Field theories of conformal and compressible quantum matter

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(Dated: July 18, 2011)

Abstract

Field theories of conformal and compressible states of matter are derived from lattice Hubbard models.

Summer School on Applications of String Theory
Institute for Nuclear Theory, Seattle July 18-20, 2011

These notes are adapted from earlier reviews [1, 2].

I. CONFORMAL QUANTUM MATTER

We will demonstrate that the superfluid-insulator quantum phase transition of the boson Hubbard model in two spatial dimensions is described by a conformal field theory (CFT).

The boson Hubbard model is

$$H_b = -w \sum_{\langle ij \rangle} \left(b_i^\dagger b_j + b_j^\dagger b_i \right) + \frac{U}{2} \sum_i n_i (n_i - 1) - \mu \sum_i n_i, \tag{1.1}$$

where b_i is the canonical boson annihilation operator, $n_i = b_i^\dagger b_i$ is the boson number operator, w is the hopping matrix element between nearest-neighbor sites, U is the on-site repulsive energy between a pair of bosons, and μ is the chemical potential. Let us assume that the average boson density is exactly n_0 per site, where n_0 is a positive integer. For $U/w \gg 1$, the ground state is simply

$$|GS\rangle = \prod_i (b_i^\dagger)^{n_0} |0\rangle, \tag{1.2}$$

where $|0\rangle$ is the empy state with no bosons. In the same limit, the lowest excited states are "particles" and "holes" with one extra or missing boson,

$$|p_i\rangle = b_i^\dagger |GS\rangle, \tag{1.3}$$

$$|h_i\rangle = b_i |GS\rangle. \tag{1.4}$$

For $w/U = 0$ strictly, the particle and hole energies (relative to the ground state) are

$$E_p^{(0)} = U n_0 - \mu, \qquad E_h^{(0)} = U(1 - n_0) + \mu. \tag{1.5}$$

For $0 < w/U \ll 1$, these states will develop dispersion. By considering the first order splitting of the degenerate manifold of particle or hole states (degeneracy associated with the site of the particle or hole), one obtains (considering the square lattice for simplicity)

$$E_p(\boldsymbol{k}) = E_p^{(0)} - 2w(n_0 + 1)(\cos k_x + \cos k_y) \approx \Delta_p + \frac{k^2}{2m_p}$$

$$E_h(\boldsymbol{k}) = E_h^{(0)} - 2w n_0 (\cos k_x + \cos k_y) \approx \Delta_h + \frac{k^2}{2m_h}, \tag{1.6}$$

where we have Taylor expanded around the minimum at $k = 0$, giving $m_p = 1/(2(n_0 + 1)w)$

and $m_h = 1/(2n_0 w)$ The excitation gaps, $\Delta_{p/h}$, are

$$\begin{aligned} \Delta_p &= U n_0 - \mu - 4w(n_0 + 1) \\ \Delta_h &= U(1 - n_0) + \mu - 4w n_0. \end{aligned} \tag{1.7}$$

to this order in w/U. As long as both of the these gaps are positive, our starting point of a Mott insulating state with an average of n_0 particles per site is stable.

When one of the gaps vanish, the Mott insulator is no longer stable, and we have a quantum transition to a superfluid state. Let us assume that it is Δ_p that vanishes first with increasing w. The transition then corresponds to a Bose-Einstein condensation of particles, with $-\Delta_p$ acting as the effective chemical potential. At $T = 0$, an increasing chemical potential implies an increasing particle density, and so the superfluid state will have a density greater than that of the Mott insulator. Similarly, if the value of μ is such that Δ_h vanishes first, the superfluid state will have a density smaller than that of the Mott insulator.

However, let us consider the special case when the density of both the Mott insulator and the superfluid are equal to n_0; this is often naturally the case under experimental conditions. Our reasoning makes it clear that this is only possible if μ is chosen so that $\Delta_p = \Delta_h \equiv \Delta$. This both gaps vanish simultaneously, the insulator-superfluid transition corresponds to condensation of both particles and holes (which can be viewed as "anti-particles"). This symmetry and particles and anti-particles is responsible for the relativistic structure of the low energy theory.

Let us now proceed to derive the effective action for the low energy theory near the insulator-superfluid transition. While it is possible to derive a field theory of this condensation from H_b, we instead just write it down based on our simple physical picture. We model the particle and hole excitations by fields $p(r,\tau), h(r,\tau)$ respectively, in the imaginary time (τ) path integral. The weight in the path integral is, as usual, the Euclidean action,

$$\mathcal{S}_b = \int d\tau d^2 r \left[p^\dagger \left(\frac{\partial}{\partial \tau} + \Delta - \frac{\nabla^2}{2m_p} \right) p + h^\dagger \left(\frac{\partial}{\partial \tau} + \Delta - \frac{\nabla^2}{2m_h} \right) h - \Lambda(p^\dagger h^\dagger + ph) + \cdots \right]. \tag{1.8}$$

Here we have included a term Λ which creates and annihilates particles and holes together in pairs, which is expected since this conserves boson number. Microscopically this term arises from the action of the hopping w on the naive ground state, which creates particle-hole pairs on neighboring sites, so $\Lambda \sim O(w)$ (the spatial dependence is unimportant for the states near $k = 0$). We have neglected – for brevity of presentation – to write a number of higher order terms involving four or more boson fields, representing interactions between particles and/or holes, and other boson number-conserving two-body and higher-body collisional processes. Note that the dependence upon w/U in Eq. (1.8) arises primarily through implicit dependence of Δ.

