the measured resistivity was the sum of the composite fermion resistivity, which is computed in zero magnetic field, and a Chern-Simons term which was purely off-diagonal. The dissipation inherent in the

former resulted in a state without quantized Hall response. Were it not for this, we would have produced a state which had an even-denominator Hall conductance. It is natural, therefore, to ask how we can arrange affairs so that the compressible state acquires a bulk gap – necessary for the precise quantizaton of σ_{xy} and dissipationless transport. In other words, how can we gap a system of fermions in zero field so that transport is dissipationless?

The answer is to destroy the composite fermion Fermi surface by forming a superconductor via the Bardeen-Cooper-Schrieffer (BCS) mechanism. The resulting superconducting state[29] will have a Meissner effect – expulsion of flux from the bulk – which, thanks to the flux-charge equivalence of Chern-Simons theory implies a charge gap. Since superconductors cannot support electric fields in the bulk, we must have $\mathbf{e} + \mathbf{E} = 0$, which leads to the necessary quantized Hall conductance[30]. Moreover, vortices of the superconducting order parameter carry electric charge thanks to the Chern-Simons term; owing to the halving of the quantum of flux in the paired state and the half-integer Hall conductance, their charge is quantized in units of $\frac{e}{4}$.

This identification of the incompressible state in a half-filled Landau level as a paired quantum Hall state of composite fermions [32, 68] has a number of close connections to the original approach based on conformal field theory (CFT), which we summarize briefly. First we note that in the simplest case of spinless fermions, the pairing necessarily occurs in the p-wave channel [88]. The resulting paired state can be in one of two phases: a weak-pairing phase (corresponding to the 'BCS' picture of long-ranged pairs in position space) and a strong-pairing phase (the 'BEC' or 'molecular' limit where the pairs are tightly bound.) In each case, we can compute first-quantized electronic wavefunctions by projecting the BCS wavefunction onto a sector with a definite number of composite fermions, and multiplying the result by the $(z_i - z_j)^{2k}$ factor required by the composite fermion flux attachment transformation.

In the weak-pairing phase, we recover (asymptotically) the 'Pfaffian' wavefunction of

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[29] Much of what we say about the superconducting properties applies equally to the composite boson theory for odd denominators.

[30] Throughout, we assume that all the action takes place in the uppermost, partially filled Landau level; thus at $\nu = \frac{5}{2}$, we are taking for granted that the underlying filled Landau levels will provide the deficit $2e^2/h$ needed to obtain the correct Hall conductance.

Moore and Read, which at $\nu = \frac{1}{2k}$ has the form

$$\Psi_{\mathrm{MR}}^{2k}[z] = \mathrm{Pf}\frac{1}{z_i - z_j}\prod_{i<j}(z_i - z_j)^{2k}e^{-\frac{1}{4}\sum_j |z_j|^2} \tag{54}$$

where the Pffafian of a $2L \times 2L$ antisymmetric matrix M is

$$\mathrm{Pf}[M] = \frac{1}{2^L L!}\sum_{\sigma \in S_{2L}}\mathrm{sign}\sigma\prod_{k=1}^{L}M_{\sigma(2k-1),\sigma(2k)}. \tag{55}$$

By solving the Bogoliubov-de Gennes (BdG) equations for a vortex in the paired state – corresponding to a quantum Hall quasiparticle– we can show that it supports a zero-energy fermionic bound state, known as a Majorana zero mode [42, 88]. As a result, the vortices (quasiparticles) acquire non-Abelian statistics: the low-energy Hilbert space with nonzero quasiparticle number is finite-dimensional, and braiding the vortices leads to a unitary evolution within in this low-energy subspace[31] – thus rederiving a striking prediction of Moore and Read. Other topological properties – such as the existence of neutral gapless chiral edge modes and the counting of ground state degeneracies – can be computed using the BdG equations and shown to match the CFT predictions.

In the strong-pairing phase, the wavefunction is no longer of the Pfaffian form, owing to the tightly bound position space pairs. Asymptotically we simply find a wavefunction describing the resulting bosonic molecules, which describes an Abelian phase. In going from weak to strong pairing, the system undergoes a topological phase transition, as they correspond to different topological orders and hence different phases of matter.

We thus have a unified picture of the two distinct sets of phenomena at even denominators, summarized as follows: at such fillings, the electrons form a composite fermion Fermi liquid state. At $\nu = \frac{1}{2}$, the inter-electron repulsion is sufficient to render pairing ineffectual in destroying the Fermi surface, and the compressible state survives. At $\nu = \frac{5}{2}$, the modified form of the electronic wavefunctions in the $n = 1$ Landau level softens the repulsion, and the Fermi surface is unstable to pairing, leading to an incompressible state. This picture has received some support from numerical studies; it naturally carries with it the presumption that the pairing strength is tunable. We make implicit use of this ability in when we show that in the weak-coupling[32], 'BCS limit' the pairing energetics enforce quasiparticle attraction, leading to a Type I quantum Hall liquid.

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[31] This has been proposed as a platform for 'topological' quantum computation. For a review, see [72] .

[32] There is a subtle and for our purposes not particularly important distinction between weak coupling,

It is possible to show that other paired states – including those with spin or other internal degrees of freedom – can be captured within the composite fermion-BCS approach. We shall not discuss these further, but direct the interested reader to [88] for details.

As a final comment before we close this section, we point out that the fact that half-filled Landau levels can support both a compressible Fermi-liquid like phase and an incompressible paired state suggests that there may be interesting avenues for exploration, in heterostructures engineered to have spatially varying interaction parameters that support pairing in some regions of the sample and suppress it in others. As the gapless state has no natural length scale, it should support pairing correlations with power-law decay[33] This leads to a quantum Hall version of the superconducting proximity effect, and closely related analogs of Andreev reflection and the Josephson effect, which may serve as yet another probe of the fractional quantum Hall regime [77]. No comparable statements can be made for odd denominator quantum Hall states, essentially because there is no competing Fermi-liquid-like state at such fillings.

