Holographic duality basics
Lecture 3

John McGreevy, MIT

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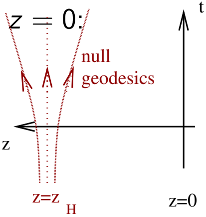
Finite temperature

AdS was scale invariant. sol'n dual to *vacuum* of CFT.
saddle point for CFT in an ensemble with a scale (some relevant perturbation) is a geometry which approaches AdS near the bdy:

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)}\right) \qquad f = 1 - \frac{z^d}{z_H^d}$$

When the emblackening factor $f \overset{z\to 0}{\to} 1$ this is the Poincaré AdS metric.
[exercise: check that this solves the same EOM as AdS.]
It has a horizon at $z = z_H$, where the emblackening factor $f \propto z - z_H$
Events at $z > z_H$ can't influence the boundary near $z = 0$:



Physics of horizons

Claim: geometries with horizons describe thermally mixed states.

Why: Near the horizon ($z \sim z_H$),

$$ds^2 \sim -\kappa^2 \rho^2 dt^2 + d\rho^2 + \frac{L^2}{z_H^2} d\vec{x}^2 \quad \rho^2 \equiv \frac{2}{\kappa z_H^2}(z - z_H) + o(z - z_H)^2$$

$\kappa \equiv \frac{4}{|f'(z_H)|} = d/2z_H$ is called the 'surface gravity'

Continue this geometry to euclidean time, $t \to i\tau$:

$$ds^2 \sim \kappa^2 \rho^2 d\tau^2 + d\rho^2 + \frac{L^2}{z_H^2} d\vec{x}^2$$

which looks like $\mathbb{R}^{d-1} \times \mathbb{R}^2_{\rho,\kappa\tau}$ with polar coordinates $\rho, \kappa\tau$.

There is a deficit angle in this plane unless we identify

$$\kappa\tau \simeq \kappa\tau + 2\pi.$$

A deficit angle would mean nonzero Ricci scalar curvature, which would mean that the geometry is *not* a saddle point of our bulk path integral.

So: $T = \kappa/(2\pi) = 1/(\pi z_H)$.

(Note: this is the temperature of the Hawking radiation.)

Static BH describes thermal equilibrium

This identification on τ also applies at the boundary. If

$$ds^2_{bulk} \overset{z\to 0}{\approx} \frac{dz^2}{z^2} + \frac{L^2}{z^2} g^{(0)}_{\mu\nu} dx^\mu dx^\nu$$

then, up to a factor, the boundary metric is $g^{(0)}_{\mu\nu}$.
This includes making the euclidean time periodic.

$$A = \int_{z=z_H,\text{fixed } t} \sqrt{g}\, d^{d-1}x = \left(\frac{L}{z_H}\right)^{d-1} V$$

The Bekenstein-Hawking entropy is

$$S = \frac{A}{4G_N} = \frac{L^{d-1}}{4G_N}\frac{V}{z_H^{d-1}} = \frac{N^2}{2\pi}(\pi T)^{d-1} V = \frac{\pi^2}{2} N^2 V T^{d-1} \quad .$$

The Bekenstein-Hawking entropy *density* is

$$s_{BH} = \frac{S_{BH}}{V} = \frac{a_{BH}}{4G_N}.$$

where $a_{BH} \equiv \frac{A}{V}$ is the 'area density' of the black hole.

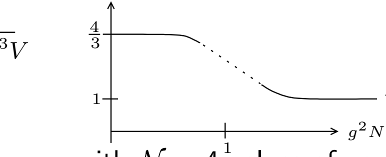
QFT thermodynamics from black holes cont'd

how to think about this:

$$Z_{CFT}(T) \approx e^{-S^{\text{eucl}}_{\text{bulk}}[\underline{g}]}$$

$\underline{g}$ is the saddle with the correct periodicity of eucl time at the bdy.

