Holographic duality basics
Lecture 2

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Recap

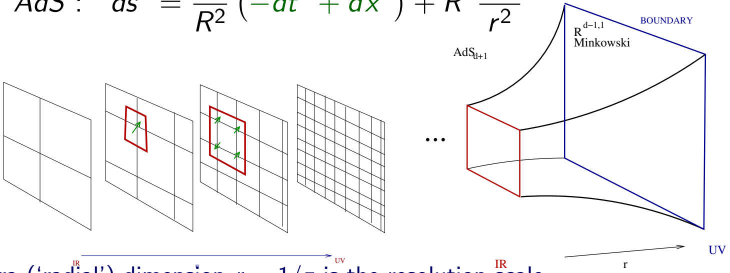
gravity in spacetimes$_{d+1}$
with timelike asymptotic boundaries $\quad \leftrightsquigarrow \quad QFT_d$

important special case:

gravity in AdS_{d+1} = d-dimensional conformal field theory (CFT)

isometries of AdS_{d+1} $\quad \leftrightsquigarrow \quad$ conformal symmetry

$$AdS: \quad ds^2 = \frac{r^2}{R^2}\left(-dt^2 + d\vec{x}^2\right) + R^2\frac{dr^2}{r^2}$$



The extra ('radial') dimension $r = 1/z$ is the resolution scale.

fields in bulk $\quad \leftrightsquigarrow \quad$ (possibly-) running couplings

$$Z_{QFT}[\text{sources}, \phi_0] \approx e^{-N^2 I_{\text{bulk}}[\text{boundary conditions at } r\to\infty]}\Big|_{\text{saddle of } I_{\text{bulk}}}$$

A word about large N^2

most prominent example: 't Hooft limit of $N \times N$ matrix fields X.
physical operators are $\mathcal{O}_k = \mathrm{tr}\, X^k$

this accomplishes several related things:

- $$\langle \mathcal{O}\mathcal{O} \rangle \sim \langle \mathcal{O} \rangle \langle \mathcal{O} \rangle + o\left(N^{-2}\right)$$

is the statement that *something* (the excitations created by $\mathcal{O}$) behaves classically.

- provides notion of single-particle states in bulk.
- makes saddle well-peaked $Z \sim e^{-N^2 I}$

important comment:

this is just the best-understood class of examples.
in other examples, the # of dofs goes like N^b, $b \neq 2$.
I'll always write N^2 as a proxy for this large number.

An example of a theory with a known gravity dual

$\mathcal{N} = 4$ SYM is a family of superconformal FTs.
The $\mathcal{N} = 4$ SYM action is schematically

$$\mathcal{L}_{\text{SYM}} \sim \mathrm{tr}\left(F^2 + (D\Phi)^2 + i\bar{\Psi}\Gamma \cdot D\Psi + g^2[\Phi,\Phi]^2 + ig\bar{\Psi}[\Phi,\Psi]\right)$$

this gauge theory comes with 2 parameters:
a coupling constant $\lambda = g^2 N$ (with $\beta_\lambda \equiv 0$)
an integer, the number of colors N.

$$\boxed{\mathcal{N} = 4 \ \mathrm{SYM}_{N,\lambda}} = \boxed{\text{IIB strings in } AdS_5 \times S^5 \text{ of size } \lambda, \hbar = 1/N}$$

[Maldacena 1997]

- large N makes gravity classical (improves saddle point, suppresses splitting and joining of strings)
- strong coupling (large λ) makes the geometry big. (improves bulk deriv. expansion)

'IIB strings in ...' specifies a list of bulk fields and interactions.
$\exists$ *infinitely many* other examples of dual pairs [*e.g.* Hanany, Vegh et al...]

More dictionary

really a ϕ_a for every $\mathcal{O}^a$ in CFT. how to match?

1. organize into reps of conformal group
2. single-trace operators correspond to 'elementary fields' in the bulk.

states from multitrace ops $(\mathrm{tr}\ X^k)^2|0\rangle$ — 2-particle states of ϕ.

