Similarities between Collective Neutrino Oscillations and the Nuclear Pairing Problem

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Table of contents

Motivation

Supernova neutrinos

- $M_{\text{progenitor}} \geq 8 M_{\odot} \Rightarrow \Delta E \sim 10^{59}$ MeV
- 99 % of this energy is carried away by neutrinos and antineutrinos with $10 \leq E_{\nu} \leq 30$ MeV $\Rightarrow 10^{58}$ neutrinos!

Neutrino Mixing

Mass and Flavor States

$$a_1(\mathbf{p}, s) = \cos\theta\, a_e(\mathbf{p}, s) - \sin\theta\, a_x(\mathbf{p}, s)$$

$$a_2(\mathbf{p}, s) = \sin\theta\, a_e(\mathbf{p}, s) + \cos\theta\, a_x(\mathbf{p}, s)$$

Flavor Isospin Operators

$$\hat{J}^+_{\mathbf{p},s} = a_e^\dagger(\mathbf{p}, s) a_x(\mathbf{p}, s) \ , \qquad \hat{J}^-_{\mathbf{p},s} = a_x^\dagger(\mathbf{p}, s) a_e(\mathbf{p}, s) \ ,$$

$$\hat{J}^0_{\mathbf{p},s} = \frac{1}{2}\left(a_e^\dagger(\mathbf{p}, s) a_e(\mathbf{p}, s) - a_x^\dagger(\mathbf{p}, s) a_x(\mathbf{p}, s) \right)$$

$$[\hat{J}^+_{\mathbf{p},s}, \hat{J}^-_{\mathbf{q},r}] = 2\delta_{\mathbf{pq}}\delta_{sr}\hat{J}^0_{\mathbf{p},s} \ , \qquad [\hat{J}^0_{\mathbf{p},s}, \hat{J}^\pm_{\mathbf{q},r}] = \pm\delta_{\mathbf{pq}}\delta_{sr}\hat{J}^\pm_{\mathbf{p},s} \ ,$$

Neutrino Hamiltonian

Vacuum Oscillation Term

$$\hat{H}_\nu = \sum_{\mathbf{p},s} \left(\frac{m_1^2}{2p} a_1^\dagger(\mathbf{p},s) a_1(\mathbf{p},s) + \frac{m_2^2}{2p} a_2^\dagger(\mathbf{p},s) a_2(\mathbf{p},s) \right) .$$

$$\hat{H}_\nu = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p$$

$$\hat{B} = (\sin 2\theta, 0, -\cos 2\theta)$$

Neutrino-Neutrino Interactions

$$\hat{H}_{\text{total}} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos \vartheta_{\mathbf{pq}}) \vec{J}_\mathbf{p} \cdot \vec{J}_\mathbf{q}$$

Neutrino Hamiltonian

Neutrino Hamiltonian with $\nu - \nu$ interactions

$$\hat{H}_{\text{total}} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos \vartheta_{\mathbf{pq}}) \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$

Single-angle approximation $\Rightarrow$

$$\hat{H}_{\text{total}} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \vec{J} \cdot \vec{J}$$

Defining $\mu = \frac{\sqrt{2} G_F}{V}$, $\tau = \mu t$, and $\omega_p = \frac{1}{\mu} \frac{\delta m^2}{2p}$ one can write

$$\hat{H} = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{J} \cdot \vec{J}$$

Conserved Quantities

Some Invariants

$$\hat{H} = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{J} \cdot \vec{J}$$

This Hamiltonian preserves the *length of each spin*

$$\hat{L}_p = \vec{J}_p \cdot \vec{J}_p \ , \qquad \left[\hat{H}, \hat{L}_p\right] = 0 \ ,$$

as well as the *total spin component* in the direction of the "external magnetic field", $\hat{B}$

$$\hat{C}_0 = \hat{B} \cdot \vec{J} \ , \qquad \left[\hat{H}, \hat{C}_0\right] = 0$$

BCS Hamiltonian

Hamiltonian in Quasi-spin basis

$$\hat{H}_{\text{BCS}} = \sum_k 2\epsilon_k \hat{t}^0_k - |G| \hat{T}^+ \hat{T}^-$$

Quasi-spin operators:

$$\hat{t}^+_k = c^\dagger_{k\uparrow} c^\dagger_{k\downarrow}, \qquad \hat{t}^-_k = c_{k\downarrow} c_{k\uparrow}, \qquad \hat{t}^0_k = \frac{1}{2}\left(c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{k\downarrow} c_{k\downarrow} - 1 \right)$$

$$[\hat{t}^+_k, \hat{t}^-_l] = 2\delta_{kl}\hat{t}^0_k \ , \qquad [\hat{t}^0_k, \hat{t}^\pm_l] = \pm\delta_{kl}\hat{t}^\pm_k \ .$$

Richardson gave a solution of this problem. Hence there exist invariants of motion.

