Viscosity in strongly interacting Fermi gases: Spectral functions and sum rules

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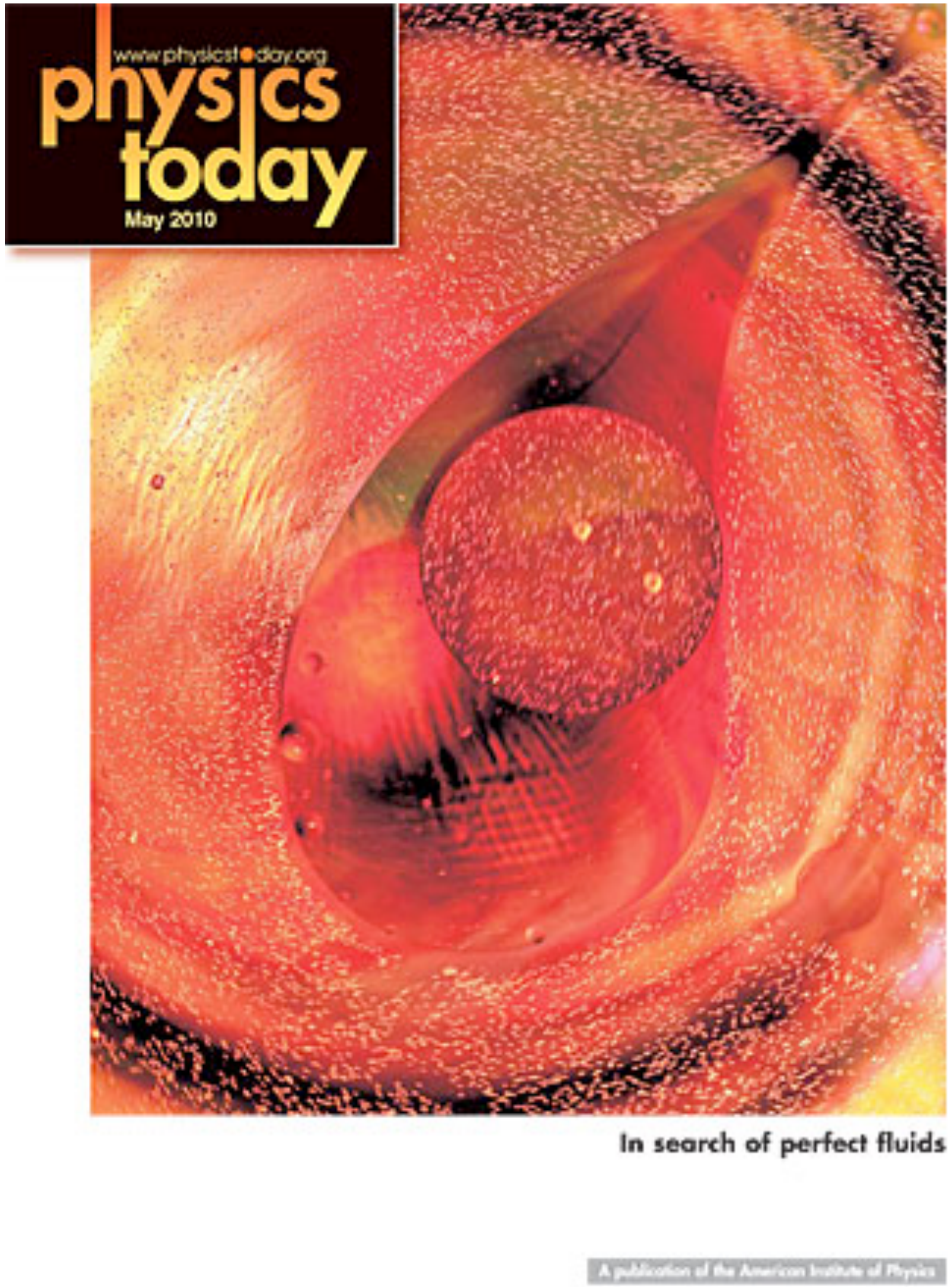
ET & Mohit Randeria, PRA **81**, 053610 (2010)

Perfect fluidity (low viscosity)

- Perfect fluids saturate *conjectured* entropy bound:

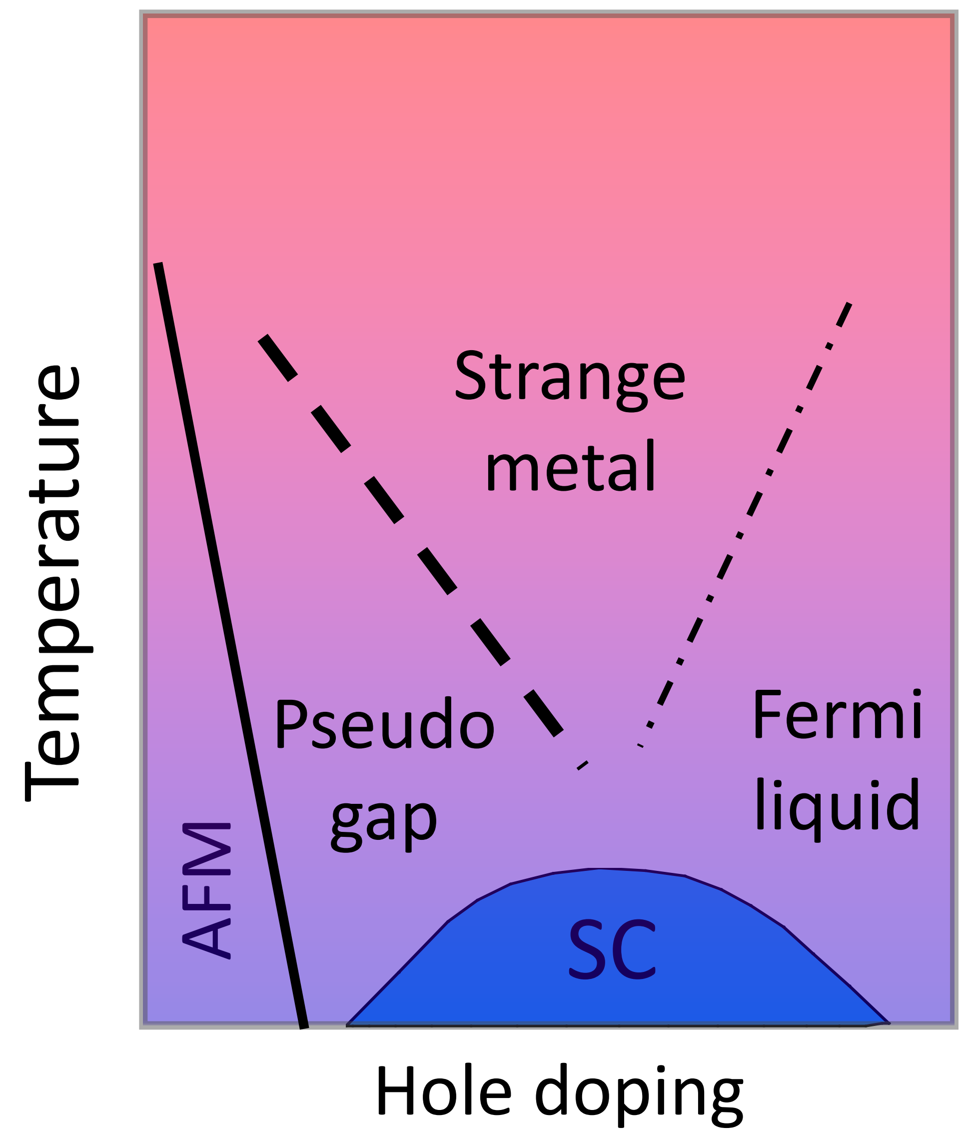
$$\eta \geq s\hbar/4\pi k_B$$

- Most perfect fluids known: quark-gluon plasma & unitary Fermi gas.
- Viscosity is small when there are no sharp quasiparticles.