3. simple egs fixed by symmetry:

- gauge fields in bulk A_μ – global currents J^μ in bdy

$S_{QFT} \ni \int A_\mu J^\mu$ (massless $A \leftrightsquigarrow$ conserved J)

- def of QFT stress tensor: response to change in metric on

boundary $S_{QFT} \ni \int \delta g_{\mu\nu} T^{\mu\nu}$

energy momentum tensor: $T^{\mu\nu}$		graviton: g_{ab}
global current: J^μ	$\leftrightsquigarrow$	Maxwell field: A_a
scalar operator: $\mathcal{O}_B$		scalar field: ϕ
fermionic operator: $\mathcal{O}_F$		fermionic field: ψ .

boundary conditions on bulk fields $\leftrightsquigarrow$ couplings in field theory

e.g.: boundary value of bulk metric $\lim_{r\to\infty} g_{\mu\nu}$

$=$ source for stress-energy tensor $T^{\mu\nu}$

different couplings in bulk action $\leftrightsquigarrow$ different field theories

Next: a few technical slides from which we can confirm our interpretation

$$u = \text{RG scale}$$

and see the machinery at work.

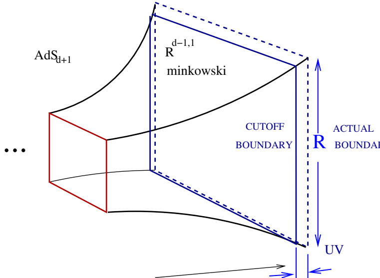
How to calculate

$$Z_{QFT}[\text{sources}] \approx e^{-N^2 I_{\rm bulk}[\text{boundary conditions at } z\to 0]}\Big|_{\text{extremum of } I_{\rm bulk}}$$

more explicitly:

$$\begin{aligned} Z_{QFT}[\text{sources}, \phi_0] &\equiv \langle e^{-\int \phi_0 \mathcal{O}} \rangle_{CFT} \\ &\approx e^{-N^2 I_{\rm bulk}[\phi | \phi(z=\epsilon) \overset{?}{=} \phi_0]}\Big|_{\phi \text{ solves EOM of } I_{\rm bulk}} \end{aligned}$$

As when counting dofs, we anticipate UV divergences
at the boundary $z \to 0$,
cut off the bulk at $z = \epsilon$
and set bc's there.



Example: scalar probe

Simple example: scalar field in the bulk. Natural (covariant) action:

$$\Delta S[\phi] = -\frac{\mathfrak{K}}{2}\int d^{d+1}x\sqrt{g}\left[g^{AB}\partial_A\phi\partial_B\phi + m^2\phi^2 + b\phi^3 + \dots\right]$$

$\mathfrak{K}$, a normalization constant: assume the theory of ϕ is weakly coupled, $\mathfrak{K} \propto N^2$.

$\left(\sqrt{g} = \sqrt{|\det g|} = \left(\frac{L}{z}\right)^{d+1}, \quad g^{AB} = \delta^{AB}z^2\right)$

We will study fluctuations around the solution $\phi = 0$, AdS.

(Recall: $\langle\mathcal{O}\mathcal{O}\rangle = \left(\frac{\delta}{\delta\phi_0}\right)^2 \ln Z|_{\underbrace{\phi_0 = 0}}$)

$\longrightarrow$ ignore interactions of ϕ for now.

Integrate by parts:

$$S = -\frac{\mathfrak{K}}{2}\int_{\partial AdS} d^d x\,\sqrt{g}\,g^{zB}\phi\,\partial_B\phi - \frac{\mathfrak{K}}{2}\int\sqrt{g}\,\phi\left(-\Box + m^2\right)\phi + o(\phi^3)$$

From this expression we learn:

- the EOM for small fluctuations of ϕ is $(-\Box + m^2)\underline{\phi} = 0$
  (An underline will indicate fields which solve the equations of motion.)
- If $\underline{\phi}$ solves the equation of motion, the on-shell action
  $S[\underline{\phi}], \quad Z \equiv e^{-S[\underline{\phi}]}$
  is just given by the boundary term.

next: relate bulk masses and operator dimensions

$$\Delta(\Delta - d) = m^2 L^2$$

by studying the AdS wave equation near the boundary.