(warning: boundary terms in action are important – see below)



$$Z_{CFT}(T) = e^{-\beta F}$$

$$-\frac{F}{V} = \frac{L^2}{16\pi G_N}\frac{1}{z_H^4} = \frac{\pi^2}{8}N^2T^4.$$

with $\mathcal{N} = 4$ values of parameters, $F(\lambda = \infty) = \frac{3}{4}F(\lambda = 0)$.

checks:

- $$\underset{\text{horizon}}{S_{BH}} = \underset{\text{integral over all spacetime}}{-\frac{\partial F}{\partial T}}$$

(relatedly: first law of thermo holds)

- $c_V > 0$ for AdS BH. (unlike schwarzchild in asymptotically flat space!)
- uniqueness of stationary BH ('no hair') ⇝ few state variables in eq thermo

Thermodynamics from gravity: boundary terms

$$Z_{CFT} \equiv e^{-\beta F} = e^{-S_{\text{bulk}}[\underline{g}]}$$

$\underline{g}$ is the euclidean saddle-point metric(s).

$$S_{\text{bulk}} = S_{EH} + S_{GH} + S_{ct} \ .$$

$$S_{EH} = -\frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{g} \left(R + \frac{d(d-1)}{L^2} \right)$$

Two kinds of boundary terms:

def of γ: $$ds^2 \overset{z \to 0}{\approx} L^2 \frac{dz^2}{z^2} + \gamma_{\mu\nu} dx^\mu dx^\nu .$$

$$S_{ct} = \int_{\partial M} d^d x \sqrt{\gamma} \frac{2(d-1)}{L} + \ldots$$

local, *intrinsic* boundary counter-term (no normal derivatives).
just like for scalar correlators. $\cdots \propto$ intrinsic curvature of bdry metric.

Gibbons-Hawking term

S_{GH}: 'Gibbons-Hawking' term is an *extrinsic* boundary term

like $\int_{\partial AdS} \phi n \cdot \partial \phi$ for scalar.

IBP in the Einstein-Hilbert term to get the EOM :

$$\delta S_{EH} = EOM + \int_{\partial AdS} \gamma^{\mu\nu} n \cdot \partial \delta \gamma_{\mu\nu},$$

but we want a Dirichlet condition on the metric: $\delta \gamma_{\mu\nu} = 0$

δS_{GH} cancels the $\partial \delta \gamma_{\mu\nu}$ bits.

$$S_{GH} = -2 \int_{\partial M} d^d x \sqrt{\gamma} \Theta$$

Θ: extrinsic curvature of the boundary

$$\Theta \equiv \gamma^{\mu\nu} \nabla_\mu n_\nu = \frac{n^z}{2} \gamma^{\mu\nu} \partial_z \gamma_{\mu\nu}.$$

n^A is an outward-pointing unit normal to the boundary $z = \epsilon$.

Stress tensor expectation value

$$\text{GKPW}: \qquad \langle T^{\mu\nu} \rangle = \frac{2}{\sqrt{\gamma}} \frac{\delta}{\delta \gamma_{\mu\nu}} S_{\text{bulk}}[\underline{g}].$$

CFT: $T_\mu^\mu = 0$ modulo scale anomaly

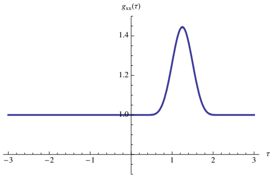
In thermal eqbm: $T_t^t = -\mathcal{E}, \quad T_x^x = P \qquad\qquad \mathcal{E} = d\ P$

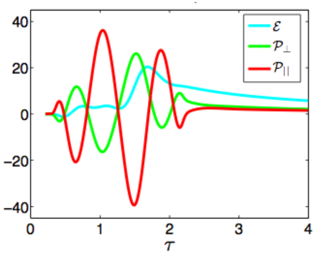
Approach to equilibrium

bulk picture: dynamics of gravitational collapse.

dissipation: energy falls into BH [Horowitz-Hubeny, 99]

- small-amplitude perturbations: quasinormal modes of BH
- far-from equilibrium processes: [Chesler-Yaffe, 08, 09] (PDEs!)

input: 

output: 

black hole forms from vacuum initial conditions.

brutally brief summary: all relaxation timescales $\tau_{th} \sim T^{-1}$.