VII.   LANDAU-GINZBURG THEORIES OF THE QUANTUM HALL EFFECT

We have argued that even-denominator quantum Hall states are usefully described as 'superconductors'of composite fermions. We mentioned, but did not explicitly demonstrate, that the presence of a Chern-Simons term for the statistical gauge field allowed us to translate various properties of the superconducting phase into the language of the quantum Hall effect. This approach is not restricted to half-filled Landau levels; we can similarly model odd-denominator quantum Hall states as superconductors with Chern-Simons electrodynamics, except that we no longer have a natural interpretation of the superconductor as a condensate of Cooper pairs. Instead, we map the original problem of electrons in a magnetic field to one of composite bosons in zero field. The condensed phase of the composite bosons then corresponds to the quantum Hall state. It is to the Chern-Simons Landau-Ginzburg theories that result from this analysis that we now turn.

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which refers to the pairing energy scale, and weak pairing, which is a statement about the size of the pair wavefunction. For a discussion of this point see [88].

[33] These should survive the inclusion of interactions, as indeed is the case in the simpler example of a repulsive Fermi liquid [80].

A. Composite Boson Chern-Simons theory

We give a brief introduction to the Chern-Simons Landau-Ginzburg theory introduced by Zhang, Hansson and Kivelson (ZHK) [126] to describe odd-denominator fractions, and by extension the Haldane-Halperin hierarchy construction. The ZHK case predates the composite fermion approach to the half-filled Landau level, and rests on the idea of statistical transmutation – a flux attachment transformation that cancels part or all of the external field, similarly to the composite fermion approach, but which leads to a bosonized description of the quantum Hall problem.

We return to the unitary transformation (40-41) , but this time attach an *odd* number of flux quanta to each electron, so that

$$\Psi \to \tilde{\Psi} \equiv e^{i(2k+1)\sum_{i<j}\operatorname{Im}\log|z_i-z_j|}\Psi$$

$$\nabla \times \mathbf{a}(\mathbf{r}_j) \equiv b(\mathbf{r}_j) = -(2k+1)\Phi_0\sum_{i\neq j}\delta(\mathbf{r}_i-\mathbf{r}_j). \tag{56}$$

Unlike in the composite fermion case where an even number of fluxes were attached, here the symmetry of the many-body wavefunction has changed: $\tilde{\Psi}$ is now symmetric in all its coordinates. The gauge-transformed electrons obey Bose statistics, and we refer to them as composite bosons. The flux attachment transformation can be captured as before within a Chern-Simons field theory, this time involving a bosonic field φ but otherwise closely resembling (42)

$$S = \int_0^\beta d\tau d^2r \left[\varphi^*(i\hbar c\partial_0 - ea_0 - \mu)\varphi + \frac{\hbar^2}{2m^*}\left|\left(-i\partial_i - \frac{e}{\hbar c}(a_i + A_i)\right)\varphi\right|^2 \right.$$
$$\left. -\frac{1}{2(2k+1)\Phi_0}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \frac{1}{2}\int d^2r' |\varphi(\mathbf{r})|^2 V(\mathbf{r}-\mathbf{r}')|\varphi(\mathbf{r}')|^2\right]. \tag{57}$$

In writing this action, we have chosen to undo the transverse gauge-fixing in (42) so that the action is manifestly gauge-invariant[34]. The equations of motion that follow are very similar to the composite fermion case, except that the boson order parameter doubles as the

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[34] Except under transformations that are do not equal the identity on the boundary. This lack of invariance under "large" gauge transformations is a standard feature of a Chern-Simons theory, that ensures that the Chern-Simons coupling is not renormalized in any perturbative, momentum-shell renormalization procedure, which will always produce gauge-invariant corrections [93].

density,

$$b \equiv \nabla \times \mathbf{a} = -(2k+1)\Phi_0|\varphi(\mathbf{r})|^2$$
$$\varepsilon^{ij}e_j \equiv \varepsilon^{ij}(\partial_0 a_j - \partial_j a_0) = -\frac{(2k+1)\Phi_0}{c}\mathbf{j}(\mathbf{r}) \tag{58}$$

and $\mathbf{j}$ is the composite boson current, which is identical to (44) but with ψ replaced by φ.

Similarly to the composite fermion case, the composite bosons experience an effective field which is the sum of the statistical and external contributions. Thus, at commensurate fillings $\nu = \frac{1}{2k+1}$, the effective field vanishes and we have a theory of interacting bosons in zero field. In order to capture the phenomenology of the Hall effect, we replace the long-range Coulomb interaction with a contact repulsion between the bosons, so that the bosonic part of the action takes on the familiar Landau-Ginzburg form [126],

$$S = \int_0^\beta d\tau d^2r \left[\varphi^*(i\hbar c\partial_0 - ea_0 - \mu)\varphi + \frac{\hbar^2}{2m^*}\left|\left(-i\partial_i - \frac{e}{\hbar c}(a_i + A_i)\right)\varphi\right|^2 \right.$$
$$\left. +\frac{\lambda}{2}\left(|\varphi(\mathbf{r})|^2 - \overline{\rho}\right)^2 - \frac{1}{2(2k+1)\Phi_0}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \right]. \tag{59}$$

while the gauge field obeys Chern-Simons electrodynamics.

At commensuration, the boson field is condensed, the system is in a superconducting phase and exhibits a Meissner effect; this translates to incompressibility of the quantum Hall system through the Chern-Simons constraints. Exactly as in the case of a paired composite fermion state, the total electric field $\mathbf{e} + \mathbf{E} = 0$ in the condensed phase, so that the measured response is appropriate to a Hall plateau at filling $\nu = \frac{1}{2k+1}$. The vortex excitations now carry an electric charge $q = \frac{e}{2k+1}$, as appropriate to Laughlin quasiparticles. For further discussion on how to derive other aspects of fractional quantum Hall effect phenomenology from the effective field theory – including potential pitfalls of this approach – we refer the reader one of the reviews on the subject [47, 125].