Wave equation in AdS

translational invariance in d dimensions, $x^\mu \to x^\mu + a^\mu$,

$$\text{Fourier}: \; \phi(z, x^\mu) = e^{ik_\mu x^\mu} f_k(z), \qquad k_\mu x^\mu \equiv -\omega t + \vec{k}\cdot\vec{x}$$

$$\begin{aligned} 0 &= (g^{\mu\nu}k_\mu k_\nu - \frac{1}{\sqrt{g}}\partial_z(\sqrt{g}g^{zz}\partial_z) + m^2)f_k(z) \\ &= \frac{1}{L^2}[z^2k^2 - z^{d+1}\partial_z(z^{-d+1}\partial_z) + m^2L^2]f_k(z), \end{aligned} \qquad (1)$$

we used $g^{AB} = (z/L)^2\delta^{AB}$, $\quad \sqrt{g} = \sqrt{|\det g|} = \left(\frac{L}{z}\right)^{d+1}$.

Near boundary ($z \to 0$), power law solns, (spoiled by the z^2k^2 term).

Try $f_k = z^\Delta$ in (1):

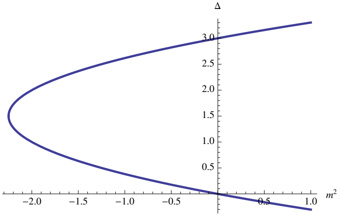
$$\begin{aligned} 0 &= k^2z^{2+\Delta} - z^{d+1}\partial_z(\Delta z^{-d+\Delta}) + m^2L^2z^\Delta \\ &= (k^2z^2 - \Delta(\Delta - d) + m^2L^2)z^\Delta, \end{aligned}$$

and for $z \to 0$ we get:

$$\Delta(\Delta - d) = m^2L^2 \qquad (2)$$

The two roots of (2) are $\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2L^2}$.

Comments



$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2 L^2}.$$

- The solution proportional to z^{Δ_-} is bigger near $z \to 0$. $\to$ usually the source ('non-normalizable')
- $\Delta_+ > 0 \;\; \forall \;\; m$: z^{Δ_+} always decays near the boundary
- $\Delta_+ + \Delta_- = d$.

We want to impose boundary conditions that allow solutions. Leading $z \to 0$ behavior of generic solution: $\phi \sim z^{\Delta_-}$, we impose

$$\phi(x, z)|_{z=\epsilon} \overset{!}{=} \phi_0(x, \epsilon) = \epsilon^{\Delta_-} \phi_0^{Ren}(x),$$

where ϕ_0^{Ren} is a renormalized source field.

Wavefunction renormalization of $\mathcal{O}$ (Heuristic but useful)

Suppose: ($g_{\mu\nu} \overset{z\approx\epsilon}{=} \frac{dz^2}{z^2} + \gamma_{\mu\nu} dx^\mu dx^\nu$ defines the boundary metric γ .)

$$
\begin{aligned}
S_{bdy} &\ni \int_{z=\epsilon} d^d x \, \sqrt{\gamma} \, \phi_0(x,\epsilon)\mathcal{O}(x,\epsilon) \\
&= \int d^d x \left(\frac{L}{\epsilon}\right)^d (\epsilon^{\Delta_-} \phi_0^{Ren}(x))\mathcal{O}(x,\epsilon),
\end{aligned}
$$

where we have used $\sqrt{\gamma} = (L/\epsilon)^d$.

Demanding that this be finite as $\epsilon \to 0$:

$$
\begin{aligned}
\mathcal{O}(x,\epsilon) &\sim \epsilon^{d-\Delta_-} \mathcal{O}^{Ren}(x) \\
&= \epsilon^{\Delta_+} \mathcal{O}^{Ren}(x),
\end{aligned}
$$

(we used $\Delta_+ + \Delta_- = d$)

The scaling dimension of $\mathcal{O}^{Ren}$ is $\Delta_+ \equiv \Delta$.