- Lesson: In these models, breakdown of hydro in this model is not set by higher-derivative terms, but from non-hydrodynamic modes.

Example: η/s

Shear viscosity is a transport coefficient like conductivity.

source: T^x_y response: T^x_y.

$$\eta = \lim_{\omega\to 0} \frac{1}{i\omega} G^R_{T^x_y T^x_y}(k=0,\omega)$$

$$\langle T^x_y \rangle = i\omega\eta\gamma^x_y \qquad \to \qquad \text{must study fluctuations of metric}$$

[compute following Iqbal-Liu 08] Assume a bulk metric of the form

$$ds^2 = g_{tt}(z)dt^2 + g_{zz}(z)dz^2 + g_{ij}(z)dx^i dx^j$$

such that

1. g_{AB} depend only on z
2. asymptotically AdS near $z \to 0$
3. Rindler horizon at $z = z_H$

$$g_{tt} \overset{z\to z_H}{\to} -2\kappa(z_H - z) \qquad g_{zz} \overset{z\to z_H}{\to} \frac{1}{2\kappa(z_H - z)}.$$

Shear fluctuations of the metric

$$\text{Consider} \quad S = S_{\text{gravity}} - \frac{1}{2}\int d^{d+1}x\sqrt{g}\frac{1}{q(z)}g^{AB}\partial_A\phi\partial_B\phi$$

Claim: fluctuations of $\phi \equiv h^x_y$ in Einstein gravity are governed by this action with $\frac{1}{q(z)} = \frac{1}{16\pi G_N}$. [lots of work by Son, Starinets, Policastro, Kovtun, Buchel, J. Liu...]

$$\text{Recall:} \quad \langle \mathcal{O}(x^\mu)\rangle_{QFT} = \lim_{z\to 0}\Pi_\phi(z, x^\mu) \qquad \text{(m=0)}$$

$$\Longrightarrow \qquad \eta = \lim_{\omega\to 0}\lim_{z\to 0}\lim_{k\to 0}\left(\frac{\Pi(z,k_\mu)}{i\omega\phi(z,k_\mu)}\right)$$

$$\Pi \equiv \frac{\partial\mathcal{L}}{\partial\left(\partial_z\phi\right)} = \frac{\sqrt{g}}{q(z)}g^{zz}\partial_z\phi.$$

Compute this in two steps:

- Find behavior near horizon.
- Use wave equation to evolve to boundary.

$$0 = \frac{\delta S_\phi}{\delta\phi(k^\mu, z)} \propto [g^{ij}k_ik_j + g^{tt}\omega^2 - \frac{1}{\sqrt{g}}\partial_z\left(g^{zz}\sqrt{g}\partial_z\right)]\phi(k^\mu, z)$$

We can safely set $\vec{k} = 0$.

Near horizon

Assumption (3) $\implies$ $z = z_H$ is a regular singular point of the wave equation.
Try $\phi(k,z) = (z - z_H)^\alpha$.

$$\phi(k,z) \simeq (z - z_H)^{\pm \frac{i\omega}{4\pi T}} \qquad \text{in/out.}$$

$$\implies \text{At horizon:} \quad \Pi(z_H, k) = \left[\frac{1}{q(z)} \sqrt{\frac{|g|}{g_{zz}|g_{tt}|}} i\omega\phi(z,k) \right]_{z=z_H}.$$

Propagate to boundary

EOM: $\partial_z \Pi \propto k_\mu k_\nu g^{\mu\nu} \phi \overset{\omega \to 0, \vec{k} \to 0}{\rightarrow} 0.$

def of Π: $\partial_z (\phi \omega) = \frac{q}{\sqrt{g} g^{zz}} \omega \Pi \overset{\omega \to 0, \omega\phi \text{ fixed}}{\rightarrow} 0.$

$$\implies \quad \frac{\Pi}{\omega \phi}|_{z=0} = \frac{\Pi}{\omega \phi}|_{z=z_H}$$ 'membrane paradigm'

$$\implies \quad \eta = \frac{1}{q(z_H)} \sqrt{\frac{|g|}{g_{zz}|g_{tt}|}} \quad .$$