Quantum transport without sharp quasiparticles

- Normal state of high T_c superconductors
- Quantum critical regions
- Quark-gluon plasma
- Unitary Fermi gas

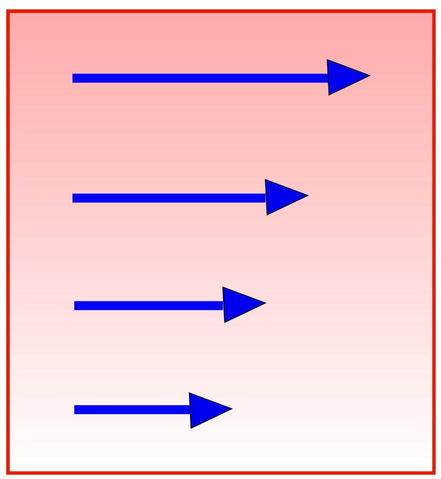


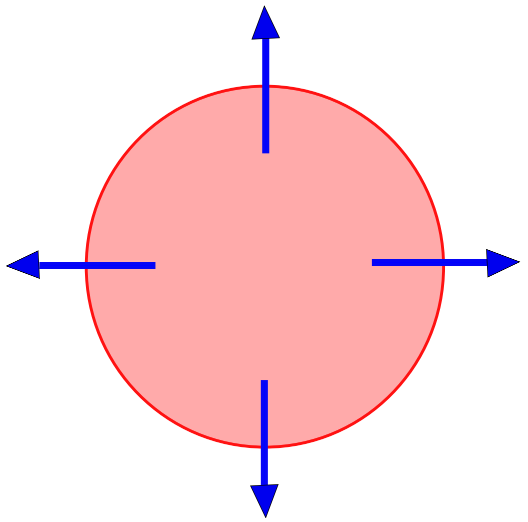
Perturbation theory fails for these systems.

Outline

- Viscosity: Preliminaries.
- Spectral functions and sum rules.
- Sum rules and bounds on DC viscosity.

Shear & bulk viscosity





$$m\frac{\partial j_x}{\partial t} = \eta \frac{\partial^2 v_x}{\partial y^2}$$

$$m\frac{\partial j_r}{\partial t} = \zeta \frac{\partial^2 v_r}{\partial r^2}$$

- Shear viscosity: Dissipation when there is a velocity gradient
- Bulk viscosity: Dissipation after uniform volume change

Kinetic theory

- Maxwell (1860): kinetic theory calculation of shear viscosity:

$$\eta \sim \rho v_{\rm rms} l = \text{mass density} \times \text{velocity} \times \text{mean free path}$$
$$\sim \frac{\sqrt{mkT}}{\sigma}$$

- Independent of density!

"Such a consequence of the mathematical theory is very startling and the only experiment I have met with on the subject does not seem to confirm it."

So he did his own:

Confirmed density independence

What kinetic theory tells us

$$\eta \sim \rho v_{\mathrm{rms}} l$$

- Strong interactions ⇒ small viscosity (small m.f. path).

- Viscosity large at low T ($l \to \infty$*) and high T ($v_{\mathrm{rms}} \to \infty$).

- Heisenberg lower bound in between? (Danielewicz & Gyulassy, PRD '85.)

$$\eta \sim n(\Delta p \Delta x) \geq n\hbar$$

- Essentially dimensional analysis. *Kinetic theory only valid when there are sharp quasiparticles.*

* For *quasi*particles

Aside: Viscosity and strong interactions

- *For most people: Strong interactions*
  ⇒ large viscosity.

- *What do we mean by "strong interactions"?*

Aside: Viscosity and strong interactions

- *For most people: Strong interactions*
  $\Rightarrow$ *large viscosity.*

- *What do we mean by "strong interactions"?*

- In gases & QGP, interactions between degrees of freedom that transport momentum (quasiparticles or molecules/atoms) are short ranged.

- Strong interactions in dilute systems = many collisions per unit time.

Kubo formulae

- Beyond kinetic theory: Kubo.
- Navier-Stokes: $\partial_t j = \text{viscosity} \times \nabla^2 v$
- Linear response to a spatially varying velocity field: $j^\alpha = \chi_J^{\alpha\beta} v_\beta$

$$\chi_J^{\alpha\beta} = \chi_L(q_\alpha q_\beta/q^2) + \chi_T(\delta_{\alpha\beta} - q_\alpha q_\beta/q^2)$$

longitudinal & transverse current correlation fncs.

- Kubo:

$$\eta(\omega) = \lim_{q\to 0} \frac{\omega}{q^2} \mathrm{Im}\chi_T(\mathbf{q},\omega) \qquad 4\eta(\omega)/3 + \zeta(\omega) = \lim_{q\to 0} \frac{\omega}{q^2} \mathrm{Im}\chi_L(\mathbf{q},\omega)$$

- In general, solved perturbatively. One exception: AdS/CFT.

AdS/CFT & viscosity

- Policastro, Kovtun, Son & Starinets (PRL '01, '05): viscosity of N=4 SSYM ("toy model" of quark-gluon plasma).