- To confirm: $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{|x|^{2\Delta}}$
- For small m^2, $\exists$ 'alternative quantization': another CFT from the same bulk theory with Neumann boundary conditions on ϕ.

Relevantness

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2 L^2}$$

• If $m^2 > 0$: $\Delta \equiv \Delta_+ > d$, so $\mathcal{O}_\Delta$ is an irrelevant operator.

$$\Delta S = \int d^d x \; (\text{mass})^{d-\Delta} \mathcal{O}_\Delta,$$

the effects of such an operator go away in the IR, at energies $E < \text{mass}$.

$\phi \sim z^{\Delta_-} \phi_0$ *is* this coupling.

It grows in the UV (small z). If ϕ_0 is a finite perturbation, it will back-react on the metric and destroy the asymptotic AdS-ness of the geometry: extra data about the UV will be required.

• $m^2 = 0 :\leftrightarrow \Delta = d$ means that $\mathcal{O}$ is marginal.

• If $m^2 < 0$: $\Delta < d$, so $\mathcal{O}$ is a relevant operator. Note that in AdS, $m^2 < 0$ is ok (*i.e.* not unstable) if m^2 is not too negative.

(Note: $\Delta(m)$ depends on the spin of the bulk field.)

Vacuum of CFT, euclidean case

Return to the scalar wave equation in momentum space:

$$0 = [z^{d+1}\partial_z(z^{-d+1}\partial_z) - m^2L^2 - z^2k^2]f_k(z)$$

If $k^2 > 0$ (spacelike or Euclidean) the general solution is (a_K, a_I, integration consts):

$$f_k(z) = a_K z^{d/2} K_\nu(kz) + a_I z^{d/2} I_\nu(kz), \quad \nu = \Delta - \frac{d}{2} = \sqrt{(d/2)^2 + m^2L^2}.$$

In the interior of AdS ($z \to \infty$), the Bessel functions behave as

$$K_\nu(kz) \overset{z\to\infty}{\approx} e^{-kz} \qquad I_\nu(kz) \overset{z\to\infty}{\approx} e^{kz} \ .$$

regularity in the interior uniquely fixes $\underline{f}_k \propto K_\nu$.

Plugging this into the action S gives $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{|x|^{2\Delta}}$

note: $\exists$ nonlinear uniqueness statement, 'Graham-Lee theorem'

Correlation functions of scalar operators from AdS

The solution with $f_k(z=\epsilon)=1$ ('the regulated bulk-to-boundary propagator'), is

$$\underline{f}_k(z)=\frac{z^{d/2}K_\nu(kz)}{\epsilon^{d/2}K_\nu(k\epsilon)} \qquad \left(\int dk\; e^{ikx}f_k(\epsilon)=\delta^d(x)\right)$$

The general position space solution can be obtained by Fourier decomposition:

$$\underline{\phi}^{[\phi_0]}(x)=\int d^dk e^{ikx}\underline{f}_k(z)\phi_0(k,\epsilon)\ .$$

'on-shell action' (*i.e.* the action evaluated on the saddle-point solution):

$$S[\underline{\phi}]=-\frac{\mathfrak{K}}{2}\int d^dx\sqrt{\gamma}\underline{\phi}n\cdot\partial\underline{\phi}$$
$$=-\frac{\mathfrak{K}L^{d-1}}{2}\int d^dk\phi_0(k,\epsilon)\mathcal{F}_\epsilon(k)\phi_0(-k,\epsilon)$$
$$\mathcal{F}_\epsilon(k)=z^{-d}\underline{f}_{-k}(z)z\partial_z\underline{f}_k(z)|_{z=\epsilon}+(k\leftrightarrow -k)$$

$$\langle\mathcal{O}(k_1)\mathcal{O}(k_2)\rangle_c^\epsilon=-\frac{\delta}{\delta\phi_0(k_1)}\frac{\delta}{\delta\phi_0(k_2)}S=(2\pi)^d\delta^d(k_1+k_2)\mathcal{F}_\epsilon(k_1)\ .$$