Without loss of generality, we assume $\Lambda > 0$, and change variables to the linear combinations

$$\psi = \frac{1}{\sqrt{2}}(p + h^\dagger) \qquad \xi = \frac{1}{\sqrt{2}}(p - h^\dagger). \tag{1.9}$$

Then the quadratic terms in the action are

$$\begin{aligned} \mathcal{S}_b \;=\; & \int d\tau d^2r \left[\xi^\dagger \frac{\partial \psi}{\partial \tau} - \xi \frac{\partial \psi^\dagger}{\partial \tau} + (\Delta - \Lambda)|\psi|^2 + (\Delta + \Lambda)|\xi|^2 \right. \\ & \left. + \left(\frac{1}{4m_p} + \frac{1}{4m_h} \right) (|\nabla \psi|^2 + |\nabla \xi|^2) + \left(\frac{1}{4m_p} - \frac{1}{4m_h} \right) (\nabla \psi^\dagger \nabla \xi + \nabla \xi^\dagger \nabla \psi) \right]. \end{aligned} \tag{1.10}$$

Notice that the quadratic form for ψ becomes unstable, before that of ξ. So let us integrate out ξ, expanding the resulting action in powers and gradients of ψ. In this manner we obtain the theory for the superfluid order parameter ψ [3]

$$\mathcal{S}_b \;=\; \int d\tau d^2r \left[\frac{1}{\Delta + \Lambda} \left| \frac{\partial \psi}{\partial \tau} \right|^2 + (\Delta - \Lambda)|\psi|^2 + \left(\frac{1}{4m_p} + \frac{1}{4m_h} \right) |\nabla \psi|^2 + u|\psi|^4 \right]. \tag{1.11}$$

This is the promised relativistic field theory. The energy gap for both particle and hole excitations is $\sqrt{\Delta^2 - \Lambda^2}$, and this vanishes at the quantum critical point.

There are a number of additional higher-order terms, not displayed above, which are not relativistically invariant. However, all of these are formally irrelevant at the Wilson-Fisher fixed point which controls the critical theory.

Finally, we note that the scale and relativistic invariance of the critical point are sufficient to establish its invariance under conformal transformations.

A. Quantum critical transport

To illustrate the general issues, we begin by computing the transport properties of the free field theory of a complex scalar with mass m, written in a Lorentz invariant notation:

$$\mathcal{S}_\psi = \int d^D r \left[|\partial_\mu \psi|^2 + m^2 |\psi|^2 \right] \tag{1.12}$$

This theory can be obtained from Eq. (1.11) at $u = 0$, after appropriate rescalings of coordinates and fields.

The conserved electrical current is

$$J_\mu = -i \left(\psi^* \partial_\mu \psi - \psi \partial_\mu \psi^* \right). \tag{1.13}$$

Let us compute its two-point correlator, $K_{\mu\nu}(k)$ at a spacetime momentum k_μ at $T = 0$.

This is given by the one-loop diagram which evaluates to

$$\begin{aligned} K_{\mu\nu}(k) &= \int \frac{d^3p}{8\pi^3}\left(\frac{(2p_\mu+k_\nu)(2p_\nu+k_\nu)}{((p+k)^2+m^2)(p^2+m^2)} - 2\frac{\delta_{\mu\nu}}{p^2+m^2}\right) \\ &= -\frac{1}{8\pi}\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\int_0^1 dx \frac{k^2(1-2x)^2}{\sqrt{m^2+k^2x(1-x)}}. \end{aligned} \tag{1.14}$$

The second term in the first equation arises from a 'tadpole' contribution which is omitted in Eq. (1.13). Note that the current correlation is purely transverse, and this follows from the requirement of current conservation

$$k_\mu K_{\mu\nu} = 0. \tag{1.15}$$

Of particular interest to us is the K_{00} component, after analytic continuation to Minkowski space where the spacetime momentum k_μ is replaced by (ω, k). The conductivity is obtained from this correlator via the Kubo formula

$$\sigma(\omega) = \lim_{k\to 0}\frac{-i\omega}{k^2}K_{00}(\omega,k). \tag{1.16}$$

In the insulator, where $m > 0$, analysis of the integrand in Eq. (1.14) shows that that the spectral weight of the density correlator has a gap of $2m$ at $k = 0$, and the conductivity in Eq. (1.16) vanishes. These properties are as expected in any insulator.