B.   A Landau-Ginzburg Theory for Paired Quantum Hall States

We have already argued that an incompressible state can be obtained in a half-filled Landau level by forming a paired state of composite fermions. Response in such paired states can be calculated by standard techniques from the theory of superconductivity [21], and verifies our claim that the low-frequency, long-wavelength response is appropriate to

that of an even-denominator quantum Hall plateau. Working within the Bogoliubov-de Gennes formulation, we can extract additional information such as the wavefunction of the N-electron ground state, the statistics – sometimes non-Abelian – of quasiparticles, and the existence of neutral edge modes.

In this section, we sketch a derivation of a Landau-Ginzburg free energy for a paired quantum Hall state, which we use in [79] to study the energetics of quasiparticles and the structure of the quantum Hall plateau around even-denominator fillings. While our derivation is formally valid only near the transition – really, a crossover – temperature T_c, for phenomenological purposes we can use it down to $T = 0$. Our Landau-Ginzburg theory does *not* take into account the non-Abelian statistics of the quasiparticles; this would necessitate an additional, non-Abelian Chern-Simons gauge field for the nontrivial braiding statistics [24, 25]. While this is an important and fascinating aspect of the theory of paired Hall states, it is somewhat peripheral to our interests and therefore we do not discuss it further.

We restrict ourselves to the spinless case and accordingly begin with the gauge-fixed composite fermion action, (42), but with an additional external vector potential A_0. We add to this a phenomenological p-wave pairing interaction 'by hand' [21]. This takes the form

$$S_{\mathrm{p}} = g \sum_{\substack{n,'n,p, \\ \mathbf{k},\mathbf{k}',\mathbf{q}}} e^{i\theta_{\mathbf{k}}} e^{-i\theta_{\mathbf{k}'}} \psi^{\dagger}\left(i\omega_n, \mathbf{k} + \frac{\mathbf{q}}{2}\right) \psi^{\dagger}\left(-i\omega_n + \eta_p, -\mathbf{k} + \frac{\mathbf{q}}{2}\right) \tag{60}$$

$$\times \psi\left(-i\omega_{n'} + \eta_p, -\mathbf{k}' + \frac{\mathbf{q}}{2}\right) \psi\left(i\omega_{n'} + \eta_p, \mathbf{k}' + \frac{\mathbf{q}}{2}\right) \tag{61}$$

where $\omega_n = \frac{2\pi(n+1)}{\beta}$ and $\eta_p = \frac{2\pi p}{\beta}$ are fermionic and bosonic Matsubara frequencies. With this term added, the derivation proceeds as follows:

1. First, we use the Chern-Simons constraint to rewrite the density-density interaction V as an interaction between Chern-Simons fluxes. Once this is done, S_{p} is the only term in the action that is quartic in the fermion operators.

2. We perform a Hubbard-Stratonovich decoupling of S_{p} in terms of a pair field Δ, so that we obtain quadratic terms of the form $\Delta \psi^{\dagger} \psi^{\dagger}$, $\Delta^* \psi \psi$, and $\frac{\Delta^2}{g}$ at the cost of an additional functional integral over Δ.

3. Next, we perform the quadratic integral over the fermionic fields, integrating them out in favor of Δ. The resulting functional integral takes the form

$$\mathcal{Z}[A_\mu] = \int \mathcal{D}\Delta\mathcal{D}a\mu \exp\left[-\frac{1}{2k\Phi_0}a_0\nabla\times\mathbf{a} + S_0[\Delta, a\mu + A_\mu] + \frac{\Delta^2}{g}\right.$$
$$\left. + \int \frac{d^2\mathbf{r}d\tau}{(2k\Phi_0)^2}\left(\nabla\times\mathbf{a}(\mathbf{r}) - 2k\Phi_0\overline{\rho}\right)V(\mathbf{r}-\mathbf{r}')\left(\nabla\times\mathbf{a}(\mathbf{r}') - 2k\Phi_0\overline{\rho}\right)\right]$$

(62)

where $\overline{\rho}$ is a uniform positive background density chosen to render the Coulomb contribution nonsingular, and $S_0[\Delta, a_\mu + A_\mu]$ is the result of integrating out the fermions.

4. We now work near $T = T_c$, and follow the usual arguments in the derivation of the Landau-Ginzburg action to focus on the zero Matsubara frequency component of S, which suffices to obtain a free energy. We can expand the result in the standard Landau-Ginzburg form in terms of a suitably rescaled order parameter $\Psi \propto \Delta$[35] [83] , i.e. $F \sim c_1\left|\left(-i\nabla + \frac{e}{\hbar c}(\mathbf{a}+\mathbf{A})\right)\Psi\right|^2 + c_2\left(|\Psi|^2 - |\Psi_0|^2\right)^2 + \chi_0 e(a_0 + A_0)^2$, where the last contribution is from the response to the external field $a_0 + A_0$, and we use the fact that the static compressibility of a superconductor is essentially the same as the underlying Fermi liquid, $\chi_0 = \frac{m^*}{2\pi\hbar^2}$.

5. Finally, we integrate out a_0; this imposes the Chern-Simons constraint on the flux, and results in a contribution of the form $\frac{1}{(4k\Phi_0)^2\chi_0} + \frac{A_0\nabla\times\mathbf{a}}{2k\Phi_0}$. The latter term provides the correct Hall response to an external electric field $\mathbf{E} = -\nabla A_0$, while the former is an effective Maxwell energy that traces its origin to the compressibility of the underlying Fermi liquid normal state. The final result, with appropriate additional rescalings and redefinitions, is a free energy whose properties are discussed in detail in [79].

Of course, we have still to perform a functional integral over $\mathbf{a}$ as well as one over the Hubbard-Stratonovich field Δ; however, we expect that while this may change details of various estimates, the qualitative features elaborated in [79]are robust to such corrections.