$$K_\nu(u) = u^{-\nu}(a_0 + a_1 u^2 + a_2 u^4 + \cdots) \qquad \text{(leading term)}$$
$$+ u^\nu \ln u (b_0 + b_1 u^2 + b_2 u^4 + \cdots) \qquad \text{(subleading term)}$$

$$\begin{aligned} \mathcal{F}_\epsilon(k) &= 2\epsilon^{-d+1}\partial_z \left(\frac{(kz)^{-\nu+d/2}(a_0+\cdots) + (kz)^{\nu+d/2}\ln kz (b_0+\cdots)}{(k\epsilon)^{-\nu+d/2}(a_0+\cdots) + (k\epsilon)^{\nu+d/2}\ln k\epsilon (b_0+\cdots)} \right. \\ &= 2\epsilon^{-d}\left[\left\{ \frac{d}{2} - \nu(1 + c_2(\epsilon^2 k^2) + c_4(\epsilon^4 k^4) + \cdots) \right\} \right. \\ &\qquad \left. + \left\{ \nu \frac{2b_0}{a_0}(\epsilon k)^{2\nu} \ln(\epsilon k)(1 + d_2(\epsilon k)^2 + \cdots) \right\} \right] \\ &\equiv \mathrm{(I)} + \mathrm{(II)} \end{aligned}$$

(I): Laurent series in ϵ with coefficients $k^{\text{even integer}}$

(*i.e.* analytic in k at $k = 0$). $\equiv$ contact terms $\equiv$ short distance goo:

$$\int d^d k e^{-ikx}(\epsilon k)^{2m}\epsilon^{-d} = \epsilon^{2m-d}\Box_x^m \delta^d(x) \qquad (m \in \mathbb{Z}_+)$$

The ϵ^{2m-d} agrees w/ ϵ is a UV cutoff for the QFT.

Checking that $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{|x|^{2\Delta}}$

The interesting bit of $\mathcal{F}(k)$, which gives the $x_1 \neq x_2$ behavior of the correlator, is non-analytic in k:

$$(\mathrm{II}) = -2\nu \cdot \frac{b_0}{a_0} k^{2\nu} \ln(k\epsilon) \cdot \epsilon^{2\nu-d}(1 + \mathcal{O}(\epsilon^2)),$$

input of Bessel: $\frac{b_0}{a_0} = \frac{(-1)^{\nu-1}}{2^{2\nu}\nu\Gamma(\nu)^2}$ for $\nu \in \mathbb{Z}$

FT of leading term: $\displaystyle \int d^d k e^{-ikx}(\mathrm{II}) = \frac{2\nu\Gamma(\Delta_+)}{\pi^{d/2}\Gamma(\Delta_+ - d/2)} \frac{1}{x^{2\Delta_+}} \epsilon^{2\nu-d}$.

- *AdS* radius appears only in overall normalization, in the combination $\mathfrak{K}L^{d-1}$.
- Multiplicative renormalization removes the $\epsilon^{2\nu-d}$.
- Holographic Renormalization: add to S_{bulk} the local, intrinsic boundary term

$$\begin{aligned} \Delta S = S_{\text{c.t.}} &= \frac{\mathfrak{K}}{2} \int_{\text{bdy}} d^d x \left(-\Delta_- L^{d-1} \epsilon^{2\Delta_- - d} \left(\phi_0^{\text{Ren}}(x)\right)^2 \right) \\ &= -\Delta_- \frac{\mathfrak{K}}{2L} \int_{\partial AdS, z=\epsilon} \sqrt{\gamma}\, \phi^2(z,x) \end{aligned}$$

Affect neither bulk EOM nor $G_2(x_1 \neq x_2)$, cancels divergences.

Real-time

In Euclidean signature (or Lorentzian signature with spacelike k^2) regularity in the IR uniquely determined the correct solution.

