Holographic duality basics
Lecture 1

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July, 2011

Preface

My goal for these lectures is to convince you that string theory may be useful for many-body problems.

The systems about which we can hope to say something using string theory have in common strong coupling.

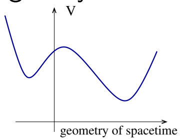
This makes our usual techniques basically useless.

goal for first lecture:

Make plausible the statement that AdS/CFT solves certain strongly-coupled quantum field theories in terms of simple (gravity) variables.

A word about string theory

String theory is a (poorly-understood) quantum theory of gravity
which has a 'landscape' of **many** groundstates
some of which look like our universe
($3 + 1$ dimensions, particle physics...)
most of which don't.



A difficulty for particle physics, a virtue for many-body physics:
by AdS/CFT, each groundstate (with $\Lambda < 0$) describes a universality class of critical behavior and its deformations
This abundance mirrors 'landscape' of many-body phenomena.

Note: tuning on both sides.
An opportunity to connect string theory and experiment.
New perspective on the structure of QFT: access to

uncalculable things	in	uncalculable situations
$G(\omega, k, T)$		at strong coupling
potentials for moving probes		far from equilibrium
entanglement entropy		in real time
		with a finite density of fermions

Outline

1. stringless motivation of the duality and basic dictionary
2. correlation functions in AdS
3. finite temperature and a little bit of transport

Some references:

JM, *Holographic duality with a view toward many-body physics*, 0909.0518

Maldacena, *The gauge/gravity duality*, 1106.6073

Polchinski, *Introduction to Gauge/Gravity Duality*, 1010.6134

Hartnoll, *Quantum Critical Dynamics from Black Holes*, 0909.3553

Horowitz, Polchinski *Gauge/gravity duality*, gr-qc/0602037

Bold assertions

[Horowitz-Polchinski, gr-qc/0602037]

a) Some ordinary quantum many-body systems are actually quantum theories of gravity in extra dimensions
($\equiv$ quantum systems with dynamical spacetime metric).

b) Some are even classical theories of gravity.

What can this mean?

Two hints:
1. The Renormalization Group (RG) is local in scale
2. Holographic Principle

Old-school universality

experimental universality (late 60s):
same critical exponents from very different systems.
Near a (continuous) phase transition (at $T = T_c$), scaling laws:
observables depend like power laws on the distance from the critical point.
e.g. ferromagnet near the Curie transition (let $t \equiv \frac{T_c - T}{T_c}$)

specific heat: $c_v \sim t^{-\alpha}$
magnetic susceptibility: $\chi \sim t^{-\gamma}$

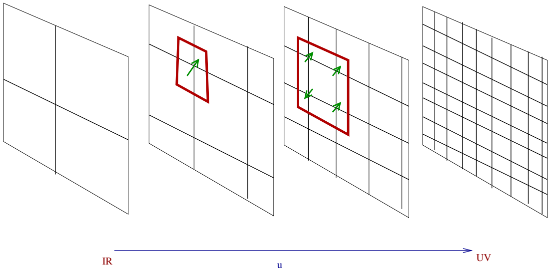
water near its liquid-gas critical point:

specific heat: $c_v \sim t^{-\alpha}$
compressibility: $\chi \sim t^{-\gamma}$

with the same α, γ!

Renormalization group idea

This phenomenon is explained via the Kadanoff-Wilson idea:



$$e.g.: \; H = \sum_{ij} J_{ij} S_i S_j$$

Idea: measure the system with coarser and coarser rulers.

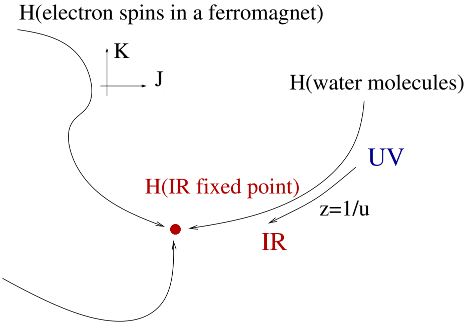
Let 'block spin' = average value of spins in block.