At the critical point, where $m = 0$, the fermionic spectrum is gapless, and so is that of the charge correlator. The density correlator in Eq. (1.14) and the conductivity in Eq. (1.16) evaluate to the simple universal results

$$\begin{aligned} K_{00}(\omega,k) &= \frac{1}{16}\frac{k^2}{\sqrt{k^2-\omega^2}} \\ \sigma(\omega) &= \frac{1}{16}. \end{aligned} \tag{1.17}$$

Going beyond the free field theory in Eq. (1.12), the effect of interactions can be accounted for order-by-order in u. In the renormalization group approach, u takes the value specified by the Wilson-Fisher fixed point at the quantum critical point. Combined with the absence of of divergencies in the perturbative expansion (which is a consequence of Eq. (1.15)), this means the only effect of interactions is to change the pre-factor in Eq. (1.17) to a different universal numerical value. So we write

$$\begin{aligned} K_{00}(\omega,k) &= \sigma_\infty\frac{k^2}{\sqrt{k^2-\omega^2}} \\ \sigma(\omega) &= \sigma_\infty, \end{aligned} \tag{1.18}$$

where σ_∞ is a universal number dependent only upon the universality class of the quantum critical point, whose value can be computed by various expansion methods.

1. *Non-zero temperatures*

We begin by repeating the above computation for the free field theory at $T > 0$. This only requires replacing the integral over the loop frequency in Eq. (1.14), by a summation over the Matsubara frequencies which are quantized by integer multiples of $2\pi T$. Such a computation, via Eq. (1.16) leads to the conductivity

$$\begin{aligned} \mathrm{Re}[\sigma(\omega)] &= \mathcal{P}\,\delta(\omega) + \frac{\theta(|\omega| - 2m)}{16}\left(\frac{\omega^2 - 4m^2}{4\omega^2}\right)\coth\left(\frac{|\omega|}{4T}\right) \\ \mathcal{P} &\equiv \frac{1}{8T}\int_0^\infty \frac{k^3 dk}{(k^2+m^2)}\frac{1}{\sinh^2\left(\sqrt{k^2+m^2}/2T\right)}; \end{aligned} \tag{1.19}$$

the imaginary part of $\sigma(\omega)$ is the Hilbert transform of $\mathrm{Re}[\sigma(\omega)] - 1/16$. Note that this reduces to Eq. (1.17) in the limit $\omega \gg T$. However, the most important new feature of Eq. (1.19) arises for $\omega \ll T$, where we find a delta function at zero frequency in the real part. Thus the d.c. conductivity is infinite at this order, arising from the collisionless transport of thermally excited carriers. This is clearly an artifact of the free field theory.

At non-zero u, collisions between carriers invalidate the form in Eq. (1.19) for the density correlation function, and we instead expect the form dictated by the hydrodynamic diffusion of charge. Thus for K_{00}, Eq. (1.18) applies only for $\omega \gg T$, while

$$K_{00}(\omega, k) = \chi \frac{Dk^2}{Dk^2 - i\omega} \quad , \quad \omega \ll T. \tag{1.20}$$

Here χ is the charge susceptibility (here it is the compressibility), and D is the charge diffusion constant. By the universality of the Wilson-Fisher fixed point, we expect that these have universal values in the quantum critical region:

$$\chi = \mathcal{C}_\chi T \quad , \quad D = \frac{\mathcal{C}_D}{T}, \tag{1.21}$$

where again $\mathcal{C}_\chi$ and $\mathcal{C}_D$ are universal numbers. For the conductivity, we expect a crossover from the collisionless critical dynamics at frequencies $\omega \gg T$, to a hydrodynamic collision-dominated form for $\omega \ll T$. This entire crossover is universal, and is described by a universal crossover function

$$\sigma(\omega) = \Sigma(\omega/T). \tag{1.22}$$

The result in Eq. (1.18) applies for $\omega \gg T$, and so

$$\Sigma(\infty) = \sigma_\infty. \tag{1.23}$$

For the hydrodynamic transport, we apply the Kubo formula in Eq. (1.16) to Eq. (1.20) and obtain

$$\Sigma(0) = \mathcal{C}_\chi \mathcal{C}_D \tag{1.24}$$

which is a version of Einstein's relation for Brownian motion.

II. COMPRESSIBLE QUANTUM MATTER

Now we will consider the Hubbard model for fermionic particles with spin $S = 1/2$ (electrons) on the triangular lattice.

For small U/w, the ground state of this model is a metal, rather than a superfluid. This is because the fermions cannot condense; instead they occupy all single particle states inside a 'Fermi surface' in momentum space, forming a Fermi liquid.

For large U/w, and with a density of one electron per site, we do expect an insulating state to form, with a gap to both particle and hole excitations, just as was the case for bosons. However, the electron localized on each site of the lattice now has a spin degeneracy, and we also have to specify the spin wavefunction in the insulator. At the largest values of U/w, it is believed that the insulator has long-range antiferromagnetic order; we will not study this ordered state here. The nature of the insulator at smaller U/w, and in particular, in the vicinity of the insulator-metal transition is still a question of some debate. In the following, we will assume that the insulating state proximate to the critical point is a particular "U(1) spin liquid", which we will describe more completely below.