In Lorentzian signature with timelike k^2 ($\omega^2 > \vec{k}^2$), $\exists$ many solutions with the same UV behavior ($z \to 0$), different IR behavior:

$$z^{d/2} K_\nu(\pm iqz) \stackrel{z\to\infty}{\approx} e^{\pm iqz} \qquad q \equiv \sqrt{\omega^2 - \vec{k}^2}$$

these modes oscillate near the Poincaré horizon.
this ambiguity reflects the multiplicity of real-time Green's f'ns.

Important example: retarded Green's function, describes causal response of the system to a perturbation.

Linear response: nothing fancy, just QM

The retarded Green's function for two observables $\mathcal{O}_A$ and $\mathcal{O}_B$ is

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,k) = -i\int d^{d-1}x dt\; e^{i\omega t - ik\cdot x}\theta(t)\langle[\mathcal{O}_A(t,x),\mathcal{O}_B(0,0)]\rangle$$

$\theta(t) = 1$ for $t > 0$, else zero.

(We care about this because it determines what $\langle\mathcal{O}_A\rangle$ does if we kick the system via $\mathcal{O}_B$.)

the source is a time dependent perturbation to the Hamiltonian:

$$\delta H(t) = \int d^{d-1}x \phi_{B(0)}(t,x)\mathcal{O}_B(x)\,.$$

$$\begin{aligned}\langle\mathcal{O}_A\rangle(t,x) &\equiv \mathrm{Tr}\;\rho(t)\,\mathcal{O}_A(x)\\ &= \mathrm{Tr}\;\rho_0\,U^{-1}(t)\mathcal{O}_A(t,x)U(t)\end{aligned}$$

in interaction picture: $U(t) = Te^{-i\int^t \delta H(t')dt'}$ (*e.g.* $\rho_0 = e^{-\beta H_0}$)

Linear response, cont'd

linearize in small perturbation:

$$\begin{aligned}\delta\langle\mathcal{O}_A\rangle(t,x) &= -i\mathrm{Tr}\ \rho_0\int^t dt'[\mathcal{O}_A(t,x),\delta H(t')] \\ &= -i\int^t d^{d-1}x'dt'\langle[\mathcal{O}_A(t,x),\mathcal{O}_B(t',x')]\rangle\phi_{B(0)}(t',x') \\ &= \int dx' G_R(x,x')\phi_B(x')\end{aligned}$$

fourier transform:

$$\delta\langle\mathcal{O}_A\rangle(\omega,k) = G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,k)\delta\phi_{B(0)}(\omega,k)$$

Linear response, an example

perturbation: an external electric field, $E_x = i\omega A_x$
couples via $\delta H = A_x J^x$ where J is the electric current ($\mathcal{O}_B = J_x$)
response: the electric current ($\mathcal{O}_A = J_x$)

$$\delta\langle\mathcal{O}_A\rangle(\omega, k) = G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega, k)\delta\phi_{B(0)}(\omega, k)$$

it's safe to assume $\langle J\rangle_{E=0} = 0$:

$$\langle\mathcal{O}_J\rangle(\omega, k) = G^R_{JJ}(\omega, k)A_x = G^R_{JJ}(\omega, k)\frac{E_x}{i\omega}$$

Ohm's law: $J = \sigma E$

$$\Longrightarrow \; Kubo\ formula: \qquad \boxed{\sigma(\omega, k) = \frac{G^R_{JJ}(\omega, k)}{i\omega}}$$

Holographic real-time prescription

Claim [Son-Starinets 2002]: corresponds to the solution which at $z \to \infty$ describes stuff falling into the horizon

- Both the retarded response and stuff falling through the horizon describe things that *happen*, rather than *unhappen*.
- You can check that this prescription gives the correct analytic structure of $G_R(\omega)$ ([Son-Starinets] and all the hundreds of papers that have used this prescription).
- It has been derived from a holographic version of the Schwinger-Keldysh prescription [Herzog-Son, Maldacena, Skenderis-van Rees].

The fact that stuff goes past the horizon and doesn't come out is what breaks time-reversal invariance in the holographic computation of G^R.