Define a Hamiltonian $H(u)$ for block spins so long-wavelength observables are the same.

$\longrightarrow$ a flow on the space of hamiltonians: $H(u)$

Fixed points of the RG are scale-invariant

This procedure (the sums) is hard to do in practice.



Many microscopic theories will flow to the same fixed-point
$\longrightarrow$ same critical scaling exponents.

The fixed point theory is scale-invariant:
if you change your resolution you get the same picture back.

Hint 1: RG is local in scale

QFT $=$ family of trajectories on the space of hamiltonians: $H(u)$

at each scale u, expand in symmetry-preserving local operators $\{\mathcal{O}_A\}$

$$H(u) = \int d^{d-1}x \sum_A g_A(u) \mathcal{O}_A(u, x)$$

[*e.g.* suppose the dof is a scalar field. then $\{\mathcal{O}_A\} = \{(\partial\phi)^2, \phi^2, \phi^4, \ldots\}$]

since $H(u)$ is determined by a step-by-step procedure,

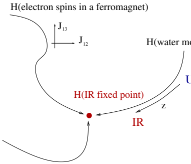
$$u\partial_u g = \beta_g(g(u)) \ .$$

for each coupling g

locality in scale: β_g depends only on $g(u)$.

Def: near a fixed point,

β_g is determined by the *scaling dimension* Δ of $\mathcal{O}$:



$$\mathcal{O}_A(x, u_1) \sim \left(\frac{u_1}{u_2}\right)^{\Delta_A} \mathcal{O}_A(x, u_2)$$

ops of large Δ $\quad$ ($> d$, "irrelevant")

become small in IR (as $u \to 0$).

Hint 2: Holographic principle

holographic principle: in a gravitating system, max entropy in region V $\propto$ area of ∂V in planck units. ['t Hooft, Susskind 1992]

recall: max entropy $S_{MAX} \sim \ln \dim \mathcal{H} \propto \# dof$.

in an ordinary system with local dofs $S_{MAX} \propto V$

to see that gravitating systems are different, we combine two facts:

fact 1: BH has an entropy $\propto$ area of horizon in planck units.

$$S_{BH} = \frac{A}{4G_N}$$

in $d+1$ spacetime dimensions, $G_N \sim \ell_p^{d-1} \quad \longrightarrow \quad S_{BH}$ dimless.

Whence fact 1?

Black holes have a temperature [Hawking] *e.g.* $T_H = \frac{1}{8\pi G_N M}$ for schwarzchild

Consistent thermodynamics requires us to assign them an entropy:

$dE_{\rm BH} = T_H dS_{\rm BH}$ for schwarzchild, $E_{\rm BH} = M$, $A = 4\pi(4M^2G^2)$ gives $(\star)$

'Generalized 2d Law': $S_{total} = S_{\rm ordinary\ stuff} + S_{\rm BH}$

Hint 2: Holographic principle, cont'd

fact 2: dense enough matter collapses into a BH
$1 + 2 \longrightarrow$ in a gravitating system,
max entropy in a region of space $=$
entropy of the biggest black hole that fits.

$$S_{max} = S_{BH} = \frac{1}{4G_N} \times \text{horizon area}$$

Idea [Bekenstein, 1976]: consider a volume V with area A in a flat region of space.

suppose the contrary: given a configuration with
$S > S_{\rm BH} = \frac{A}{4G_N}$ but $E < E_{\rm BH}$ (biggest BH fittable in V)
then: throw in junk (increases S and E) until you make a BH.
S decreased, violating 2d law.

punchline: gravity in $d + 2$ dimensions has the same number of degrees of freedom as a QFT in *fewer* $(d + 1)$ dimensions.