Invariants

Invariants

The collective neutrino Hamiltonian given has the following constants of motion:

$$\hat{h}_p = \hat{B} \cdot \vec{J}_p + 2 \sum_{q(\neq p)} \frac{\vec{J}_p \cdot \vec{J}_q}{\omega_p - \omega_q}.$$

The individual neutrino spin-length discussed before in an independent invariant. However $\hat{C}_0 = \sum_p \hat{h}_p$. The Hamiltonian itself is also a linear combination of these invariants.

$$\hat{H} = \sum_p w_p \hat{h}_p + \sum_p \hat{L}_p \ .$$

Eigenvalues and Eigenstates

Eigenstates of the system

- $J_{\max} = N/2$ $\qquad$ N, the total number of neutrinos
- A state with all electron neutrinos: $|\nu_e \, \nu_e \, \nu_e \ \ldots\rangle = |J_{\max} \ \ J_{\max}\rangle_f$
- Matter and flavor bases are connected with a unitary transformation: $|J_{\max} \ \ J_{\max}\rangle_f = \hat{U}^\dagger |J_{\max} \ \ J_{\max}\rangle_m$
- $|J_{\max} \ \ J_{\max}\rangle_m = \prod_{\mathbf{p},s} a_1^\dagger(\mathbf{p},s) \ |0\rangle$
  $|J_{\max} \ -J_{\max}\rangle_m = \prod_{\mathbf{p},s} a_2^\dagger(\mathbf{p},s) \ |0\rangle$
  $E_{(+J_{\max})} = -\sum_p \frac{n_p}{2}\omega_p + J_{\max}\left(J_{\max}+1\right)$
  $E_{(-J_{\max})} = \sum_p \frac{n_p}{2}\omega_p + J_{\max}\left(J_{\max}+1\right)$

Eigenvalues and Eigenstates

Other states

$$\mathcal{Q}^{\pm}(\xi) == \sum_p \frac{1}{\omega_p - \xi}\left(\cos^2\theta \hat{J}_p^{\pm} + \sin 2\theta \hat{J}_p^0 - \sin^2\theta \hat{J}_p^{\mp}\right)$$

$$\begin{aligned}\hat{H}\mathcal{Q}^{+}(\xi)|J \; -J\rangle_m &= \left(E_{(-J)} - 2J - \xi\right)\mathcal{Q}^{+}(\xi)|J \; -J\rangle_m \\ &+ \underbrace{\left(1 + 2\sum_p \frac{-j_p}{w_p - \xi}\right)\mathcal{Q}^{+}|J \; -J\rangle_m}_{\text{should be zero if eigenstate}}\end{aligned}$$

This gives us the Bethe ansatz equation $\Rightarrow \sum_p \frac{-j_p}{w_p - \xi} = -\frac{1}{2}$

Eigenvalues and Eigenstates

Most General Eigenstate

$$|\xi_1, \xi_2, \dots \xi_\kappa\rangle \equiv \mathcal{Q}^+(\xi_1)\mathcal{Q}^+(\xi_2)\dots \mathcal{Q}^+(\xi_\kappa)|J \quad -J\rangle_m$$

$$E(\xi_1, \xi_2, \dots, \xi_\kappa) = E_{(-J)} - \sum_{\alpha=1}^{\kappa} \xi_\alpha - \kappa(2J - \kappa + 1) \; ,$$

$$\underbrace{\sum_p \frac{-j_p}{\omega_p - \xi_\alpha} = -\frac{1}{2} + \sum_{\substack{\beta=1 \\ (\beta \neq \alpha)}}^{\kappa} \frac{1}{\xi_\alpha - \xi_\beta} \; .}_{\text{Bethe ansatz equations}}$$

An RPA-like approximation

An RPA-inspired approximation when $[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2] = 0$. Approximate the operator product as

$$\hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 \sim \hat{\mathcal{O}}_1 \langle \hat{\mathcal{O}}_2 \rangle + \langle \hat{\mathcal{O}}_1 \rangle \hat{\mathcal{O}}_2 - \langle \hat{\mathcal{O}}_1 \rangle \langle \hat{\mathcal{O}}_2 \rangle \ ,$$

where the expectation values should be calculated with respect to a state $|\Psi\rangle$ which satisfies the condition $\langle \hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 \rangle = \langle \hat{\mathcal{O}}_1 \rangle \langle \hat{\mathcal{O}}_2 \rangle$.

$$\hat{H} \sim \hat{H}^{\mathrm{RPA}} = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{P} \cdot \vec{J}$$

Polarization vector: $\vec{P}_{\mathbf{p},s} = 2\langle \vec{J}_{\mathbf{p},s} \rangle$. Use SU(2) coherent states for the expectation value.