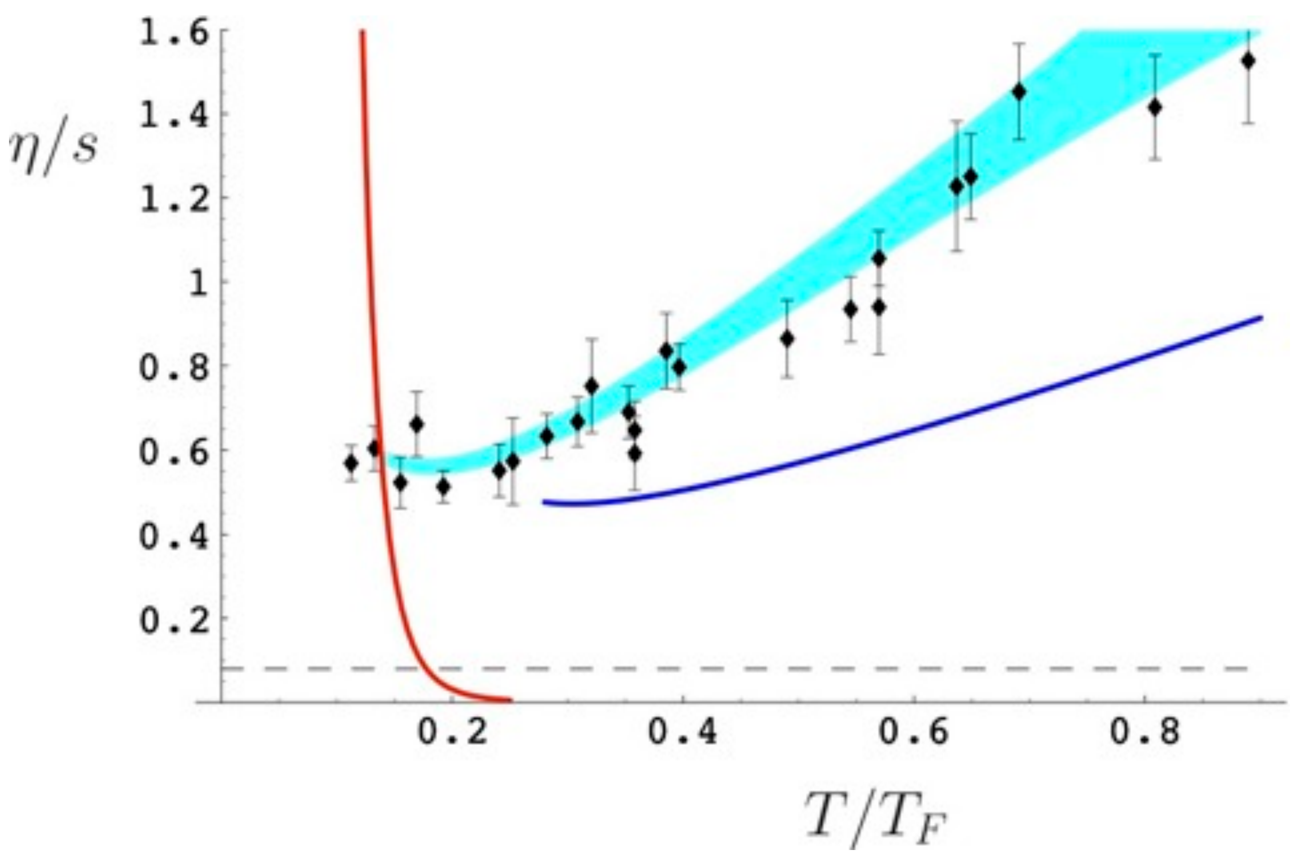
$$\eta/s = \hbar/4\pi k_B$$

- Kovtun, Starinets, and Son (2005) conjecture: (shear viscosity)/(entropy density) for any fluid.

$$\eta/s \geq \hbar/4\pi k_B$$

- Around the same time, experiments at RHIC on QGP & unitary Fermi gases reveal that these fluids come close to saturation.

Viscosity in unitary Fermi gases

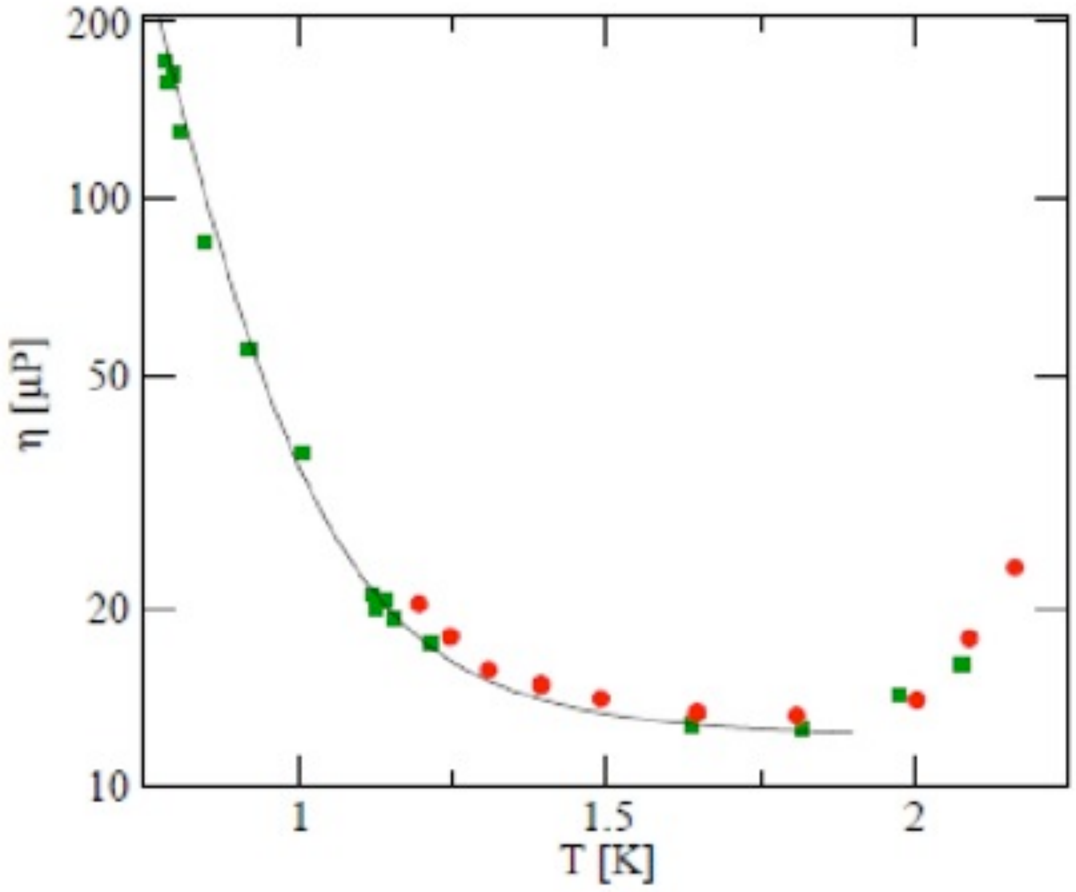


(Schaefer, arXiv:0808.0734)

- Early experiments: damping of collective modes (Kinast *et al*, '06).
- Kinetic theory: Bruun & Smith '05, Rupak & Schaefer '07:
  - High T: atoms $\eta \sim T^{3/2}$
  - Low T: phonons $\eta \sim T^{-5}$

- Viscosity minimum close to superfluid transition $T \sim 0.2T_F$.

Viscosity in superfluid ^{4}He



(Hollis-Hallett, 1955)

- Viscosity of *normal fluid.*
- Low-T phonon divergence; minimum close to $T_c \sim 2.17$K.