Here, the ingoing choice is $\phi(t, z) \sim e^{-i\omega t + iqz}$:

as t grows, the wavefront moves to larger z.

(the solution which computes causal response is $z^{d/2} K_{+\nu}(iqz)$.)

The same prescription, adapted to the black hole horizon, works in the finite temperature case.

What to do with the solution

determining $\langle \mathcal{O}\mathcal{O} \rangle$ is like a scattering problem in QM

The solution of the equations of motion, satisfying the desired IR bc, behaves near the boundary as

$$\underline{\phi}(z,x) \approx \left(\frac{z}{L}\right)^{\Delta_-} \phi_0(x) \left(1 + \mathcal{O}(z^2)\right) + \left(\frac{z}{L}\right)^{\Delta_+} \phi_1(x) \left(1 + \mathcal{O}(z^2)\right);$$

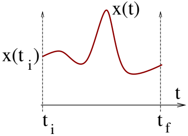
this formula defines the coefficient ϕ_1 of the subleading behavior of the solution.

All the information about G is in ϕ_0, ϕ_1.

recall: $Z[\phi_0] \equiv e^{-W[\phi_0]} \simeq e^{-S_{\text{bulk}}[\underline{\phi}]}\Big|_{\phi \overset{z\to 0}{\to} z^{\Delta_-}\phi_0}$

confession: this is a euclidean eqn. next: a nice general trick. [Iqbal-Liu]

classical mechanics interlude: consider a particle in 1d with action $S[x] = \int_{t_i}^{t_f} dtL$. The variation of the action with respect to the initial value of the coordinate is the initial momentum:



$$\Pi(t_i) = \frac{\delta S}{\delta x(t_i)}, \quad \Pi(t) \equiv \frac{\partial L}{\partial \dot{x}} \quad . \tag{3}$$

Thinking of the radial direction of AdS as time, a mild generalization of (3): [Iqbal-Liu]

$$\langle \mathcal{O}(x) \rangle = \frac{\delta W[\phi_0]}{\delta \phi_0(x)} = \lim_{z \to 0} \left(\frac{z}{L} \right)^{\Delta_-} \Pi(z, x)|_{\mathrm{finite}},$$

where $\Pi \equiv \frac{\partial \mathcal{L}}{\partial(\partial_z \phi)}$ is the bulk field-momentum with z treated as time.

two minor subtleties:

(1) the factor of z_-^Δ arises because of our renormalization of ϕ: $\phi \sim z^{\Delta_-} \phi_0$, so $\frac{\partial}{\partial \phi_0} = z^{-\Delta_-} \frac{\partial}{\partial \phi(z=\epsilon)}$.

(2) Π itself in general has a term proportional to the source ϕ_0

Linear response from holography

With these caveats, away from the support of the source:

$$\langle \mathcal{O}(x) \rangle = \mathfrak{K} \frac{2\Delta - d}{L} \phi_1(x).$$

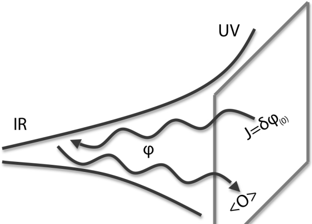
linearize in the size of the perturbing source:

$$\langle \mathcal{O}(x) \rangle = G_R \cdot \phi_0$$

summary: The leading behavior of the solution encodes the source *i.e.* the perturbation of the *action* of the QFT.
The coefficient of the subleading falloff encodes the response

[Balasubramanian et al, 1996].

$$\boxed{G \propto \frac{\phi_1}{\phi_0}}$$



[figure: Hartnoll, 0909.3553]

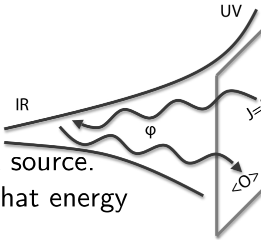
(Quasi)normal modes

determining $\langle \mathcal{O}\mathcal{O} \rangle$ is like a scattering problem in QM