Entropy density: $s = \frac{a}{4G_N} = \frac{1}{4G_N} \sqrt{\frac{|g|}{g_{zz}|g_{tt}|}}$

$$\implies \quad \boxed{\frac{\eta}{s} = \frac{1}{4\pi}}.$$

Fluid/gravity duality

Here we've computed the value of a hydro transport coeff of the CFT plasma.

More generally: perturb BH horizon by local boost $u^\mu(x)$, slowly varying.

[Janik-Peschanski,Bhattacharyya et al...]: In an expansion in derivatives of $T(x), u^\mu(x)$,

sol'ns of Einsten eqns of this form	$\leftrightarrow$	soln's of Navier-Stokes eqns with particular transport coeffs

personal disappointment: holographic duality doesn't average over turbulent flows.

Finite Density States

To describe low-temperature states of matter, we need more ingredients.

Suppose the CFT has a conserved $U(1)$ current.

$\to$ massless gauge field A_μ in bulk.

Wilson-natural starting point: $\Delta S_{bulk} = -\frac{1}{4g_F^2} \int d^{d+1}x \sqrt{g} F_{AB} F^{AB}$.

$$\text{Max eqn}: \quad 0 = \frac{\delta S_{bulk}}{\delta A_C} \propto \frac{1}{\sqrt{g}} \partial_A \left(\sqrt{g} g^{AB} g^{CD} F_{BD} \right)$$

Max eqn near AdS bdy: $\underline{A} \sim A^{(0)}(x) + \left(\frac{z}{L}\right)^{d-2} A^{(1)}(x)$

$$\text{in particular,} \quad A_t \sim \mu + \left(\frac{z}{L}\right)^{d-2} \rho.$$

$$\Pi_{A_t} = \frac{\partial \mathcal{L}}{\partial \left(\partial_z A_t\right)} = E_z = A^{(1)} = \rho.$$

Charged black holes in AdS

saddle point w this BC (and no other matter): AdS Reissner-Nördstrom.

$$ds^2 = \frac{L^2}{z^2}\left(-f\,dt^2 + d\vec{x}^2 + \frac{dz^2}{f}\right), \qquad A_t = \mu - \left(\frac{z}{z_0}\right)^{d-2}\mu$$

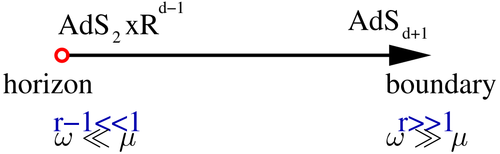
$$f(z) = 1 - Mz^d + Qz^{2d-2}$$ note: multiple zeros

At $T \ll \mu$ the near-horizon geometry of black hole is $AdS_2 \times \mathbb{R}^{d-1}$.

$$ds^2 \overset{z_0 \to z_1}{\sim} -a(z-z_0)^2 dt^2 + b\frac{dz^2}{(z-z_0)^2} + \frac{d\vec{x}^2}{z_0^2} = \overbrace{-\zeta^2 dt^2 + \frac{d\zeta^2}{\zeta^2}} + \frac{d\vec{x}^2}{z_0^2}$$

The conformal invariance of this metric is **emergent**.

The bulk geometry is a picture of the RG flow from the CFT_d to this NRCFT.



AdS/CFT: low-ω physics determined by dual **IR CFT**.

[Much more on this in Tom Faulkner's lectures]

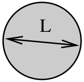
Other observables, other models

So far: thermodynamics, correlators of local ops.
Other observables have natural holographic realizations:
gauge-theory-specific: Wilson loops, external quarks
very universal: entanglement entropy

So far: CFTs and their relevant deformations (*e.g.* by T, μ).
We can realize holographically different UV behavior:
Galilean CFTs, Lifshitz theories.