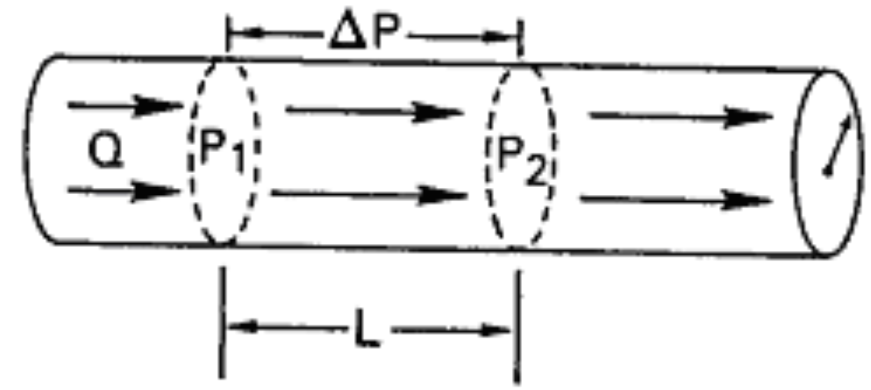
Experimental summary

Fluid	T[K]	η [Pa · s]	η/n [h]	η/s [h/k]
Water	370	2.9×10^{-4}	85	8.2
Helium-4	2	1.2×10^{-6}	0.5	1.9
Lithium-6 (unitarity)	23×10^{-6}	$\leq 1.7\times10^{-15}$	≤ 1	≤ 0.5
QGP	2×10^{12}	$\leq 5\times10^{11}$	-	≤ 0.4

Adapted from Schaefer & Teaney, Rep. Prog. Phys. '09

Poiseuille's law and Washington trees

UW campus tree



$$Q = \text{ flow rate } = \frac{\pi r^4}{8L} \frac{\Delta P}{\eta}$$

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- KSS: $\eta/s \sim 0.08$

Adapted from Schaefer & Teaney, Rep. Prog. Phys. '09

Viscosity bound?

- Known *theory* violations of bound: (Cohen '05; Brigante *et al.* '08, ...)
- May be unphysical (& baroque), though. No *known* physical systems violate bound.
- Our approach: use exact sum rules to learn about the spectral function. Make connections with long-standing problems in strongly correlated electronic systems.

Frequency-dependent transport: spectral functions & sum rules

Kubo II: Current vs. stress tensor correlators

$$\eta(\omega) = \lim_{q\to 0} \frac{\omega}{q^2} \mathrm{Im}\chi_T(\mathbf{q},\omega)$$

$$4\eta(\omega)/3 + \zeta(\omega) = \lim_{q\to 0} \frac{\omega}{q^2} \mathrm{Im}\chi_L(\mathbf{q},\omega)$$

- For non-relativistic applications, current correlation fnc. much more convenient than stress-energy tensor correlation fnc.

$$\eta(\omega) = \lim_{q\to 0} \mathrm{Im}\chi_\Pi^{xy,xy}(\mathbf{q},\omega)/\omega$$

$$\zeta(\omega) + 4\eta(\omega)/3 = \lim_{q\to 0} \mathrm{Im}\chi_\Pi^{xx,xx}(\mathbf{q},\omega)/\omega$$

- *j* is a simple operator. Π is not ($im[j_\alpha, H] = \partial_\beta \Pi_{\alpha\beta}$).

Relation between viscosity and normal fluid density

- At low *T*, η/ρ_n diverges. η goes to zero as ρ_n does however.

$$\eta = \lim_{\omega\to 0}\lim_{q\to 0}\frac{\omega}{q^2}\mathrm{Im}\chi_T(\mathbf{q},\omega) \qquad \rho_n = \lim_{q\to 0}\lim_{\omega\to 0} m^2\mathrm{Re}\chi_T(\mathbf{q},\omega)$$

- Spectral representation:

$$\rho_n = \frac{\pi m^2}{Z}\sum_{a,b}[e^{-\beta E_a} - e^{-\beta E_b}](E_b - E_a)\frac{|\langle b|\hat{j}^x_{\mathbf{q}}|a\rangle_T|^2}{q^2}\delta(\omega - E_b + E_a)$$

$$\eta = \frac{m^2}{Z}\sum_{a,b}\left(\frac{e^{-\beta E_a} - e^{-\beta E_b}}{E_b - E_a}\right)|\langle b|\hat{j}^x_{\mathbf{q}}|a\rangle_T|^2$$

$$\rho_n = 0 \Rightarrow \eta = 0$$

Viscosity sum rules

$$\int_0^\infty d\omega\, \omega^n\, \eta(\omega)$$

- **Sum rules:** exact results useful for:
  - Understanding experiments (e.g., neutron scattering, optical conductivity).
  - Constraining approximate calculations.
  - Proving rigorous results.

Deriving viscosity sum rules

- From Kramers-Kronig

$$\frac{1}{\pi}\int_0^\infty d\omega\eta(\omega) = \lim_{q\to 0} \frac{m^2\langle[\hat{j}^x_{-\mathbf{q}}, [\hat{H}, \hat{j}^x_{\mathbf{q}}]]\rangle_T}{2q^2}$$

$$\frac{1}{\pi}\int_0^\infty d\omega\left[\zeta(\omega)+\frac{4\eta(\omega)}{3}\right] = \lim_{q\to 0} \frac{m^2\langle[\hat{j}^x_{-\mathbf{q}}, [\hat{H}, \hat{j}^x_{\mathbf{q}}]]\rangle_L}{2q^2} - \frac{\rho c_s^2}{2}$$

- We obtain general sum rules for the shear and bulk viscosity in any non-relativistic system.

General viscosity sum rules

- General sum rules for any non-relativistic system:

$$\frac{1}{\pi}\int_0^\infty d\omega \eta(\omega) = \frac{\varepsilon}{3} - \frac{\langle \hat{V} \rangle}{3} + \frac{2\overline{V}'}{15} + \frac{\overline{V}''}{30}$$

$$\frac{1}{\pi}\int_0^\infty d\omega \zeta(\omega) = \frac{5\varepsilon}{9} + \frac{4\langle \hat{V} \rangle}{9} + \frac{5\overline{V}'}{9} + \frac{\overline{V}''}{18} - \frac{\rho c_s^2}{2}$$

- ε = energy density, <V> = potential energy, and

$$\overline{V}' \equiv \langle\!\langle p\,(\partial V/\partial p)\rangle\!\rangle \quad \overline{V}'' \equiv \langle\!\langle p^2\left(\partial^2 V/\partial p^2\right)\rangle\!\rangle$$

$$\langle\!\langle Q\rangle\!\rangle \equiv \frac{1}{2}\sum_{\substack{\mathbf{k}\mathbf{k}'\mathbf{p}\\ \sigma\sigma'}} Q\langle \hat{c}^\dagger_{\mathbf{k}+\mathbf{p}\sigma}\hat{c}^\dagger_{\mathbf{k}'-\mathbf{p}\sigma'}\hat{c}_{\mathbf{k}'\sigma'}\hat{c}_{\mathbf{k}\sigma}\rangle$$

Viscosity sum rules for dilute Fermi gases

- For arbitrary interactions ($k_F a$) and temperatures, in zero-range limit ($r_0 << k^{-1}_F$), one finds

$$\int_0^\infty d\omega\ \eta(\omega,\Lambda)/\pi = \frac{\varepsilon}{3} - \frac{C}{10\pi m a} + \frac{C\Lambda}{5\pi^2 m}$$

$$\int_0^\infty d\omega\ \zeta(\omega)/\pi = \frac{1}{72\pi m a^2}\left(\frac{\partial C}{\partial a^{-1}}\right)_s$$

- a = s-wave scat. length. Λ = momentum cutoff.