1+2

Combining these hints, we conjecture:

$$\begin{array}{c}\text{gravity}\\ \text{in a space with an extra dim}\\ \text{whose coord is the energy scale}\end{array} \quad \stackrel{?}{=} \quad \text{QFT}$$

To make this more precise, we consider a simple case
(AdS/CFT) [Maldacena, 1997]
in more detail.

AdS/CFT

A relativistic field theory, scale invariant ($\beta_g = 0$ for all nonzero g)

$$x^\mu \to \lambda x^\mu \quad \mu = 0...d-1, \quad u \to \lambda^{-1} u$$

u is the energy scale, RG coordinate

with d-dim'l Poincaré symmetry: Minkowski $ds^2 = -dt^2 + d\vec{x}^2$

Most gen'l $d+1$ dim'l metric w/ Poincaré plus scale inv.

$$AdS_{d+1}: \quad ds^2 = \frac{u^2}{L^2}\left(-dt^2 + d\vec{x}^2\right) + L^2 \frac{du^2}{u^2} \qquad L \equiv \text{‘}AdS \text{ radius'}$$

If we rescale space and time and move in the radial dir,

the geometry looks the same (isometry).

copies of minkowski space of varying ‘size'.

(Note: this metric also has conformal symmetry $SO(d, 2)$

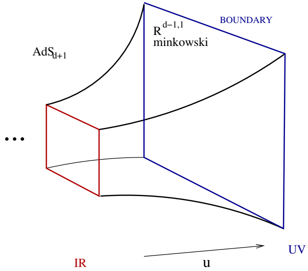
$\exists$ gravity dual $\implies$ “Polchinski's Theorem” for any d.)

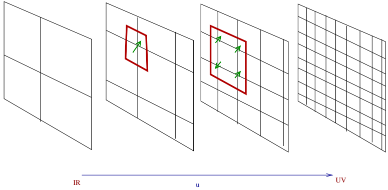
another useful coordinate:

$$z \equiv \frac{L^2}{u} \qquad ds^2 = L^2 \frac{-dt^2 + d\vec{x}^2 + dz^2}{z^2}$$

$[u] =$ energy, $[z] =$ length ($c = \hbar = 1$ units).

Geometry of *AdS* continued





The extra ('radial') dimension is the resolution scale.

(The bulk picture is a hologram.)

preliminary conjecture:

$$\text{gravity on } AdS_{d+1} \text{ space} \stackrel{?}{=} \mathrm{CFT}_d$$

crucial refinement:

in a gravity theory the metric fluctuates.

$\longrightarrow$ what does 'gravity in AdS' mean ?!?

Geometry of *AdS* continued

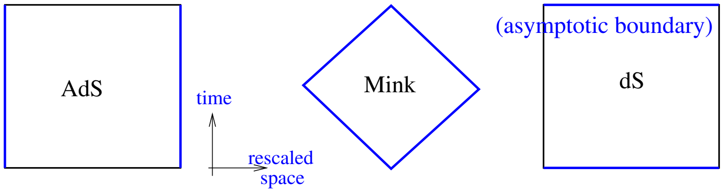
AdS has a boundary (where $u \to \infty, z \to 0$, 'size' of Mink blows up).

massless particles reach it in finite time.

$\implies$ must specify boundary conditions there.

the fact that the geometry is *AdS* near there is one of these boundary conditions.

different from Minkowski space or (worse) de Sitter:



so: some $\mathrm{CFT}_d \overset{?}{=}$ gravity on *asymptotically* AdS_{d+1} space

(we will discuss the meaning of this '=' much more)

Preview of dictionary

"bulk" $\leftrightsquigarrow$ "boundary"

fields in AdS_{d+1} $\leftrightsquigarrow$ operators in CFT

(Note: operators in CFT don't make particles.)