Comment on entanglement entropy

If $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\overline{A}}$ (*e.g.* in local theory, A is a region of space)
If ignorant of $\overline{A} \to \rho_A = \mathrm{tr}_{\mathcal{H}_{\overline{A}}} \rho$ *e.g.* $\rho = |\Omega\rangle\langle\Omega|$.
$S_A \equiv -\mathrm{tr}_A \rho_A \ln \rho_A$. (notoriously hard to compute)



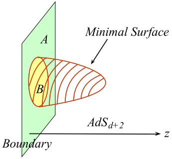
- 'order parameter' for topologically ordered states

in 2+1d, $S(L) = \gamma \frac{L}{a} + S_{\text{top}}$ [Levin-Wen, Preskill-Kitaev 05]

- scaling with region-size characterizes simulability:[Verstraete,Cirac,Eisert...]

boundary law $\leftrightarrow$ matrix product state ansatz (DMRG) will work.

[Ryu-Takayanagi]
$$S_A = \text{extremum}_{\partial M = \partial A} \frac{\text{area}(M)}{4G_N}$$



outcome from holography:
which bits are universal in CFT? in d space dims,

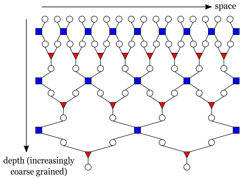
$$S_A = p_1 \left(\frac{L}{a}\right)^{d-1} + p_3 \left(\frac{L}{a}\right)^{d-3} \cdots + \begin{cases} p_{d-1} \frac{L}{a} + \tilde{c} \ , & d: \text{ even} \\ p_{d-2} \left(\frac{L}{a}\right)^2 + \tilde{c} \log(L/a) \ , & d: \text{ odd} \end{cases}$$

In fact, the area law coeff is also a universal measure of # of dofs, can be extracted from mutual information $S_A + S_B - S_{A \cup B}$ for colliding regions. [Swingle]

Other observables, other models

So far: thermodynamics, correlators of local ops.
Other observables have natural holographic realizations:
gauge-theory-specific: Wilson loops, external quarks
very universal: entanglement entropy

• entanglement RG [G. Vidal]:
a real space RG which keeps track of entanglement
builds an extra dimension
$ds^2 \stackrel{?}{=} dS^2$ [Swingle 0905.1317, Raamsdonk 0907.2939]



So far: CFTs and their relevant deformations (*e.g.* by T, μ).
We can realize holographically different UV behavior: Galilean CFTs, Lifshitz theories.

Strongly-coupled NRCFT

The fixed-point theory ("fermions at unitarity") is a strongly-coupled nonrelativistic CFT ('Schrödinger symmetry')

[Mehen-Stewart-Wise, Nishida-Son].

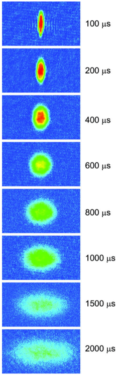
Universality: it also describes neutron-neutron scattering.

Two-body physics is completely solved.

Many body physics is mysterious.

Experiments: very low viscosity, $\frac{\eta}{s} \sim \frac{\text{few}}{4\pi}$ [Thomas, Schafer...]

$\longrightarrow$ strongly coupled.



AdS/CFT?

Clearly we can't approximate it as a *relativistic* CFT.

Different hydro: conserved particle number.

A holographic description?

Method of the missing box

AdS : relativistic CFT

"*Schrodinger spacetime*" : galilean-invariant CFT

A metric whose isometry group is the Schrödinger group:

[Son; K Balasubramanian, JM 0804]

$$L^{-2}ds^2 = \frac{2d\xi dt + d\vec{x}^2 + dr^2}{r^2} - 2\beta^2 \frac{dt^2}{r^4}$$

This metric solves reasonable equations of motion.
Holographic prescription generalizes naturally.

But: the vacuum of a galilean-invariant field theory is extremely boring:
no antiparticles! no stuff!
How to add stuff?