The approach we have taken to analyzing the composite fermion pairing problem has been entirely phenomenological: we asserted that pairing occured, and inserted an *ad hoc* pairing

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[35] In general the order parameter can have more components, but our problem is simplified by working in the spinless case with a coupling that is attractive for only one of the two independent p-wave chiralities.

interaction S_{p}. Ideally, we would like to derive the pairing physics directly from the Coulomb interaction, but this is fraught with difficulties [12]. Recent progress on understanding the pairing transition within the renormalization group approach, whose preliminary results appear consistent with the results of [79], may shed additional light on these issues [66].

C. Off-Diagonal Long Range Order in the lowest Landau level

The central appeal of the effective field theory approach to the fractional quantum Hall effect is that it allows us to understand the topologically ordered quantum Hall phase as arising from the condensation of a bosonic order parameter. This simplifcation is achieved by means of the nonlocal flux attachment transformation. Unfortunately, by its very nature of partitioning the magnetic field into fluxes attached to charges, the flux attachment approach discards any information about the structure of Landau levels; the resulting theory is thus purely phenomenological, and various parameters have to be taken as physical inputs. For instance, since the effective field theory has to simultaneously (a) reproduce Kohn's theorem on the cyclotron resonance frequency and (b) give the correct scale ($\sim \frac{e^2}{\varepsilon \ell_B}$) to collective modes in the lowest Landau level, the question of an appropriate choice for the effective mass remains a thorny one.

It is natural to ask, in light of such issues, if we can construct a version of the composite boson/paired composite fermion construction that builds in Landau level physics at the outset. In important early work in this direction, Girvin and MacDonald [29] produced a lowest-Landau level operator – equivalent to the bosonic order parameter φ of the Zhang-Hansson-Kivelson theory – that exhibited 'algebraic off-diagonal long-range order (ODLRO)', i.e. its two-point correlations exhibits power-law rather than exponential decay with distance. This was later extended by Read [86] to construct an operator that exhibits true off-diagonal long-range order – the two-point correlator tends to a constant at infinite separation. We briefly describe how the 'Read operator' is constructed for the Laughlin fractions; similar arguments can be made for paired Hall states.

We begin with the N-particle Laughlin ground state at $\nu = \frac{1}{m}$ which we shall denote by $|0_L; N\rangle$; this has the usual (unnormalized) coordinate representation $\prod_{i<j}(z_i - z_j)^m e^{-\frac{1}{4}\sum_k |z_k|^2}$. It will prove useful to define $\psi(z) \equiv \sum_{n=0}^{\infty} a_n \varphi_n(z)$, where a_n the second-quantized field operator for the nth single-particle lowest Landau level basis state $\varphi_n(z)$

(see Section V A), as well as the Laughlin quasihole operator, which in first quantization is simply $U(z) = \prod_{i=1}^{N}(z_i - z)$.

Read observed that the two-point correlator

$$\langle 0_L; N|\tilde{U}^\dagger(z)^m \psi(z)\psi^\dagger(z')\tilde{U}(z')^m|0_L; N\rangle = \rho^{-1}\langle 0_L; N+1|\rho(z)\rho(z')|0_L; N+1\rangle$$

$$\to \rho_0 \tag{63}$$

as $|z - z'| \to \infty$, with $|z|, |z'| > N$. Here, we have defined a normalized version of the quasihole operator $\tilde{U}(z)^m|\alpha\rangle \equiv (\langle\alpha||U(z)|^2|\alpha\rangle)^{-1/2}U(z)^m|\alpha\rangle$.

The result (63) follows more or less immediately from the observation that the simultaneous addition of m fluxes and one electron to an N-electron Laughlin state must have nonzero overlap with the $N + 1$ electron Laughlin state, and the fact that the Laughlin states have liquid-like density correlations. Note that since the expectation value of the Read operator $\langle \mathcal{O}_R(z)\rangle \equiv \langle\psi^\dagger\tilde{U}^m\rangle$ vanishes in a Laughlin state while its square-expectation is nonzero, the Laughlin state is *not* a pure state. An example of the latter, in which $\psi^\dagger\tilde{U}^m$ also has an expectation value, can be constructed by superposing Laughlin states of various particle numbers[36]

$$|0_L; \theta\rangle = \sum_{N=1}^{\infty} \alpha_N e^{-iN\theta}|0_L; N\rangle \tag{64}$$

where α_N is real and squares to a binomial distribution on N with mean $\bar{N} \gg 1$ and variance of order $\bar{N}$, and θ is an arbitrary parameter. For this state, and arbitrary z, it is easily verified that

$$\langle\mathcal{O}_R(z)\rangle_\theta \to \rho_0^{1/2}e^{i\theta} \tag{65}$$

as $\bar{N} \to \infty$. Thus the Read operator serves as an order parameter for the fractional quantum Hall state. In exact analogy with a BCS superconductor, the fixed-particle-number Laughlin wavefunction follows as

$$|0_L; N\rangle = \left[\int d^2z\, \psi^\dagger(z)U(z)^m e^{-|z^2|/4}\right]^N |0\rangle \tag{66}$$

While not quite local as it involves a flux insertion, the Read operator *can* be uniquely associated with a single point, and therefore it can be used to construct a lowest-Landau-level field theory for the fractional quantum Hall states [85, 86]. However, the Chern-Simons

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[36] This should be familiar to readers well-versed in the theory of superconductivity; this is exactly how one constructs states in which the Cooper pair operator has a definite expectation value.

Landau-Ginzburg formalism is significantly easier to work with; we therefore adopt this view, under the assumption that the long-wavelength predictions of both approaches are morally the same.