mass $\leftrightsquigarrow$ scaling dimension

$$m^2 L^2 = \Delta(\Delta - d)$$

a simple bulk theory with a small # of light fields	$\leftrightsquigarrow$	CFT with a small # of ops of small Δ (like rational CFT)

What to calculate

some observables of a QFT (Euclidean for now):
vacuum correlation functions of local operators:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle$$

standard trick: make a generating functional $Z[J]$ for these correlators by perturbing the action of the QFT:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \sum_A J_A(x)\mathcal{O}_A(x) \equiv \mathcal{L}(x) + \mathcal{L}_J(x)$$
$$Z[J] = \langle e^{-\int \mathcal{L}_J}\rangle_{CFT}$$

$J_A(x)$: arbitrary functions (sources)

$$\langle \prod_n \mathcal{O}_n(x_n)\rangle = \prod_n \frac{\delta}{\delta J_n(x_n)} \ln Z\Big|_{J=0}$$

Hint: $\mathcal{L}_J$ is a UV perturbation – near the boundary, $z \to 0$

Holographic duality made quantitative

[Witten; Gubser-Klebanov-Polyakov (GKPW)]

$$\begin{aligned} Z_{QFT}[\text{sources}] &= Z_{\text{quantum gravity}}[\text{boundary conditions at } u \to \infty] \\ &\approx e^{-S_{\text{bulk}}[\text{boundary conditions at } u \to \infty]}\Big|_{\text{saddle of } S_{\text{bulk}}} \\ J = \phi_0 & \qquad \text{"}\phi \overset{u\to\infty}{\to} \phi_0\text{"} \end{aligned}$$

What's S_{bulk}? AdS solves the EOM for

$$S_{\text{bulk}} = \frac{1}{\# G_N} \int d^{d+1}x \sqrt{g} \left(\mathcal{R} - 2\Lambda + \ldots\right)$$

(... = fields which vanish in groundstate, more irrelevant couplings.)

expansion organized by decreasing relevance

$\Lambda = -\frac{d(d-1)}{2L^2}$ note tuning!

$\mathcal{R} \sim \partial^2 g \implies G_N \sim \ell_p^{d-1}$

gravity is classical if $L \gg \ell_p$.

This is what comes from string theory (when we can tell) at low E and for $\frac{1}{L} \ll \frac{1}{\sqrt{\alpha'}} \equiv \frac{1}{\ell_s}$ ($\frac{1}{\alpha'}$ = string tension)

(One basic role of string theory here: fill in the dots.)

Conservation of evil

large AdS radius $L \leftrightsquigarrow$ strong coupling of QFT

(avoids an immediate disproof – obviously a perturbative QFT isn't usefully an extra-dimensional theory of gravity.)

a special case of a

Useful principle (Conservation of evil):
different weakly-coupled descriptions
have non-overlapping regimes of validity.

strong/weak duality: hard to check, very powerful

Info goes both ways: once we believe the duality, this is our best definition of string theory.

Holographic counting of degrees of freedom

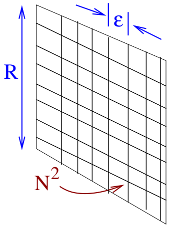
[Susskind-Witten]

$$S_{max} = \frac{\text{area of boundary}}{4G_N} \stackrel{?}{=} \# \text{ of dofs of QFT}$$

$$\text{yes} : \qquad \infty = \infty$$

need to regulate two divergences: dofs at every point in space (UV) ($\#$ dofs $\equiv N^2$) ,
spread over $\mathbb{R}^{d-1}$ (IR).
counting in QFT$_d$:

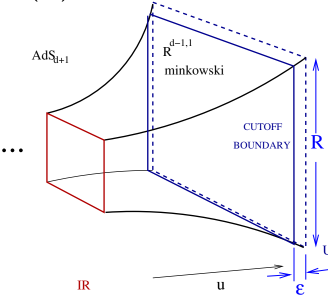
$$S_{max} \sim \left(\frac{R}{\epsilon}\right)^{d-1} N^2$$



counting in AdS$_{d+1}$: at fixed time: $ds^2_{AdS} = L^2 \frac{dz^2 + d\vec{x}^2}{z^2}$

$$A = \int_{bdy,\ z\ \ fixed} \sqrt{g} d^{d-1}x = \int_{\mathbb{R}^{d-1}} d^{d-1}x \left(\frac{L}{z}\right)^{d-1} |_{z\to 0}$$

$$A = \int_0^R d^{d-1}x \frac{L^{d-1}}{z^{d-1}} |_{z=\epsilon} = \left(\frac{RL}{\epsilon}\right)^{d-1}$$



The holographic principle

then says that the maximum entropy in the bulk is

$$\frac{A}{4G_N} \sim \frac{L^{d-1}}{4G_N} \left(\frac{R}{\epsilon}\right)^{d-1} .$$

$$\boxed{\frac{L^{d-1}}{G_N} = N^2}$$

lessons:

1. parametric dependence on R checks out.
2. gravity is classical if QFT has lots of dofs/pt: $N^2 \gg 1$

$$Z_{QFT}[\text{sources}] \approx e^{-N^2 I_{\text{bulk}}[\text{boundary conditions at } r\to\infty]}|_{\text{extremum of } I_{\text{bulk}}}$$

classical gravity (sharp saddle) $\leftrightsquigarrow$ many dofs per point, $N^2 \gg 1$

Confidence-building measures

Why do we believe this enough to try to use it to do physics?

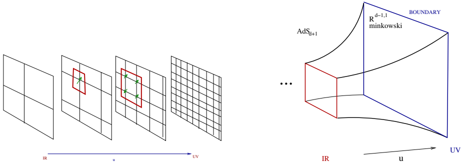
- 1. **Many** detailed checks in special examples
  examples: relativistic gauge theories (fields are $N \times N$ matrices), with extra symmetries (conformal invariance, supersymmetry)
  checks: 'BPS quantities,' integrable techniques, some numerics
- 2. Sensible answers for physics questions
  rediscoveries of known physical phenomena: *e.g.* color confinement, chiral symmetry breaking, thermo, hydro, thermal screening, entanglement entropy, chiral anomalies, superconductivity, ...
  Gravity limit, when valid, says who are the correct variables.
  Answers questions about thermodynamics, transport, RG flow, ...
  in terms of geometric objects.
- 3. Applications to quark-gluon plasma (QGP)
  benchmark for viscosity, hard probes of medium, approach to equilibrium

Simple pictures for hard problems, an example

Bulk geometry is a spectrograph separating the theory by energy scales.

$$ds^2 = w(z)^2 \left(-dt^2 + d\vec{x}^2\right) + \frac{dz^2}{z^2}$$

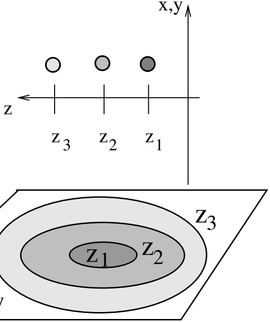
CFT: bulk geometry goes on forever, warp factor $w(z) = \frac{L}{z} \to 0$:



$t, \vec{x}$ are the field theory time and space coordinates.

$$(\text{size})_{FT} = \frac{1}{w(z)} (\text{proper size})$$

$$E_{FT} \sim i\partial_t = w(z) E_{\text{proper}}$$



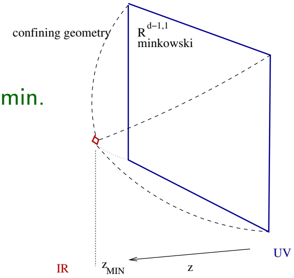
The role of the warp factor, cont'd

Model with a gap:

geometry ends smoothly, warp factor $w(z)$ has a min.

if IR region is missing,

no low-energy excitations, energy gap.



large N counting

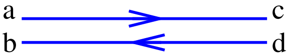
consider a matrix field theory
Φ_a^b is a matrix field. $a, b = 1..N$. other labels (*e.g.* spatial position, spin) are suppressed.