- C is the *contact:* probability of finding two fermions of opposite spin close to each other (Tan, ’05/’08, Braaten & Platter, ’08, Zhang & Leggett ’09):

$$\langle \hat{\psi}^\dagger_\uparrow(r)\hat{\psi}^\dagger_\downarrow(0)\hat{\psi}_\downarrow(0)\hat{\psi}_\downarrow(r)\rangle \simeq C\left(\frac{1}{r}-\frac{1}{a}\right)^2 \qquad r_0 \lesssim r \ll k_F^{-1}$$

Diverging η sum rule & High-frequency tails

$$\int_0^\infty d\omega\ \eta(\omega,\Lambda)/\pi = \frac{\varepsilon}{3} - \frac{C}{10\pi m a} + \frac{C\Lambda}{5\pi^2 m}$$

- Assuming that $\eta(\omega,\Lambda) = f(\omega)\Theta(\Lambda^2/m - \omega)$ gives

$$\int_0^{\Lambda^2/m} d\omega \left[\eta(\omega,\Lambda)/\pi - \frac{C}{10\pi\sqrt{m\omega}}\right] = \frac{\varepsilon}{3} - \frac{C}{10\pi m a}$$

$$\Rightarrow \eta(\omega\to\infty) = \frac{C}{10\pi\sqrt{m\omega}} \ \ \&\ \ S(\mathbf{q},\omega\to\infty) = \frac{2q^4 C}{15\pi^2 m^{1/2}\omega^{7/2}}$$

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- But this is wrong! (Thompson & Son, 2010 using OPE):

$$S(\mathbf{q},\omega\to\infty) = \frac{4q^4C}{45\pi^2 m^{1/2}\omega^{7/2}}$$

Contact & high-frequency tails

- $\omega^{n/2}$ tails: quantum effect of short-range (high-energy physics):

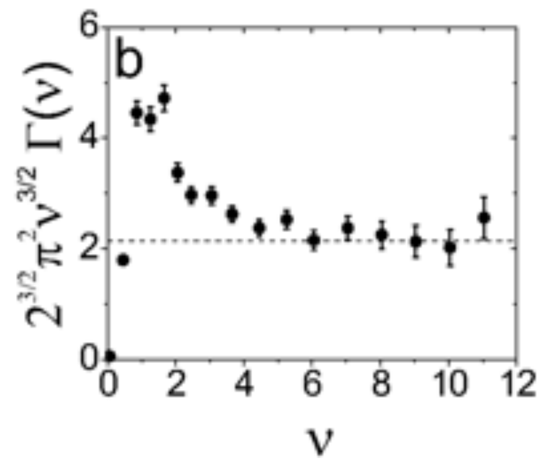
$$\frac{\hbar^2 k_F^2}{2m} \lesssim \hbar\omega \lesssim \frac{\hbar^2}{2m r_0^2}$$

- At large energies, momentum conservation ⇒ *virtual pair* excitation.

  Response ∝ prob. of finding two particles close to each other (*contact.*)

- Also RF response (Pieri et al, '09; Schneider & Randeria '10)

$$I_{\rm RF}(\omega \to \infty) = \frac{C}{4\pi^2 m^{1/2} \omega^{3/2}}$$



(Stewart *et al.* PRL '10)

- Generic to *any* probe (radio-frequency, Bragg, neutron scattering,...) in *any* quantum liquid as long as there is *some* separation of length scales.

S(q,ω) tail in ^{4}He

- *Even in ^{4}He!* (A. Griffin, *Excitations in a Bose condensed liquid, Cambridge*):

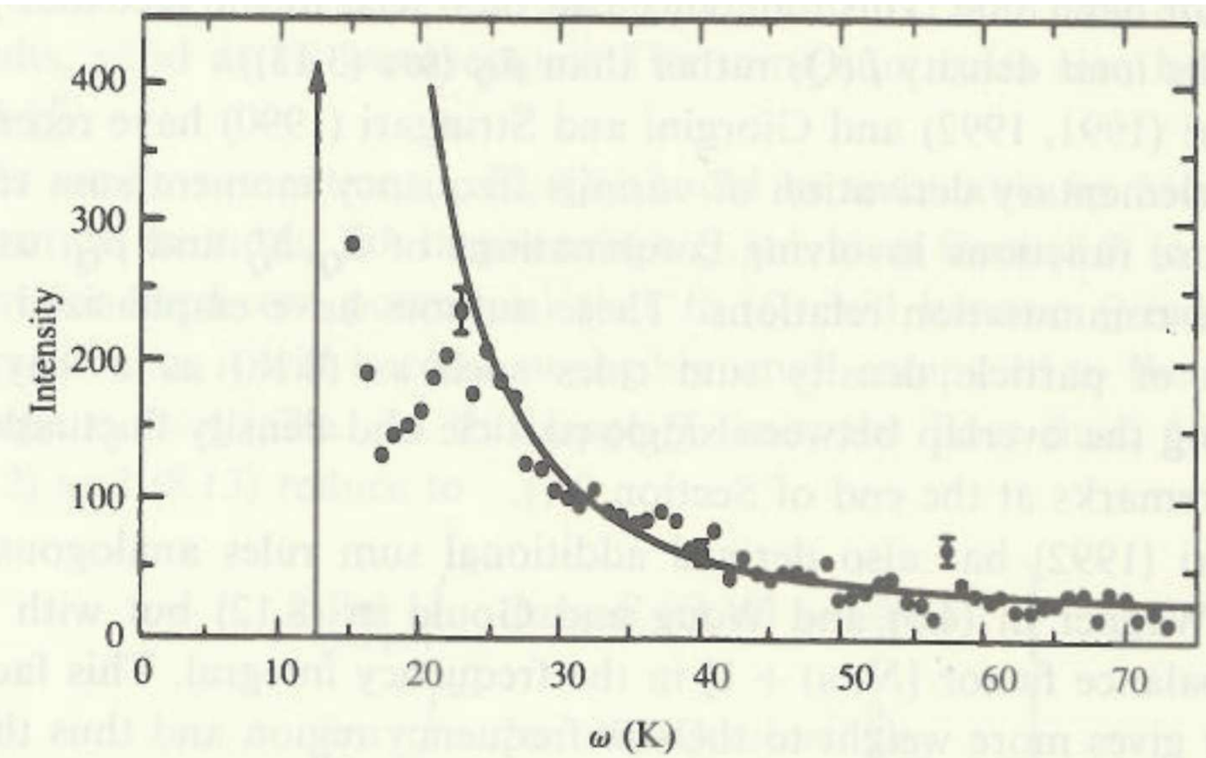


Fig. 8.1. Fit of the high-frequency tail of $S(\mathbf{Q},\omega)$ to the predicted $\omega^{-7/2}$ line shape, for $Q = 0.8$ Å^{-1}, $T = 1.2$ K. The neutron data are from Woods *et al.*, 1972 [Source: Wong, 1977].