VIII.   TYPE I AND TYPE II QUANTUM HALL LIQUIDS

Superconductors famously come in two varieties, which differ in their response to external magnetic fields: Type I superconductors phase separate into superconducting and normal regions, with flux concentrated in the latter, while Type II superconductors form an Abrikosov lattice of vortices, each carrying a single flux quantum. The analogy between superconductors and fractional quantum Hall phases suggests that there is a similar distinction between Type I and Type II quantum Hall liquids, manifested in dramatically different patterns of charge organization upon doping with quasiparticles. While Type II quantum Hall liquids exhibit the Wigner crystallization of fractionally charged quasiparticles traditionally assumed to occur about commensurate fillings in the clean limit, their Type I cousins exhibit either phase separation or for sufficiently long-ranged interactions, its frustrated mesoscopic analogs. Surprisingly, this quite general dichotomy was pointed out only recently, when it was argued – as recounted in [79]– that Type I behavior occurs in paired quantum Hall states, in the 'BCS limit', i.e. when the pairing scale is weak [79]. While the focus of that work was the Pfaffian phase seen in the vicinity of filling factor $\nu = 5/2$, the results generalize implicitly to all paired states.

In [78],we show how to obtain Type I behavior at several other fractions, both within effective field theory as well as more microscopically in terms of Hamiltonians projected into the lowest Landau level. In both approaches, we rely crucially on the existence of special points in the space of parameters, at which the interaction between quasiparticles [37] vanishes, and the underlying quantum Hall fluid is simultaneously gapped. Adding a weak attractive perturbation then renders the quasiparticles attractive without closing the gap required for their existence, which leads immediately to Type I behavior. Further modifications – such as the introduction of interactions of varying range and competing signs – can frustrate phase separation, leading to a variety of mesoscale phases upon doping.

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[37] More precisely, quasiholes but not quasielectrons, as elaborated in [78].

The distinction between Type I and Type II liquids thus adds an additional layer of complexity to the characterization of various fractional quantum Hall phases. While they share topological quantum numbers – such as ground state degeneracies on nontrivial manifolds, and charge and statistics of fractionalized excitations – they have significantly different quasiparticle energetics, reflected in the structure of their Hall plateaus. Other aspects of their phenomenology, such as their response to disorder, may further distinguish the two regimes.

IX. $\nu = 1$ IS A FRACTION TOO: QUANTUM HALL FERROMAGNETS

So far, our discussion of the Hall effect has not included spin, or other internal degrees of freedom. What happens when we add these to the problem?

In the case of spin, if we consider the lowest Landau level of a two-dimensional electron gas in free space, the answer is: not much. This is because the Zeeman energy $g\mu_B B$ which characterizes the gap between the different spin polarization states, is exactly equal to the cyclotron gap $\hbar\omega_c$, for $g = 2$ as appropriate to free space. The lowest Landau level has spins aligned with the magnetic field; every other spin-up Landau level corresponding to the nth oscillator level is degenerate with the spin-down Landau level of the $(n - 1)$th oscillator level. The gap to spin excitations is the same as the gap to inter-level transitions, and so once we're in a regime where the lowest Landau level approximation may be made, the spin degrees of freedom are 'frozen out', and therefore don't significantly change the physics at $\nu = 1$[38].

In real materials, however, two things conspire to alter this situation. First of all, the effective mass in these systems is much smaller than the physical electron mass ($m^*/m \approx 0.068$ for the conduction band of GaAs), and second, spin-orbit scattering reduces the effective g factor (g ≈ 0.4 in GaAs.) The first effect increases the cyclotron gap, whereas the second reduces the Zeeman splitting; the net result is that the ratio of the two energy scales is reduced from 1 to about 1/70. This means that at sufficiently low temperatures, the kinetic energy is quenched and the system may be considered confined to the lowest Landau level, but the spin degrees of freedom remain free to fluctuate. In fact, it is reasonable to ignore

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[38] At higher fillings, the two-dimensional electron gas in free space is an example of Landau level coincidence, which can also lead to a quantum Hall ferromagnet!

the Zeeman splitting at leading order, and consider the two spin states to be degenerate.

The first and immediate conclusion is that without interactions, there can be no $\nu = 1$ quantized Hall state seen in any realistic experimental conditions. The problem is once again massively degenerate – we have enough electrons to fill *one* Landau level, but are presented with *two* degenerate levels – and therefore our only recourse is to interactions as a means of resolving the degeneracy to produce a gapped (i.e., incompressible) state. In this sense, $\nu = 1$ is very similar to the case of fractional filling, although we can for the most part treat the interactions within Hartree-Fock theory and use Slater determinant trial states, unlike the case of the more 'fractional' fractions.

It remains to ask what ground state results, once interactions are included. When the latter are purely repulsive – a reasonable assumption in the systems of interest – we argue that the system must be ferromagnetically ordered at $\nu = 1$. This is because a spin-polarized state (with $S_{tot}^z = \frac{N\hbar}{2}$) has a wavefunction totally symmetric in its spin indices; the exclusion principle demands that the spatial part of the wavefunction is antisymmetric in every pair of electron coordinates, and thus the charge density has a node when any two electrons approach each other, which minimizes the interaction energy – essentially the same effect that leads to Hund's rule in atomic physics. At $\nu = 1$ the ground state can be written as a simple Slater determinant [39]; in second-quantized form, we have

$$|\Psi\rangle = \prod_X c_{X,\downarrow}^\dagger |0\rangle \tag{67}$$

where $c_{X,\sigma}^\dagger$ is the creation operator for a single-particle state in the lowest Landau level at guiding center coordinate X and spin σ. Our choice of polarization is appropriate to the case of GaAs, where the Zeeman term while small, is nevertheless nonzero and negative, thereby providing a weak symmetry breaking that picks a down-spin ground state.