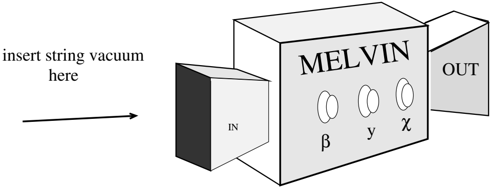
A holographic description of more than zero atoms?

A black hole (BH) in Schrödinger spacetime.

[A. Adams, K. Balasubramanian, JM; Maldacena, Martelli, Tachikawa; Herzog, Rangamani, Ross 0807]

Here, string theory was extremely useful:

A solution-generating machine named Melvin [Ganor et al]



IN: $AdS_5 \times S^5$ OUT: Schrödinger $\times S^5$

IN: AdS_5 BH $\times S^5$ OUT: Schrödinger BH $\times$ squashed S^5

[since then, many other stringy realizations: Hartnoll-Yoshida, Gauntlett, Colgain, Varela, Bobev, Mazzucato...]

Results so far

This black hole gives the thermo and hydro of some NRCFT
('dipole theory' [Ganor et al 05]).

$$\text{Einstein gravity} \implies \frac{\eta}{s} = \frac{1}{4\pi} \quad .$$

Satisfies laws of thermodynamics, correct scaling laws, correct kubo relations.

[Rangamani-Ross-Son 09, McEntee-JM-Nickel, unpublished]

But it's a different class of NRCFT from unitary fermions:

$$F \sim -\frac{T^4}{\mu^2}, \quad \mu < 0$$

This is because of an

Unnecessary assumption: all of Schröd must be realized geometrically.

We now know how to remove this assumption, can find more realistic models.

Concluding comments

Remarks on the role of supersymmetry (susy)

- susy constrains the form of interactions.
  fewer candidates for dual.
- in susy theories, $\exists$ more coupling-independent quantities, hence $\exists$ more checks.
- susy allows *lines* of fixed points (*e.g.* $\mathcal{N} = 4$ SYM)
  coupling = dimensionless parameter
- for these applications, susy is broken by finite T, μ, anyway. it's not clear what influence it has on the resulting states.
  one implication: a phonino pole

  [Lebedev-Smilga, Kovtun-Yaffe, seen holographically by Gauntlett-Sonner-Waldram]

Remarks on the role of string theory

1. What are consistent ways to UV complete our gravity model?
   - So far, no known constraints that aren't visible from EFT. And if we can't find the physics we want in *any* gravity model ...
   - Suggests interesting resummations of higher-derivative terms, protected by stringy symmetries.
     e.g. the DBI action $L_{DBI} \sim \sqrt{1 - F^2}$ is 'natural' in string theory because its form is protected by the T-dual Lorentz invariance.
2. What is a microscopic description of the dual QFT?
   - Such a description is crucial for the detailed checks that make us believe the duality.
   - A weak coupling limit needn't exist (isolated fixed points are generic).
   - A Lagrangian description needn't exist
     (*e.g.* minimal models) gravity plus matter in *AdS* provides a much more direct construction of CFT.
   - Honesty: Any L_{micro} that we would get from string theory is so far from $L_{Hubbard}$ anyway that it isn't clear how it helps.

Public service announcement

Please practice holography responsibly.

Please Practice Holography Responsibly

Holography gives us tractable toy models of strongly correlated systems.

Toy models are only useful if we ask the right questions.

- critical exponents depend on ‘landscape issues’
  (parameters in bulk action)
- thermodynamics doesn’t distinguish weak and strong coupling
  (in examples: $\mathcal{N} = 4$ SYM, lattice QCD)
- transport is very different
  transport by weakly-interacting quasiparticles is less effective

$$\left(\frac{\eta}{s}\right)_{\rm weak} \sim \frac{1}{g^4 \ln g} \quad \gg \quad \left(\frac{\eta}{s}\right)_{\rm strong} \sim \frac{1}{4\pi}.$$

- far from equilibrium physics: ?
- source of optimism: Weisskopf story.

The end

Thanks for listening.