Mean-neutrino field

Polarization vectors

$$\hat{H} \sim \hat{H}^{\mathrm{RPA}} = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{P} \cdot \vec{J}$$

$$\vec{P}_{\mathbf{p},s} = 2\langle \vec{J}_{\mathbf{p},s} \rangle$$

Eqs. of motion: $$\frac{d}{d\tau}\vec{J}_p = -i[\vec{J}_p, \hat{H}^{\mathrm{RPA}}] = (\omega_p \hat{B} + \vec{P}) \times \vec{J}_p$$

RPA Consistency requirement $\Rightarrow \frac{d}{d\tau}\vec{P}_p = (\omega_p \hat{B} + \vec{P}) \times \vec{P}_p$

Invariants $$I_p = 2\langle \hat{h}_p \rangle = \hat{B} \cdot \vec{P}_p + \sum_{q(\neq p)} \frac{\vec{P}_p \cdot \vec{P}_q}{\omega_p - \omega_q} \Rightarrow \frac{d}{d\tau} I_p = 0$$

Total Hamiltonian

Hamiltonian with both ν's and $\bar{\nu}$'s

$$
\begin{aligned}
\hat{H}_{\text{total}} &= \sum_p \frac{\delta m^2}{2p}\left(-\cos 2\theta\, \hat{J}^0_p + \sin 2\theta\, \frac{\hat{J}^+_p + \hat{J}^-_p}{2}\right) \\
&+ \sum_{\bar{p}} \frac{\delta m^2}{2\bar{p}}\left(\cos 2\theta\, \hat{\bar{J}}^0_{\bar{p}} + \sin 2\theta\, \frac{\hat{\bar{J}}^+_{\bar{p}} + \hat{\bar{J}}^-_{\bar{p}}}{2}\right) \\
&+ \frac{\sqrt{2}G_F}{V}\left(\sum_{\mathbf{p},\mathbf{q}}(1-\cos\vartheta_{\mathbf{pq}})\vec{J}_{\mathbf{p}}\cdot\vec{J}_{\mathbf{q}} + \sum_{\bar{\mathbf{p}},\bar{\mathbf{q}}}(1-\cos\vartheta_{\bar{\mathbf{p}}\bar{\mathbf{q}}})\vec{\bar{J}}_{\bar{\mathbf{p}}}\cdot\vec{\bar{J}}_{\bar{\mathbf{q}}}\right. \\
&+ \left.\sum_{\mathbf{p},\bar{\mathbf{q}}}(1-\cos\vartheta_{\mathbf{p}\bar{\mathbf{q}}})\left(2\hat{J}^0_{\mathbf{p}}\hat{\bar{J}}^0_{\bar{\mathbf{q}}} - \hat{J}^+_{\mathbf{p}}\hat{\bar{J}}^-_{\bar{\mathbf{q}}} - \hat{J}^-_{\mathbf{p}}\hat{\bar{J}}^+_{\bar{\mathbf{q}}}\right)\right) .
\end{aligned}
$$

Including antineutrinos

Single angle approximation

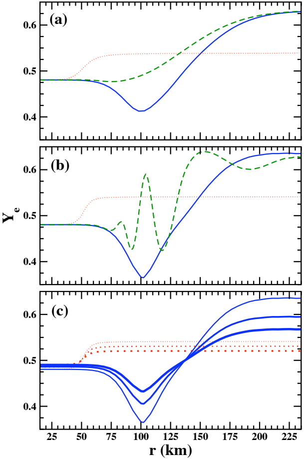
$$H_{\text{total}} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p - \sum_{\bar{p}} \frac{\delta m^2}{2\bar{p}} \hat{B} \cdot \vec{\tilde{J}}_p + \frac{\sqrt{2} G_F}{V} \left(\vec{J} + \vec{\tilde{J}} \right) \cdot \left(\vec{J} + \vec{\tilde{J}} \right)$$

Defining $\omega_{\bar{p}} = -\frac{1}{\mu} \frac{\delta m^2}{2\bar{p}}$, one writes