The $\nu = 1$ state in GaAs is thus an itinerant quantum ferromagnet. In many ways, this is the simplest itinerant ferromagnet: the usual competition between the increase in kinetic energy and the decrease in interaction energy in a polarized state is rendered moot as the kinetic energy at $\nu = 1$ is quenched by the quantizing magnetic field. Thus, the exchange gain prevails, leading to a polarized state. We note in passing that there is an

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[39] This is the case even for the case of short-ranged interactions, which is often useful as a first approximation to the problem. We shall, however, discuss the Coulomb case as it is not too much more complicated.

interesting intermediate example, of a *three* dimensional electron gas in high fields and small Zeeman coupling where the kinetic energy is only partially quenched; the question of itinerant magnetism in such a system to our knowledge remains an open problem.

A. Spin Waves

In the absence of spin, the only neutral collective modes at $\nu = 1$ correspond to quasielectron-quasihole pairs, labeled by their momentum q. It is easy to show [27] that these necessarily have a gap set by the cyclotron frequency, $\Delta(q) \to \hbar\omega_c$ as $q \to 0$. When spin is included, however, there is a new branch of neutral excitations: spin waves (or magnons); these also involve quasielectron-quasiholes pairs, but now with opposite spin. Since (as $g \to 0$) a flipped spin is degenerate with the original one in the noninteracting problem, it is clear that the characteristic energy scale of the spin wave excitations is set, not by the cyclotron frequency but by the interaction scale[40] $\frac{e^2}{\varepsilon\ell_B}$. The dispersion of these modes can be shown to be [27, 46]

$$\Delta_{sw}(q) = g\mu_B B + \sqrt{\frac{\pi}{2}}\frac{e^2}{\varepsilon\ell_B}\left[1 - e^{-k^2\ell_B^2/4}I_0\left(\frac{k^2\ell_B^2}{4}\right)\right] \tag{68}$$

where I_0 is the modified Bessel function. That this dispersion is gapless and quadratic as $q \to 0$ is unsurprising, since we require such an excitation branch by Goldstone's theorem.

B. Skyrmions

What is the lowest-energy charged excitation in the quantum Hall ferromagnet? Naively, we would expect that we should simply remove a down spin or add a up spin, without disturbing the remainder. An estimate for the gap to creating such a quasiparticle-quasihole pair at infinite separation is to simply take the $k \to \infty$ limit of (68), giving a gap $\Delta_{\text{p-h}} = g\mu_B B + \sqrt{\frac{\pi}{2}}\frac{e^2}{\varepsilon\ell_B}$.

We can, however, do better by taking advantage of the ability to produce 'spin textures', topologically nontrivial configurations of the ferromagnetic order, while varying the charge density. This rests on the observation that the exchange gain for any given electron is essentially a local contribution from its interaction with those within a few magnetic lengths

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[40] This can be most easily understood by noting that a flipping a spin involves a loss of exchange energy.

of it. Thus, if while adding an up spin (removing a down spin) electron we simultaneously rotate the other spins around the added (removed) electron, we pay a far lower exchange energy. We can then slowly relax the spin configuration back to the down spin background over several magnetic lengths in a circularly symmetric fashion. A moment's thought suffices to realize that since the spin should interpolate smoothly between 'up' near the center and 'down' far away, there is necessarily a rotation in the local spin orientation as we encircle the added (removed) electron, leading to a topologically stable configuration of the ferromagnetic order parameter, commonly referred to as a skyrmion (see Fig. 7). While a skyrmion enjoys a significantly lower exchange contribution to the energy, it has an increased Zeeman cost; the competition between this and the Hartree energy of the nonuniform charge distribution sets the size and the energy gap of the resulting excitation. We can estimate the size of a skyrmion (λ) and the cost of a skyrmion-antiskyrmion pair (Δ_{sk}) using the nonlinear sigma model of the next subsection. To logarithmic accuracy at small Zeeman coupling we find

$$
\begin{aligned}
\left(\frac{\lambda}{\ell_B}\right)^3 &= \left(\frac{9\pi^2}{2^8}\right)\frac{\ell_B}{\varepsilon a}(g|\log g|)^{-1} \\
\Delta_{\mathrm{sk}}(g) &= \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{e^2}{\varepsilon \ell_B}\left[1 + \frac{3\pi}{4}\left(\frac{18}{\pi}\right)^{1/6}\left(\frac{\varepsilon a}{\ell_B}\right)^{1/3}(g|\log g|)^{1/3}\right]
\end{aligned}
\tag{69}
$$

where $a = \hbar^2/m^* e^2$ is the Bohr radius [99]. The cost of a skyrmion-anti skyrmion pairs is thus – in the limit of vanishing Zeeman coupling – one-half the cost of the simple spin-flip pair; below, we verify that the skyrmions carry an electrical charge. This continues to hold for small but nonzero vaues of g, so that in this limit the lowest-energy charged excitations of the $SU(2)$ symmetric quantum Hall ferromagnet are charged skyrmions. The associated spin textures have various observable consequences for tranport and magnetic resonance experiments.

C.   Low-energy Dynamics

An elegant treatment of the dynamics of the quantum Hall ferromagnet may be derived [99] within the Chern-Simons Landau-Ginzburg approach with appropriate modifications to include the spin index [59]. Without elaboration, we simply present the low-energy effective
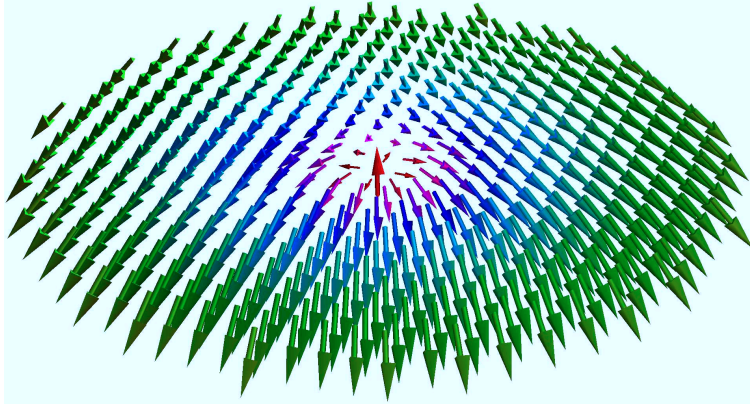
FIG. 7: Skyrmion spin configuration.