- Small but non-zero separation of length scales ($n^{-1/3} \geq r_0$).

Regularizing shear sum rule

$$\int_0^\infty d\omega\, \eta(\omega,\Lambda)/\pi = \frac{\varepsilon}{3} - \frac{C}{10\pi m a} + \frac{C\Lambda}{5\pi^2 m}$$

- We assumed that $\eta(\omega,\Lambda) = f(\omega)\Theta(\Lambda^2/m - \omega)$
- It turns out that $\eta(\omega \to \infty, \Lambda) = \frac{CF(\omega/\Lambda)}{\sqrt{m\omega}}\Theta(\Lambda^2/m - \omega)$
- Enss, Haussmann, & Zwerger '11: Sum rule for zero-range correlator:

$$\int_0^\infty d\omega\, \eta(\omega,\Lambda \to \infty)/\pi = \frac{\varepsilon}{3} - \frac{C}{12\pi m a} + \frac{2C\Lambda}{15\pi^2 m}$$

- Diff. between sum rule for 0-range correlator and 0-range limit of sum rule?
- Either way, **Low-ω integral = energy density/3.**

Bulk viscosity sum rule

$$\int_0^\infty d\omega \, \zeta(\omega)/\pi = \frac{1}{72\pi m a^2}\left(\frac{\partial C}{\partial a^{-1}}\right)_s$$

- Positive definite: $\zeta(\omega) \geq 0 \Rightarrow$
  - Contact is monotonically increasing through the crossover (Also, Werner & Castin, '10 ?):

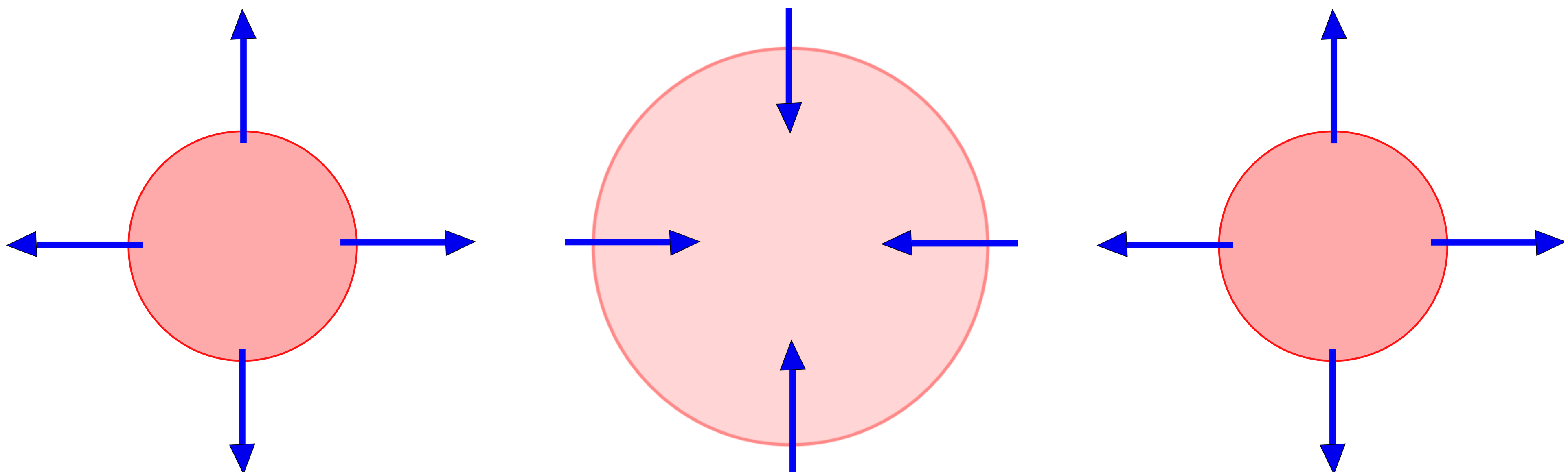
$$\partial C/\partial a^{-1} \geq 0 \;\; \forall a$$

  - Bulk viscosity is zero at unitarity at all frequencies:

$$\zeta(\omega) = 0 \;\; \forall \omega \;\; (|a| = \infty)$$

  - Generalizes Son, PRL '07: $\zeta(0)=0$ at unitarity

Vanishing bulk viscosity & scale invariance



- Bulk viscosity = dissipation after uniform dilation

- *Scale invariance* at unitarity (Werner & Castin, PRA , '06):

$$r \to \lambda r, \quad \psi(r_1, ..., r_N) \to \psi(\lambda r_1, ..., \lambda r_N)/\lambda^{3N/2}$$

remains an eigenstate of the Hamiltonian

- Unitary Fermi gas never leaves equilibrium under uniform dilation at unitarity: $\zeta(\omega) = 0$.

Other sum rules

- Unitary Fermi gas (*F* contains non-universal physics):

$$\int_0^\infty d\omega\, [\eta(\omega)/\pi - CF(\omega)/\sqrt{m\omega}] = \varepsilon/3 \qquad \int_0^\infty d\omega\, \zeta(\omega)/\pi = P - \varepsilon/9 - \rho c_s^2/2 = 0$$

- Unitary Fermi gas, kinetic theory (Braby, Chao, & Schaefer '11):

$$\int_0^\infty d\omega\, \eta(\omega)/\pi = \varepsilon/3$$

- N=4 SS Yang-Mills (Romatschke & Son, '09)

$$\int_0^\infty d\omega\, [\eta(\omega) - \eta_{T=0}(\omega)]/\pi = \varepsilon/5$$

- Yang-Mills (Romatschke & Son, '09)

$$\int_0^\infty d\omega\, [\zeta(\omega) - \zeta_{T=0}(\omega)]/\pi = (3\varepsilon + P)(1 - 3c^2) - 4(\varepsilon - 3P) \qquad c^2 = \partial P/\partial\varepsilon$$

Sum rules and bounds on DC viscosity: Lessons from strongly correlated electronic systems

Viscosity bound?