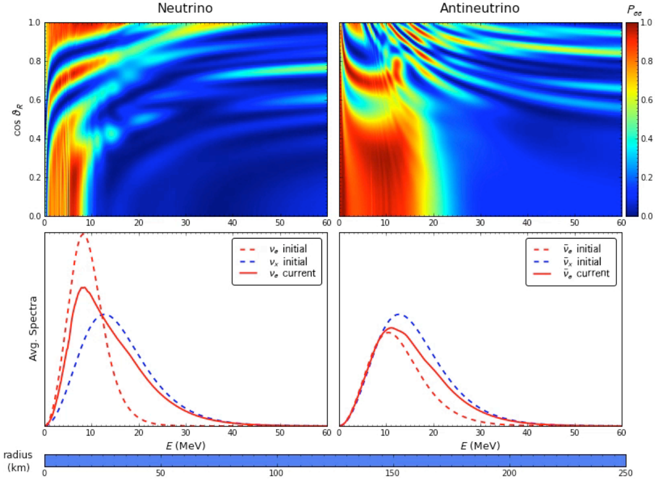
$$H = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \sum_{\bar{p}} \omega_{\bar{p}} \hat{B} \cdot \vec{\tilde{J}}_p + \left(\vec{J} + \vec{\tilde{J}} \right) \cdot \left(\vec{J} + \vec{\tilde{J}} \right)$$

Examples of mean-field calculations

- With ν luminosity $L^{51} =$ 0.001 (blue), 0.1 (green), 50 (red)
- Balantekin and Yüksel, New J. Phys. **7** 51 (2005).



Examples of mean-field calculations



Fuller, Qian, Carlson, Duan,...

Invariants

Invariants

Conserved quantities for each neutrino energy mode p:

$$\hat{h}_p = \hat{B} \cdot \vec{J}_p + 2 \sum_{q(\neq p)} \frac{\vec{J}_p \cdot \vec{J}_q}{\omega_p - \omega_q} + 2 \sum_{\bar{q}} \frac{\vec{J}_p \cdot \vec{\tilde{J}}_{\bar{q}}}{\omega_p - \omega_{\bar{q}}}$$

Conserved quantity $\hat{h}_{\bar{p}}$ for each antineutrino energy mode:

$$\hat{h}_{\bar{p}} = \hat{B} \cdot \vec{\tilde{J}}_p + 2 \sum_{\bar{q}(\neq \bar{p})} \frac{\vec{\tilde{J}}_{\bar{p}} \cdot \vec{\tilde{J}}_{\bar{q}}}{\omega_{\bar{p}} - \omega_{\bar{q}}} + 2 \sum_{q} \frac{\vec{\tilde{J}}_{\bar{p}} \cdot \vec{J}_q}{\omega_{\bar{p}} - \omega_q} .$$

Invariants

Mean-field Invariants

$$I_p = 2\langle \hat{h}_p \rangle = \hat{B} \cdot \vec{P}_p + \sum_{q(\neq p)} \frac{\vec{P}_p \cdot \vec{P}_q}{\omega_p - \omega_q} + \sum_{\bar{q}} \frac{\vec{P}_p \cdot \vec{\tilde{P}}_{\bar{q}}}{\omega_p - \omega_{\bar{q}}}$$

$$I_{\bar{p}} = 2\langle \hat{h}_{\bar{p}} \rangle = \hat{B} \cdot \vec{\tilde{P}}_{\bar{p}} + \sum_{\bar{q}(\neq \bar{p})} \frac{\vec{\tilde{P}}_{\bar{p}} \cdot \vec{\tilde{P}}_{\bar{q}}}{\omega_{\bar{p}} - \omega_{\bar{q}}} + \sum_{q} \frac{\vec{\tilde{P}}_{\bar{p}} \cdot \vec{P}_q}{\omega_{\bar{p}} - \omega_q}$$

Conclusions

Conclusions

- We examined the many-neutrino gas both from the exact many-body perspective and from the point of view of an effective one-body description formulated with the application of the RPA method. In the limit of the single angle approximation, both the many-body and the RPA pictures possess many constants of motion manifesting the existence of associated dynamical symmetries in the system.
- The existence of constants of motion offer practical ways of extracting information even from exceedingly complex systems. Even when the symmetries which guarantee their existence is broken, they usually provide a convenient set of variables which behave in a relatively simple manner depending on how drastic the symmetry breaking factor is.

Conclusions

Conclusions - continued

- The existence of such invariants naturally lead to associated collective modes in neutrino oscillations. However, symmetries alone do not guarantee the stability of such collective behavior. An extensive numerical study of the collective neutrino phenomena associated with our invariants would shed light on the question of stability.