Lagrangian that results from this analysis[41]:

$$\mathcal{L}_{\text{eff}} = \alpha \mathcal{A}[\mathbf{n}(\mathbf{r})] \cdot \partial_t \mathbf{n}(\mathbf{r}) + \frac{\rho_s}{2} \left(\nabla \mathbf{n}(\mathbf{r}) \right)^2 + g\bar{\rho}\mu_B \mathbf{n}(\mathbf{r}) \cdot \mathbf{B}$$
$$- \frac{1}{2} \int d^2 r' \, V(\mathbf{r} - \mathbf{r}') q(\mathbf{r}) q(\mathbf{r}') \tag{70}$$

with the constraint $[\mathbf{n}(\mathbf{r})]^2 = 1$. Here, α and ρ_s are couplings that depend on the interaction scale $\frac{e^2}{\varepsilon \ell_B}$ and $\bar{\rho}$ is the average density. $\mathcal{A}$ is the vector potential of a unit monopole at the origin of the spin-space Bloch sphere, i.e. $\nabla \times \mathcal{A} = \mathbf{n}$, and is chosen to give the precessional dynamics to $\mathbf{n}$ required for the quantum-mechanical equations of motion for a spin. The first three terms of $\mathcal{L}_{\text{eff}}$ are the standard terms in the nonlinear sigma model treatment of a ferromagnet [23]; the new ingredient in the quantum Hall ferromagnet is the final term, which represents an interaction between topological or Pontryagin densities

$$q(\mathbf{r}) = \frac{1}{8\pi} \varepsilon^{ij} \varepsilon^{abc} n^a \partial_i n^b \partial_j n^c. \tag{71}$$

The spatial integral of q gives the Pontryagin index (Chern number) of $\mathbf{n}(\mathbf{r})$ thought of as a map from the plane (suitably compactified by including the point at infinity) to the

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[41] In writing this Lagrangian we have followed the authors of [99] in neglecting a Hopf term, which is the transcription of the Chern-Simons term required in all long-wavelength theories of the quantum Hall effect to the sigma-model description; while important to obtain fermionic statistics for the skyrmionic quasiparticles, for our purposes this is not essential.

spin sphere. This is the topological invariant that underlies the stability of a skyrmionic configuration to smooth deformations of the order paramter. To see that q is also the density of *electric* charge of a skyrmion configuration – and thus explain the inclusion of the Coulomb interaction term in (70) – consider the following argument, from [27]. An electron described by position coordinate x^μ, moving in a static background spin configuration n^ν can be described by the Lagrangian

$$\mathcal{L}_0 = -\frac{e}{c}\dot{x}_\mu A^\mu + \frac{\hbar}{2}\dot{n}^\mu \mathcal{A}_\mu[\mathbf{n}] = -\frac{e}{c}\dot{x}_\mu \left(A^\mu + a_B^\mu\right) \tag{72}$$

where the first term is the usual coupling to the electromagnetic gauge field, and the second is a contribution coming from the Berry phase from the changing local field $\dot{n}^\mu = \dot{x}^\nu \partial_\nu n^\mu$. This defines a Berry vector potential for transport in the spin background, $a_S^\mu \equiv -\frac{\Phi_0}{2}\partial_\mu n^\nu \mathcal{A}^\nu[\mathbf{n}]$. In writing $\mathcal{L}_0$ in this form, we have assumed that the exchange coupling is strong enough to force the electron spin to follow the local orientation, and the $\frac{\hbar}{2}$ factor is appropriate to a spin-$\frac{1}{2}$ particle. The additional Berry potential produces a pseudo-magnetic field b_S, which is easily verified to be simply $b_S(\mathbf{r}) = -\Phi_0 q(\mathbf{r})$ where q is the Pontryagin density defined previously.

If we adiabatically deform the spin configuration $\mathbf{n}$, the electronic degrees of freedom see this as an added Berry flux. Since the Berry potential couples to the electrons in identical fashion as the physical electric field, it follows by the same argument as for the charge of a Laughlin quasiparticle that the adiabatic deformation produces a change in the charge density

$$\delta\rho(\mathbf{r}) = \frac{\sigma_{xy}}{c}b_S(\mathbf{r}) = -\nu e q(\mathbf{r}) \tag{73}$$

The integral of the Pontryagin index over all space vanishes, *unless* the spin configuration $\mathbf{n}$ has nonzero topological index (skyrmion number). Thus, skyrmions in a quantum hall ferromagnet at filling factor ν carry νe units of electric charge, which is identical to the charge of the Laughlin quasiparticle. The spin of the skyrmion is a somewhat more delicate issue; for a discussion, see [74].

An alternative approach which allows a direct evaluation of terms in the ferromagnetic energy functional was developed by [67]. In its essence, the method involves explicitly computing the energy for long-wavelength fluctuations about the ferromagnetic ground state using the algebra [30] of operators projected into the lowest Landau level. The relation between the topological and electrical charges can also be derived microscopically in this

fashion. The results are consistent with [99] and are readily extended to cases where the symmetry of the low-energy theory is not immediately obvious, such as those in [1].

D. Other Examples

We focused above on the electron spin, as it is an illustrative example and the best studied to date. However, there are various other internal degrees of freedom – such as the semiconductor valley pseudospin, layer index in double quantum wells, and Landau level index when different Landau levels are brought into coincidence in tilted fields, to name a few. The symmetry of the resulting ferromagnet depends on details of the interaction, which can introduce various anisotropies; for instance, in [1] we study an example where owing to an anisotropic effective mass tensor the ferromagnet has a strong easy-axis (Ising) anisotropy, while in bilayer systems the tendency to favor equal fillings in both layers leads to an easy-plane (XY) system [67]. Each symmetry class has distinctive features; we defer discussion of the easy-axis and easy-plane cases to [1]and to the cited reference, respectively. A general classification of quantum Hall ferromagnets into different pseudospin anisotropy categories based on the symmetries of their interactions may be found in [45].