The solution of the equations of motion, satisfying the desired IR bc, behaves near the boundary as

$$\underline{\phi}(z,x) \overset{z\to 0}{\approx} \left(\frac{z}{L}\right)^{\Delta_-} \phi_0(x)\left(1+\mathcal{O}(z^2)\right) + \left(\frac{z}{L}\right)^{\Delta_+} \phi_1(x)\left(1+\mathcal{O}(z^2)\right);$$

$$\boxed{G \propto \frac{\phi_1}{\phi_0}}$$



[figure: Hartnoll, 0909.3553]

G has poles when $\phi_1 \neq 0, \phi_0 = 0$: response without source.
this means that the system has an actual mode at that energy
(if $\omega \in \mathbb{C}$, 'quasinormal mode')

Important conceptual point: the Hilbert spaces are the same.

A useful visualization: 'Witten diagrams'

e.g. consider 3-point function, $\langle \mathcal{O}\mathcal{O}\mathcal{O}\rangle = \left(\frac{\delta}{\delta\phi_0}\right)^3 \ln Z|_{\phi_0=0}$.

cubic coupling matters:

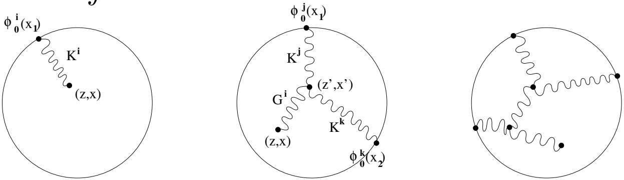
$$(\Box - m_1^2)\phi_1(z,x) = b\phi_2\phi_3 \quad \text{and perms.}$$

Solve perturbatively in ϕ_0: (K, G are Green's f'ns for $\Box - m_i^2$)

$$\underline{\phi}^1(z,x) = \int d^dx_1 K^{\Delta_1}(z,x;x_1)\phi_0^1(x_1)$$

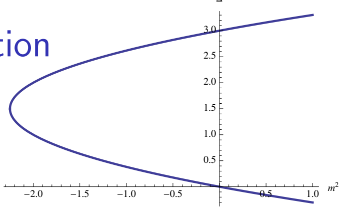
$$+ \quad b\int d^dx'dz'\sqrt{g}\,G^{\Delta_1}(z,x;z',x')$$

$$\times \quad \int d^dx_1 \int d^dx_2 K^{\Delta_2}(z',x';x_1)\phi_0^2(x_1)K^{\Delta_3}(z',x';x_2)\phi_0^3(x_2) + o(b^2\phi_0^3$$



external legs $\leftrightarrow$ sources ϕ_0, vertices $\leftrightarrow$ bulk interactions

Comment on alternative quantization



$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2 L^2}.$$

Lowest Δ_+ is $d/2$. Unitarity bound is $\frac{d-2}{2}$.

Idea [Klebanov-Witten 99]: If under $\delta\phi$, $\delta S \neq \infty$, maybe we can let $\delta\phi$ fluctuate.

$$\text{With} \quad S_{\text{above}} = \int_{\text{bulk}} \left[(\partial\phi)^2 + \phi^2 \right]$$

$\delta\phi \sim \delta\phi_1 z^\Delta$ gives $\delta S_{\text{above}} = \infty$ for $\Delta > d/2$.

$$\text{With} \quad S = S_{\text{above}} + \# \int_{\text{bdy}} \sqrt{\gamma}\phi n \cdot \partial\phi$$

the fluctuation with $\phi \sim z^\Delta$ is normalizible for $\Delta < \frac{d-2}{2}$.

Result: can treat $\phi_1 z^{\Delta_-}$ as source, $\phi_0 z^{\Delta_+}$ as response: $G_{\text{alt}} = \frac{\phi_0}{\phi_1} = G_{\text{usual}}^{-1}$.

Interpretation: alternative quantization is a CFT with a relevant double trace operator $\Delta\left(\mathcal{O}^2\right) = 2\Delta_-$

Perturbation (by $\Delta S_{\text{alt}} = \int_{\text{bdy}} \sqrt{\gamma}\phi^2$)leads back to ordinary quantization.

Next: thermal equilibrium