The Hubbard Hamiltonian is

$$H = -w \sum_{\langle ij \rangle} \left(c^\dagger_{i\alpha} c_{j\alpha} + c^\dagger_{j\alpha} c_{i\alpha} \right) + \sum_i \left[-\mu \left(n_{i\uparrow} + n_{i\downarrow} \right) + U \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right) \right]. \tag{2.1}$$

Here $c_{i\alpha}$, $\alpha = \uparrow, \downarrow$ are annihilation operators on the site i of a triangular lattice. The density of electrons is controlled by the chemical potential μ which couples to the total electron density, with

$$n_{i\uparrow} \equiv c^\dagger_{i\uparrow} c_{i\uparrow} \quad , \quad n_{i\downarrow} \equiv c^\dagger_{i\downarrow} c_{i\downarrow}. \tag{2.2}$$

For completeness, we also note the algebra of the fermion operators:

$$\begin{aligned} c_{i\alpha} c^\dagger_{j\beta} + c^\dagger_{j\beta} c_{i\alpha} &= \delta_{ij} \delta_{\alpha\beta} \\ c_{i\alpha} c_{j\beta} + c_{j\beta} c_{i\alpha} &= 0. \end{aligned} \tag{2.3}$$

Let us begin by considering the case $U = 0$. Then the ground state is a metal at all densities, with a Fermi surface separating occupied and empty states in momentum space. Landau's Fermi liquid (FL) theory describes how the free-electron model of a metal can be extended to non-zero U. For our purposes, we need only two basic facts: (i) the fermionic

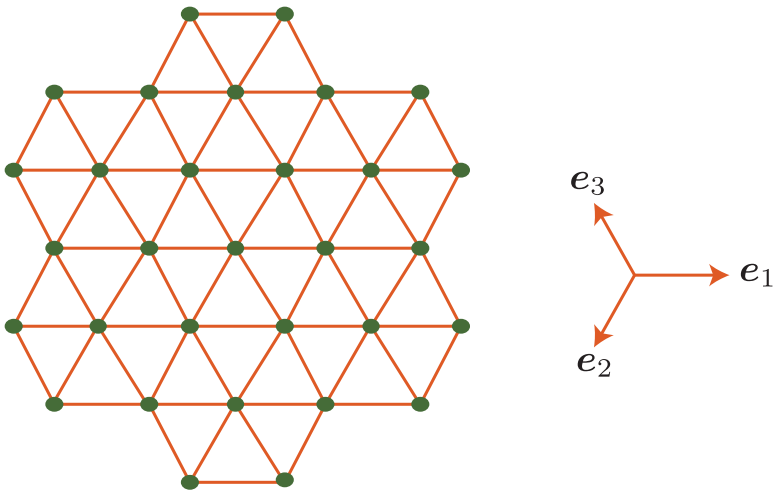

FIG. 1: The triangular lattice

excitations near the Fermi surface are essentially non-interacting electrons, and (*ii*) the area enclosed by the Fermi surface is equal to the electron density—this is Luttinger's theorem, which we state more explicitly below.

At $U = 0$, the Hamiltonian of the FL metal is

$$H_0 = \sum_{\boldsymbol{k}} c^\dagger_\alpha(\boldsymbol{k}) \Big[-\mu - 2t \Big(\cos(\boldsymbol{k}\cdot\boldsymbol{e}_1) + \cos(\boldsymbol{k}\cdot\boldsymbol{e}_2) + \cos(\boldsymbol{k}\cdot\boldsymbol{e}_3) \Big) \Big] c_\alpha(\boldsymbol{k}), \tag{2.4}$$

where the $\boldsymbol{e}_i$ are as shown in Fig. 1. The reciprocal lattice consists of the vectors $\boldsymbol{G} = \sum_i n_i \boldsymbol{G}_i$, where

$$\boldsymbol{G}_1 = \frac{4\pi}{3}(\boldsymbol{e}_1 - \boldsymbol{e}_2) \quad , \quad \boldsymbol{G}_2 = \frac{4\pi}{3}(\boldsymbol{e}_2 - \boldsymbol{e}_3) \quad , \quad \boldsymbol{G}_3 = \frac{4\pi}{3}(\boldsymbol{e}_3 - \boldsymbol{e}_1). \tag{2.5}$$

The electronic dispersion in Eq. (2.4) is plotted in Fig. 2: it only has simple parabolic minima at $\boldsymbol{k} = 0$, and its periodic images at $\boldsymbol{k} = \boldsymbol{G}$, and there are no Dirac points. At any chemical potential, the negative energy states are occupied, leading to a Fermi surface bounding the set of occupied states, as shown in Fig. 3. Luttinger theorem states that the total area of the occupied states, the shaded region of the first Brillouin zone in Fig. 3 occupies an area, $\mathcal{A}$, given by

$$\frac{\mathcal{A}}{2\pi^2} = \mathcal{N}, \tag{2.6}$$

where $\mathcal{N} = \sum_\alpha c^\dagger_\alpha c_\alpha$ is the total electron density. This relationship is obviously true for free electrons simply by counting occupied states, but it also remains true for interacting electrons.

Now we turn up the strength of the interactions, U, at a density of one electron per site. By the same argument as that for bosons, an insulator will appear for sufficiently large

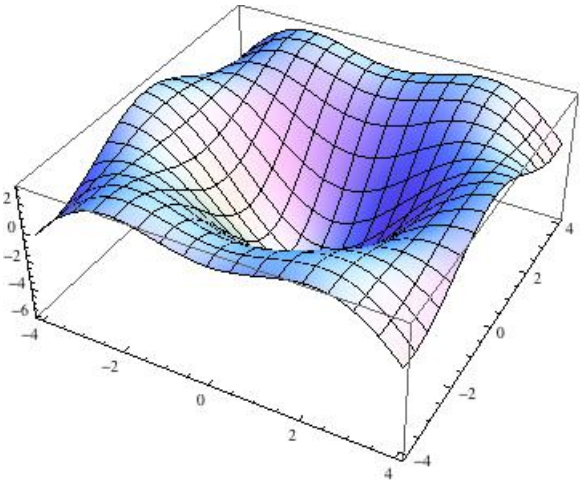

FIG. 2: The electronic dispersion in Eq. (2.4) for $\mu = 0$ and $t = 1$.