Quantum Hall ferromagnetism is not restricted to integer Landau levels with interactions, but can be generalized to other fillings, for instance $\nu = \frac{1}{3}$ [99]. Indeed, the question of whether such ferromagnetic behavior occurs at $\nu = \frac{5}{2}$ is a central issue in determining whether it is in fact a non-Abelian quantum Hall state [17].

X. ANTIFERROMAGNETIC ANALOGS AND AKLT STATES

It is clear that the fractional quantum Hall states are extraordinary from the conventional Landau-Ginzburg-Wilson perspective of broken-symmetry phases of matter: they break no symmetries; they exhibit fractionalization of quantum numbers; they have a nontrivial ground state degeneracy on a torus; and so on. This complex of phenomena was soon recognized as characteristic of a new kind of order emergent in the low-energy description of strongly correlated quantum matter, commonly termed topological order. Unlike traditional broken-symmetry phases where the natural low-energy description is a sigma model in terms of a local parameter, topological phases are described by an emergent gauge symmetry;

while on occasion a particular gauge may be found in which a description in terms of a local order parameter obtains, such examples are fortuitous exceptions to the general rule that no such description is possible. There are by now many different theoretical examples of topological phases, although the quantum Hall states remain the only ones on firm experimental footing; this is because their topological order manifests itself in transport and is thus readily measurable.

Quantum antiferromagnets in low dimensions have proven to be an abundant source of strongly-correlated phases, since they naturally have strong fluctuations and, in addition, can be readily frustrated by competing interactions or geometry. Topological phases are no exception, as evidenced by the multitude of topologically ordered 'spin liquid' states proposed on various lattices in $d = 2$ and 3. Many, if not all, of these phases can be captured within mean-field constructions where the spins are decomposed into fermionic spinons or Schwinger bosons[42], and standard techniques from the study of Fermi or Bose gases can then be applied to the new variables; since the fractionalization into the emergent degrees of freedom artificially enlarges the Hilbert space, an emergent gauge field is introduced to constrain calculations to the physical subspace [118]. For a review of experimental and theoretical developments in the study of spin liquids, we direct the reader to [61]; there are also recent numerical results [64, 124] that have received much attention. We shall not, however, discuss these further.

Instead, we focus on a somewhat simpler set of antiferromagnetic phases, that are *not* topologically ordered – they have no nontrivial groundstate degeneracies and host no fractionalized excitations. However, they do not break any lattice or rotational symmetries, their ground states lack order owing to quantum fluctuations and (in some cases) geometric frustration, and they are the *exact* ground states of local – indeed, nearest-neighbor – Hamiltonians. These are the quantum paramagnetic valence bond solid states originally proposed by Affleck, Kennedy, Lieb and Tasaki (AKLT) as ground states for one-dimensional spin-1 chains [2, 3]. The motivation of those authors was to construct a rigorous example of an integer-spin system that was in accord with Haldane's conjecture [35] that half-integer spin chains support gapless spinon excitations and power law-correlated ground states, while in-

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[42] In this sense, the fractionalization is manifest at the outset. This is closely related to the parton construction of quantum Hall phases [44, 115, 117].

teger chains are gapped and exhibit exponential correlations. The essential insight of the AKLT approach is to build a wavefunction that incorporates quantum fluctuations from the ground up through the following steps: (a) expand the Hilbert space by decomposing every spin into spin-$\frac{1}{2}$ constituents (b) build a quantum-disordered state by placing these into pairwise singlets along bonds; and (c) impose a constraint, of symmetrization on each site, to project back to the physical degrees of freedom.

While they are not topologically ordered, the fact that AKLT states build in 'good' correlations at the outset and have model Hamiltonians places them on conceptually similar footing with Laughlin's trial wavefunctions for the fractional quantum Hall effect. These similarities were crystallized in work by Arovas, Auerbach, and Haldane, who showed how to construct families of AKLT states on arbitrary lattices through the Schwinger boson approach[43] [5]; the construction constrains the spin on a site to be a half-integer multiple of the coordination number. These states were shown to be exact ground states of Hamiltonians that, in the original variables, could be written in terms of projectors onto states of definite total angular momentum of pairs of spins. Finally, it was noted that the ground state correlations could be calculated in terms of a finite-temperature classical model on the same lattice, by working in the basis of coherent states of spin in which the wavefunction has a Jastrow form – closely paralleling Laughlin's plasma analogy.

Are the AKLT states really quantum disordered? This is a less trivial question than it might seem at first glance – recall that the Laughlin wavefunctions eventually evolve crystal-like correlations for large enough m; a similar complication could in principle occur in the AKLT approach. Here, the fact that the classical model is at finite temperature comes to our rescue, since it would necessarily have to break a continuous symmetry to order, which is precluded in $d = 1$ and $d = 2$ by the Mermin-Wagner theorem [44] [65]. The three-dimensional case is not so straightforward, and here the question must be settled by explicit study of the ground state correlations, which is the subject of [81].

We close by noting that in $d = 1$ the spin-1 AKLT state is the exact ground state for the Heisenberg model with an additional, nearest-neighbor biquadratic term. By varying

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[43] While a two-dimensional example on the honeycomb lattice was already known to AKLT, Arovas *et. al.* were the first to systematically give a prescription for *all* lattices.

[44] Note that this does *not* rule out the Wigner crystal transition in the classical plasma corresponding to the Laughlin state, since the presence of long-range interactions invalidates the assumptions of [65].

the strength of the biquadratic term, the ground state can be studied numerically; these show that the AKLT state is adiabatically connected to the pure Heisenberg model, and therefore captures universal properties of the phase. Whether this remains the case in higher dimensions remains, to our knowledge, an open question that warrants further study.

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