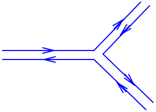
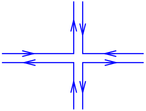
$$\mathcal{L} \sim \frac{1}{g^2} \mathrm{Tr}\ \left((\partial \Phi)^2 + \Phi^2 + \Phi^3 + \Phi^4 + \ldots \right)$$

here *e.g.* $(\Phi^2)_a^c = \Phi_a^b \Phi_b^c$,
the interactions are invariant under the $U(N)$ symmetry
$\Phi \to U^{-1} \Phi U$

't Hooft counting

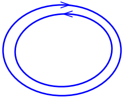
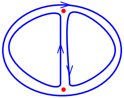
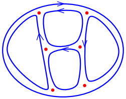
double-line notation:

$$\langle \tilde{\Phi}^a_b \tilde{\Phi}^d_c \rangle \propto g^2 \delta^a_c \delta^d_b \equiv g^2$$ 

 $\propto g^{-2}$  $\propto g^{-2}$

$$\text{diagram} \sim \left(\frac{\lambda}{N}\right)^{\text{no. of prop.}} \left(\frac{N}{\lambda}\right)^{\text{no. of int. vert.}} N^{\text{no. of index loops}}$$

't Hooft limit: take $N \to \infty, g \to 0$ holding $\lambda \equiv g^2 N$ fixed.

 $\propto N^2$  $\propto \lambda N^2$  $\propto \lambda^3 N^2$

(a) (b) (c)

Figure: planar graphs that contribute to the vac$\to$vac ampl.

topology of graphs

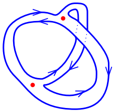
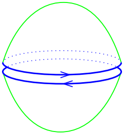
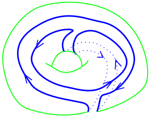
 $\propto g^2 N = \lambda N^0$

Figure: Non-planar (but still oriented!) graph that contributes to the vacuum→vacuum amplitude.

 $\sim S^2$  $\sim T^2$

If $E = \#$of propagators, $V = \#$ of vertices, and $F = \#$ of index loops,
a diagram contributes $N^{F-E+V}\lambda^{E-V}$.

$F - E + V = \chi(\text{surface}) = 2 - 2h - b$

(h = number of handles, b = number of boundaries)

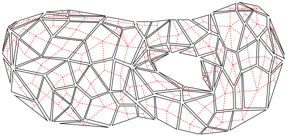
topology of graphs, cont'd

the effective action (the sum over connected vacuum-to-vacuum diagrams) has the expansion:

$$\ln Z = \sum_{h=0}^{\infty} N^{2-2h} \sum_{\ell=0}^{\infty} c_{\ell,h} \lambda^{\ell} = \sum_{h=0}^{\infty} N^{2-2h} \mathfrak{F}_h(\lambda)$$

['t hooft]:$1/N$ as a small parameter, string expansion. $1/N$ suppresses splitting and joining of strings.

e.g.



concrete point: at large N, $\ln Z \sim N^2$.

N counting for correlation functions

$$\mathcal{O}(x) = c(k, N)\mathrm{Tr}(\Phi_1(x)...\Phi_k(x))$$

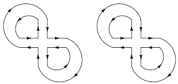


Figure: Disconnected diagram contributing to the correlation function $\langle \mathrm{Tr}(\Phi^4)\mathrm{Tr}(\Phi^4)\rangle \sim N^2$

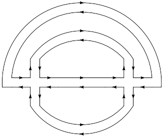


Figure: Connected diagram contributing to the correlation function $\langle \mathrm{Tr}(\Phi^4)\mathrm{Tr}(\Phi^4)\rangle$ goes like N^0