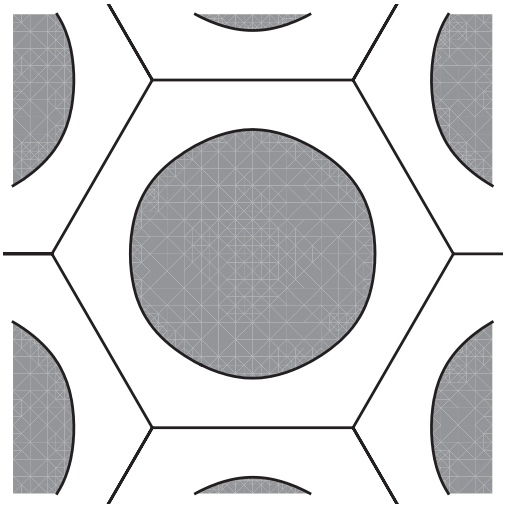

FIG. 3: The Fermi surface of Eq. (2.4) for $\mu = 1/2$ and $t = 1$; the occupied states are shaded. Also shown are the periodic images of the Fermi surface in their respective Brillouin zones.

U. As stated above, we will focus particular route to the destruction of the small U Fermi liquid, one which reaches directly to an insulator which is a 'spin liquid' [5–7]. The spin liquid insulator is a phase in which the spin rotation symmetry is preserved, and there is a gap to all charged excitations. However, there are gapless spin excitations, and an emergent compact U(1) gauge field in a deconfined phase.

The key to the description of this metal-insulator transition is an exact rewriting of the Hubbard model in Eq. (2.1) as a compact U(1) lattice gauge theory. To derive this, let us proceed with using the same strategy as that used in Section I for the boson Hubbard model. So to represent charged excitations of the insulator, we introduce bosonic operators p_i and h_i, representing the doubly-occupied and empty sites respectively. However, the singly-occupied site cannot be treated as a featureless vacuum, as we were able to in Section I.

Now we need a fermionic operator f_α (the 'spinon') to represent the spin orientation of the singly-occupied site. For every site, we make the following correpondences for the four states in the Fock space

$$\begin{aligned} |0\rangle &\quad \Leftrightarrow \quad h^\dagger|0\rangle \\ c_\alpha^\dagger|0\rangle &\quad \Leftrightarrow \quad f_\alpha^\dagger|0\rangle \\ c_\uparrow^\dagger c_\downarrow^\dagger|0\rangle &\quad \Leftrightarrow \quad p^\dagger f_\uparrow^\dagger f_\downarrow^\dagger|0\rangle \end{aligned} \tag{2.7}$$

It is now easy to verify that Eq. (2.7) is equivalent to the operator identification

$$c_\alpha = (h^\dagger + p) f_\alpha, \tag{2.8}$$

provided we always project to states which obey the constraint

$$f_\alpha^\dagger f_\alpha - p^\dagger p + h^\dagger h = 1 \tag{2.9}$$

on every site. All physical observables are operators which stay within the subspace defined by Eq. (2.9): such operators are invariant under the following compact U(1) gauge transformation

$$f_\alpha \to f_\alpha e^{i\zeta} \quad , \quad p \to p e^{-i\zeta} \quad , \quad h \to h e^{i\zeta}. \tag{2.10}$$

As we will show below, there will is an emergent gauge field B_μ in the effective theory associated with this gauge transformation. The constraint in Eq. (2.9) will be the Gauss law of this gauge theory. These operator identities are related to those of the 'slave rotor' representation [8].

First, let us rewrite the Hubbard model in terms of these new bosonic and fermionic operators. The Hubbard Hamiltonian in Eq. (2.1) is now exactly equivalent to

$$\begin{aligned} H[f,p,h] \;=\; & -w \sum_{\langle ij \rangle} f_{i\alpha}^\dagger f_{j\alpha} (h_i + p_i^\dagger)(h_j^\dagger + p_j) \;+\; \text{H.c.} \\ & + \sum_i \left(-\mu (p_i^\dagger p_i - h_i^\dagger h_i + 1) + \frac{U}{2} \left(p_i^\dagger p_i + h_i^\dagger h_i - \frac{1}{2} \right) \right), \end{aligned} \tag{2.11}$$

provided our attention is restricted to the set of states which obey the constraint in Eq. (2.9) on every lattice site; note that the Hamiltonian in Eq. (2.11) commutes with constraints in (2.9), and so these can be consistently imposed. In the on-site terms Eq. (2.11) we have used the p and h operators to measure the electron density on each site