$$\eta \geq s\hbar/4\pi k_B$$

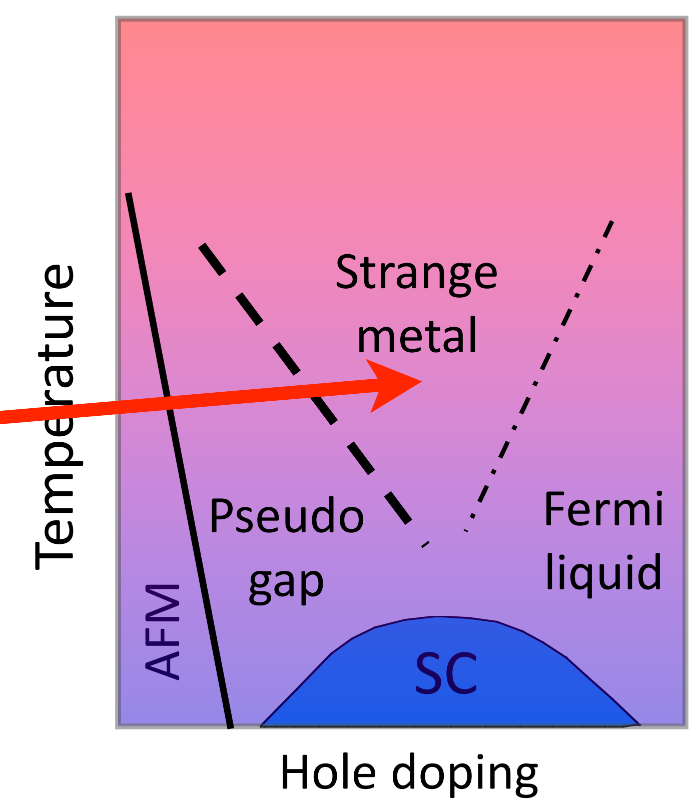
- Enormous theoretical and experimental effort devoted to determining value of η/s.
- Arguably, the most remarkable feature of this result is not the smallness of η/s, it is the fact that the viscosity is proportional to an equilibrium thermodynamic quantity.
- Sum rules suggest that this is a consequence of having *maximally incoherent quasiparticles*.

Maximally incoherent quasiparticles

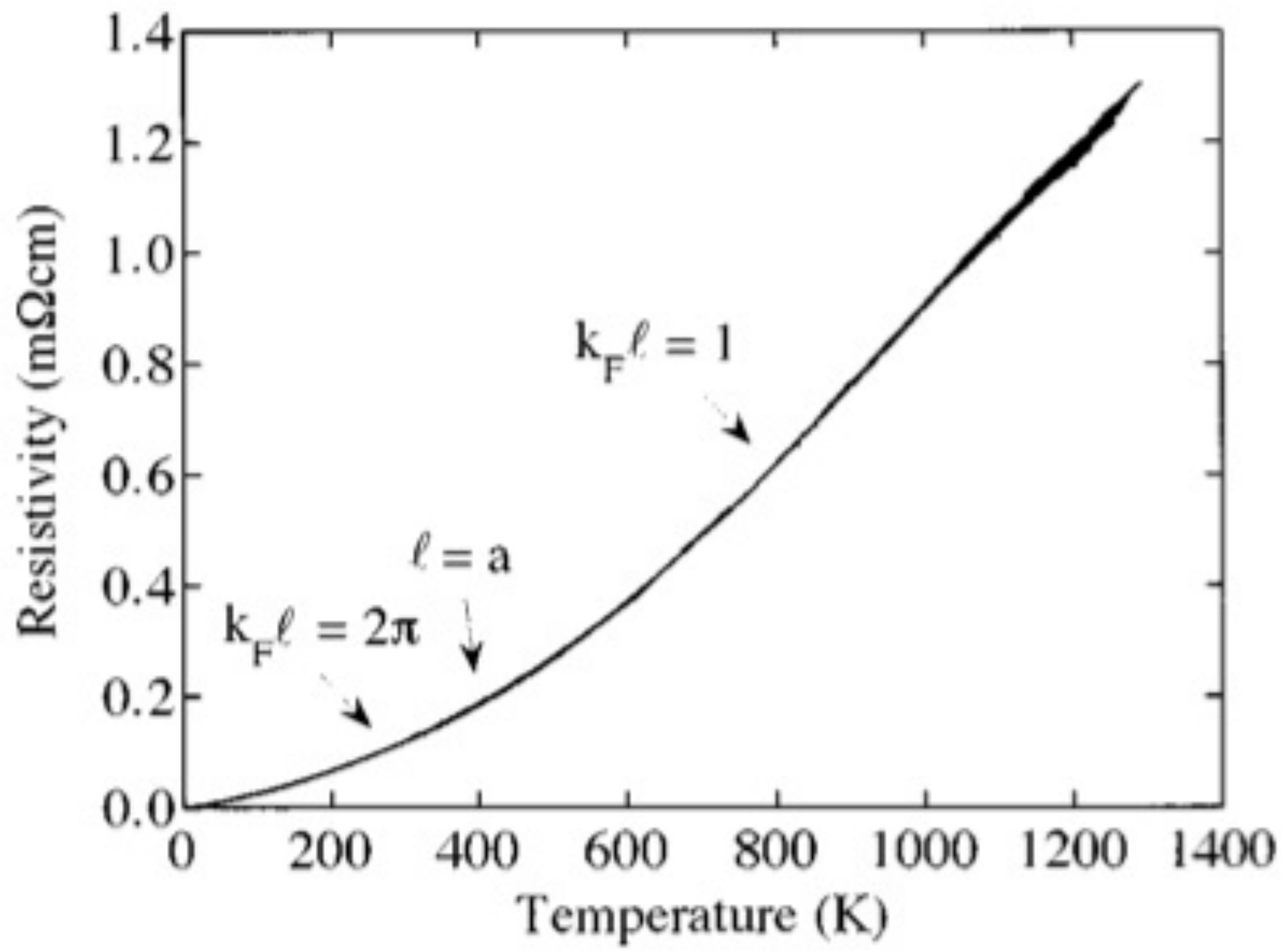
- Strong interactions ⇒ short-lived quasiparticles.
- *How short-lived?* Experience suggests $1/\tau \geq T$:
  - E.g., above quantum critical point (Damle, Sachdev, 98)
  - Normal phase of cuprates. ("Marginal Fermi liquid"; Varma et al, 1989)