We can now implement the commutation relations, the Hamiltonian, and the constraint

in a coherent state path integral

$$\mathcal{Z} = \int \mathcal{D}f_{i\alpha}(\tau)\mathcal{D}p_i(\tau)\mathcal{D}h_i(\tau)\mathcal{D}\lambda_i(\tau)\exp\Bigg(-\int d\tau\, H[f,p,h] - \int d\tau \sum_i \Bigg[f^\dagger_{i\alpha}\frac{\partial f_{i\alpha}}{\partial\tau} + p^\dagger_i\frac{\partial p_i}{\partial\tau} + h^\dagger_i\frac{\partial h_i}{\partial\tau} + i\lambda_i(f^\dagger_{i\alpha}f_{i\alpha} - p^\dagger_i p_i + h^\dagger_i h_i - 1)\Bigg]\Bigg), \quad (2.12)$$

where the constraint in Eq. (2.9) is implemented using an auxilliary field $\lambda_i(\tau)$ which acts as a Lagrange multiplier.

A key observation now is that the partition function in Eq. (2.12) is invariant under a site, i, and τ-dependent U(1) gauge transformation $\zeta_i(\tau)$ where the fields transform as in Eq. (2.10), and λ transforms as

$$\lambda_i \to \lambda_i - \frac{\partial\zeta_i}{\partial\tau}. \quad (2.13)$$

In other words, λ transforms like the temporal component of a U(1) gauge field.

How do we obtain the spatial components of the gauge field? For this, we apply a "Hubbard-Stratonovich transformation" to the hopping term in Eq. (2.11). For this, we introduce another auxiliary complex field $Q_{ij}(\tau)$ which lives on the links of the triangular lattice and replace the hopping term by

$$\sum_{\langle ij\rangle}\left(\frac{|Q_{ij}(\tau)|^2}{w} - Q_{ij}(\tau)f^\dagger_{i\alpha}f_{j\alpha} - Q^*_{ij}(\tau)(h_i + p^\dagger_i)(h^\dagger_j + p_j) + \text{H.c.}\right) \quad (2.14)$$

We now see from Eq. (2.10), that Q_{ij} transforms under the gauge transformation in Eq. (2.10) as

$$Q_{ij} \to Q_{ij}e^{i(\zeta_i - \zeta_j)}. \quad (2.15)$$

In other words, $\arg(Q_{ij})$ is the needed spatial component of the compact U(1) gauge field.

So far, we have apparently only succeeded in making our analysis of the Hubbard model in Eq. (2.1) more complicated. Instead of the functional integral of the single complex fermion $c_{i\alpha}$, we now have a functional integral over the complex fermions $f_{i\alpha}$, the bosons p_i, h_i, and the auxilliary fields λ_i and Q_{ij}. How can this be helpful? The point, of course, is that the new variables help us access new phases and critical points which were inaccessible using the electron operators, and these phases have strong correlations which are far removed from those of weakly interacting electrons.

The utility of the new representation is predicated on the assumption that the fluctuations in the auxiliary fields Q_{ij} and λ_i are small along certain directions in parameter space. So let us proceed with this assumption, and describe the structure of the phases so obtained. We parameterize

$$Q_{ij} = \overline{Q}_{ij}e^{B_{ij}} \quad , \quad \lambda_i = -i\overline{\lambda} - B_{i\tau} \quad (2.16)$$

and ignore fluctuations in the complex numbers $\overline{Q}_{ij}$, and the real number $\overline{\lambda}$. With these definitions, it is clear from Eqs. (2.13) and (2.15) that B_{ij} and B_τ form the spatial and temporal components of a U(1) gauge field, and so must enter into all physical quantities in a gauge invariant manner. The values of $\overline{Q}_{ij}$ and $\overline{\lambda}$ are determined by a suitable saddle-point analysis of the partition function, and ensure that the constraint (2.9) is obeyed. With these assumptions, the partition function separates into separate fermionic and bosonic degrees of freedom interacting via their coupling to a common U(1) gauge field $(B_{i\tau}, B_{ij})$. In the continuum limit, the gauge fields become a conventional U(1) gauge field $B_\mu = (B_\tau, \boldsymbol{B})$. The partition function of the gauge theory is

$$\begin{aligned}
\mathcal{Z} \;=\; & \int \mathcal{D}f_{i\alpha}(\tau)\mathcal{D}p_i(\tau)\mathcal{D}h_i(\tau)\mathcal{D}B_{i\tau}(\tau)\mathcal{D}B_{ij}(\tau) \\
& \qquad \exp\left(-\int d\tau\Big[\mathcal{L}_f + \mathcal{L}_b + i\sum_i B_{i\tau}\Big]\right) \\
\mathcal{L}_f = & \sum_i f^\dagger_{i\alpha}\left(\frac{\partial}{\partial\tau} + \overline{\lambda} - iB_{i\tau}\right) f_{i\alpha} - \sum_{\langle ij\rangle}\overline{Q}_{ij} f^\dagger_{i\alpha} e^{iB_{ij}} f_{j\alpha} + \text{H.c.} \\
\mathcal{L}_b = & \sum_i p^\dagger_i\left(\frac{\partial}{\partial\tau} - \overline{\lambda} - \mu + \frac{U}{2} + iB_{i\tau}\right) p_i + \sum_i h^\dagger_i\left(\frac{\partial}{\partial\tau} + \overline{\lambda} + \mu + \frac{U}{2} - iB_{i\tau}\right) h_i \\
& - \sum_{\langle ij\rangle}\overline{Q}^*_{ij}(h_i + p^\dagger_i)e^{-iB_{ij}}(h^\dagger_j + p_j) + \text{H.c.}.
\end{aligned} \tag{2.17}$$

Thus we have fermions $f_{i\alpha}$ moving in a band structure which is roughly the same as that of the electrons in Eq. (2.4), the bosons p_i, h_i Hubbard-like Hamiltonian essentially identical in form to that in Section I, and all particles are minimally coupled to a compact U(1) gauge field.

We begin by neglecting the gauge fields, and computing the separate phase diagrams of $\mathcal{L}_f$ and $\mathcal{L}_b$.

The fermions are free, and so occupy the negative energy states determined by the chemical potential $\overline{\lambda}$.

The phase diagram of $\mathcal{L}_b$ is more interesting: it involves strong interactions between the p and h bosons. It can be a analyzed in a manner similar to that of the boson Hubbard model (see Chapter 9 of Ref. 4), leading to the familiar "Mott lobe" structure shown in Fig. 4.

At large values of $\overline{Q}/U$ we have the analog of the superfluid states of the boson Hubbard model, in which there is a condensate of the same operator as that in Eq. (1.9):

$$\psi = p + h^\dagger. \tag{2.18}$$

Note that ψ is the ladder operator for the number operator $n = p^\dagger p - h^\dagger h$ used to characterize the insulating phases in Fig. (4). The ψ operator carries unit charge under the U(1) gauge field (from Eq. (2.10), and so the superfluid phase, with $\langle\psi\rangle \neq 0$, does not break any

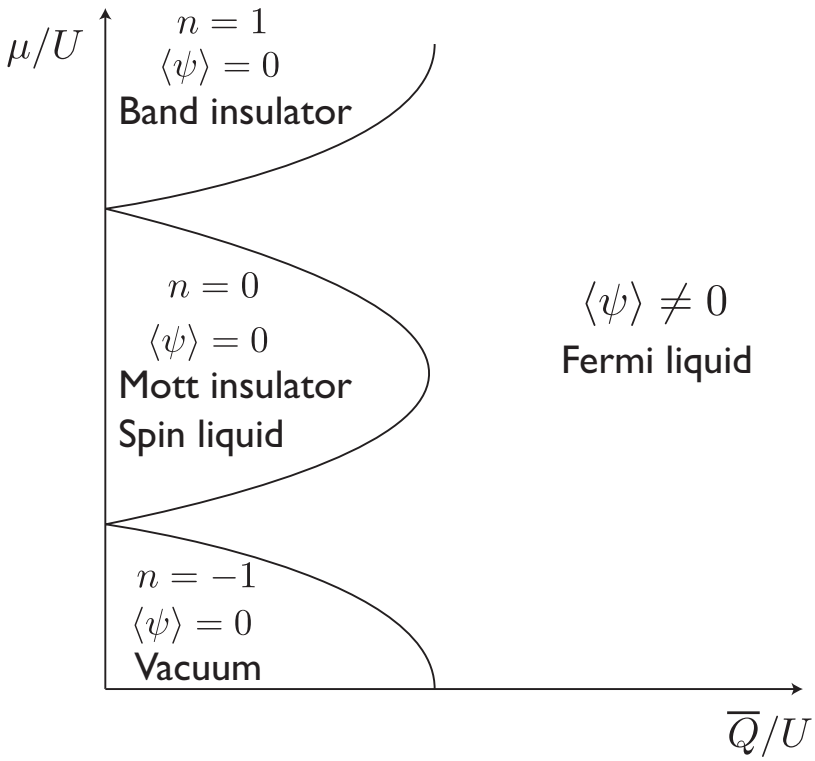

FIG. 4: Possible phase diagram of the electron Hubbard model in Eq. (2.1) on the triangular lattice. This phase diagram is obtained by a mean-field analysis of the theory $\mathcal{L}_b$ in Eq. (2.17), similar to that for the boson Hubbard model in Chapter 9 of Ref. 4. We use the number operator $n = p^\dagger p - h^\dagger h$, which commutes with the boson hopping term, to characterize the Mott insulating states. Only the Mott insulating lobes with $n = -1, 0, 1$ are compatible with the constraint in Eq. (2.9); these Mott insulating lobes have fermion density $\left\langle f_\alpha^\dagger f_\alpha \right\rangle = n + 1$.

global symmetries (unlike the boson model of Section I). Instead it is a "Higgs" phase. In the presence of the Higgs condensate, the operator relation in Eq. (2.8) implies that $c_\alpha \sim f_\alpha$, and so the f_α fermions carry the same quantum numbers as the physical electron. Consequently, the f_α Fermi surface is simply an electron Fermi surface. Furthermore, the Higgs condensate quenches the B_μ fluctuations, and so there are no singular interactions between the Fermi surface excitations. This identifies the present phase as the familiar Fermi liquid, as identified in Fig. 4.

Having reproduced a previously known phase of the Hubbard model in the U(1) gauge theory, let us now examine the new phases within the 'Mott lobes' of Fig. 4. In these states, the boson excitations are gapped, and number operator $n = p^\dagger p - h^\dagger h$ has integer expectation values. The constraint in Eq. (2.9) implies that only $n = -1, 0, 1$ are acceptable values, and so only these values are shown. It is clear from the representation in Eq. (2.7) that any excitation involving change in electron number must involve a p or h excitation, and so the gap to the latter excitations implies a gap in excitations carrying non-zero electron number. This identifies the present phases as insulators. Thus the phase boundary out of the lobes

in Fig. 4 is a metal-insulator transition.

The three insulators in Fig. 4 have very different physical characteristics.

Using the constraint in Eq. (2.9) we see that the $n = -1$ insulator has no f_α fermions. Consequently this is just the trivial empty state of the Hubbard model, with no electrons.

Similarly, we see that the $n = 1$ insulator has 2 f_α fermions on each site. This is the just the fully-filled state of the Hubbard model, with all electronic states occupied. It is a band insulator.

Finally, we turn to the most interesting insulator with $n = 0$. Now the electronic states are half-filled, with $\langle f_\alpha^\dagger f_\alpha \rangle = 1$. Thus there is an unpaired fermion on each site, and its spin is free to fluctuate. There is a non-trivial wavefunction in the spin sector, realizing an insulator which is a 'spin liquid'. In our present mean field theory, the spin wavefunction is specified by Fermi surface state of the f_α fermions. Going beyond mean-field theory, we have to consider the fluctuations of the B_μ gauge field, and determine if they destabilize the spin liquid. The f_α fermions carry the B_μ gauge charge, and these fermions form a Fermi surface. The gapless fermionic excitations at the Fermi surface prevent the proliferation of monopoles in the compact U(1) gauge field: the low energy fermions suppress the tunneling event associated with global change in B_μ flux[9, 10]. Thus the emergent U(1) gauge field remains in a deconfined phase, and this spin liquid state is stable. These gapless gauge excitations have strong interactions with the f_α fermions, and this leads to strong critical damping of the fermions at the Fermi surface which is described by a strongly-coupled field theory[11–13]. The effect of the gauge fluctuations is also often expressed in terms of an improved trial wavefunction for the spin liquid [5]: we take the free fermion state of the f_α fermions, and apply a projection operator which removes all components which violate the constraint in Eq. (2.9). This yields the 'Gutzwiller projected' state

$$|\text{spin liquid}\rangle = \left(\prod_i \left[\frac{1 - (-1)^{\sum_\alpha f_{i\alpha}^\dagger f_{i\alpha}}}{2} \right] \right) \left(\prod_{\boldsymbol{k} < k_F} f_\uparrow^\dagger(\boldsymbol{k}) f_\downarrow^\dagger(\boldsymbol{k}) \right) |0\rangle, \tag{2.19}$$

where the product over $\boldsymbol{k}$ is over all points inside the Fermi surface.

Finally, we turn to an interesting quantum phase transition in Fig. 4. This is the transition between the spin liquid and the Fermi liquid at total electron density $\mathcal{N} = 1$, which occurs at the tip of the $n = 0$ Mott lobe. From the boson sector, this looks like a Higgs transition, of the condensation of a complex scalar ψ as in Section I, but in the presence of a fluctuating U(1) gauge field. However, the fermionic sector is crucial in determining the nature of this transition. Indeed, in the absence of the Fermi surface, this transition would not even exist beyond mean field theory: this is because the U(1) gauge field is compact, and the scalar carries unit charge, and so the confining and Higgs phases of this gauge theory are smoothly connected. So we have to combine the Higgs theory of a complex scalar with the gapless Fermi surface excitations. We can obtain the field theory for this metal-insulator transition by applying the methods of Section I to Eq. (2.17). The analog of the condition $\Delta_p = \Delta_h$

needed to obtain a density of one electron per site is now $\overline{\lambda} + \mu = 0$. In this manner, we find the field theory [7, 15]

$$\begin{aligned} \mathcal{L} &= |(\partial_\mu + iB_\mu)\psi|^2 + s|\psi|^2 + u|\psi|^4 + iB_\tau \mathcal{N} \\ &+ f_\alpha^\dagger \left[\frac{\partial}{\partial \tau} - \varepsilon_F - iB_\tau - \frac{1}{2m}(\boldsymbol{\nabla} - i\boldsymbol{A})^2 \right] f_\alpha, \end{aligned} \tag{2.20}$$

where the energy ε_F is to be adjusted to yield total fermion density $\mathcal{N} = 1$. The transition is accessed by tuning s, and we move from a spin liquid for $s < s_c$, to a Fermi liquid for $s > s_c$. The critical properties of the theory in Eq. (2.20) have been studied [7, 14], and an interesting result is obtained: the Fermi surface excitations damp the gauge bosons so that they become ineffective in coupling to the critical b fluctuations. Consequently, the gauge bosons can be ignored in the ψ fluctuations, and the transition is in the universality class of the 2+1 dimensional XY model. In other words, quite unexpectedly, the critical theory is the same as that of the superfluid-insulator transition of Section I. There are additional gapless excitations associated with the gauge field and the Fermi surface, but these are irrelevant for the values of certain critical exponents.

Acknowledgements

This research was supported by the National Science Foundation under grant DMR-0757145 and by a MURI grant from AFOSR.

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