$$\mathrm{Im}\Sigma(\mathbf{k},\omega) \sim \tau^{-1} \sim \max(\omega, T)$$

- Primary manifestation: *Linear resistivity in "strange metal" phase.*

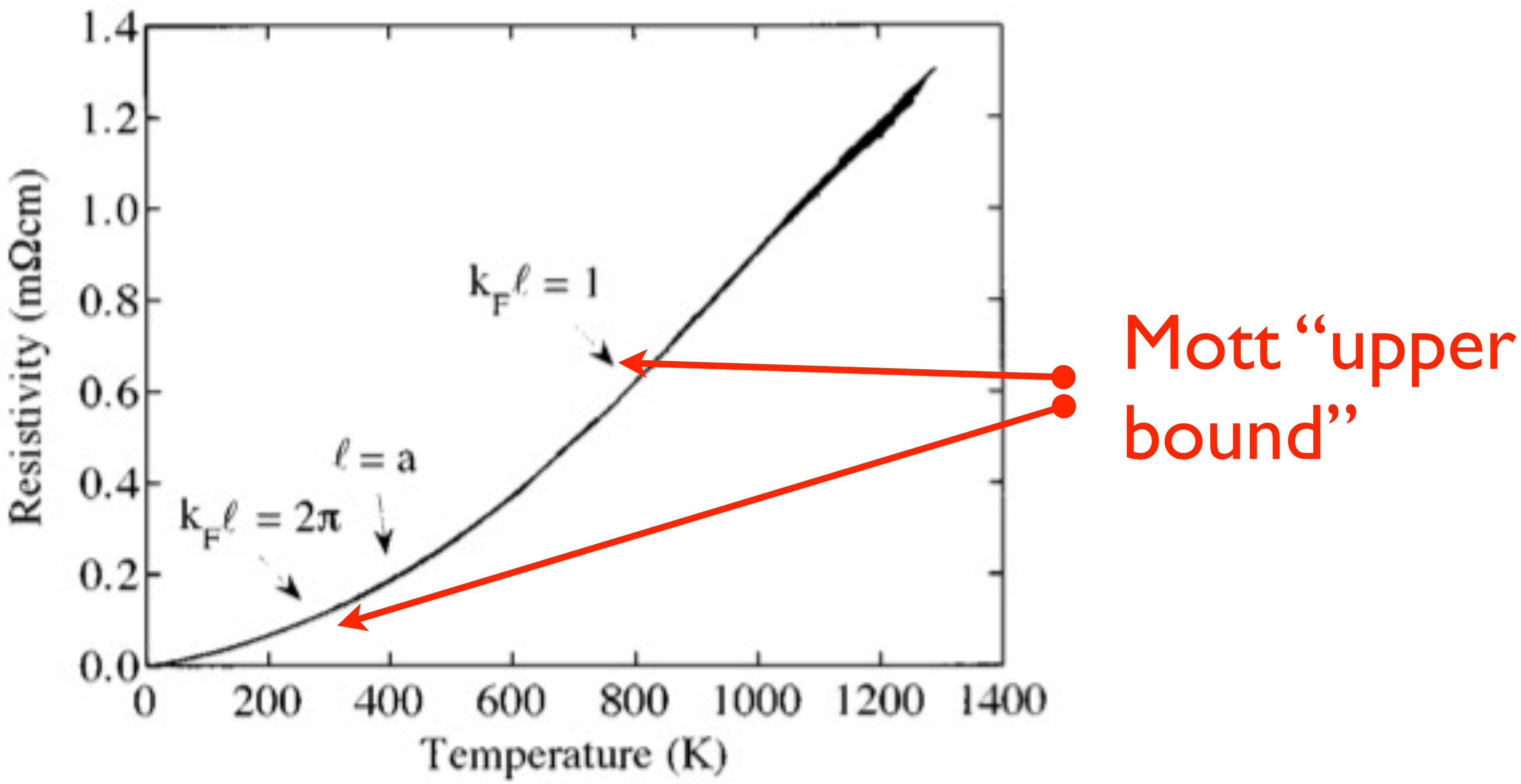


Linear resistivity



Linear resistivity Sr_2Ru0_4 (Tyler *et al*, PRB '98)

Linear resistivity



Linear resistivity Sr_2Ru0_4 (Tyler *et al*, PRB '98)

- Mott minimum conductivity (1/ρ):"mean-free path"*
  l ≤ inter-particle spacing.

* lattice spacing

Conductivity: Kubo formulae & sum rules

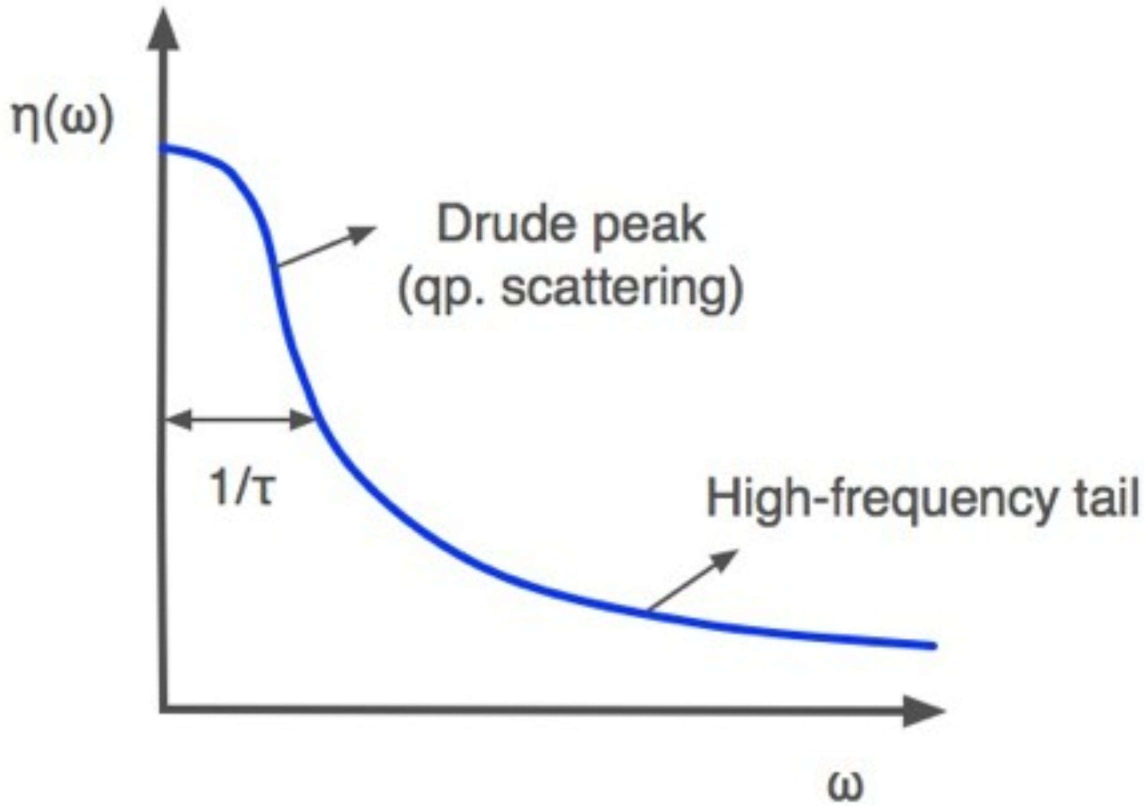
- Optical conductivity has similar structure as viscosity

$$\eta(\omega) = \lim_{q\to 0} \frac{\omega}{q^2} \mathrm{Im}\chi_T(\mathbf{q},\omega) \qquad 4\eta(\omega)/3 + \zeta(\omega) = \lim_{q\to 0} \frac{\omega}{q^2} \mathrm{Im}\chi_L(\mathbf{q},\omega)$$

$$\sigma(\omega) = \lim_{q\to 0} \frac{e^2}{\omega} \mathrm{Im}\chi_{T,L}(\mathbf{q},\omega)$$

$$\int_{-\infty}^{\infty} \sigma(\omega)/\pi = ne^2/m$$

Spectral phenomenology above T_c



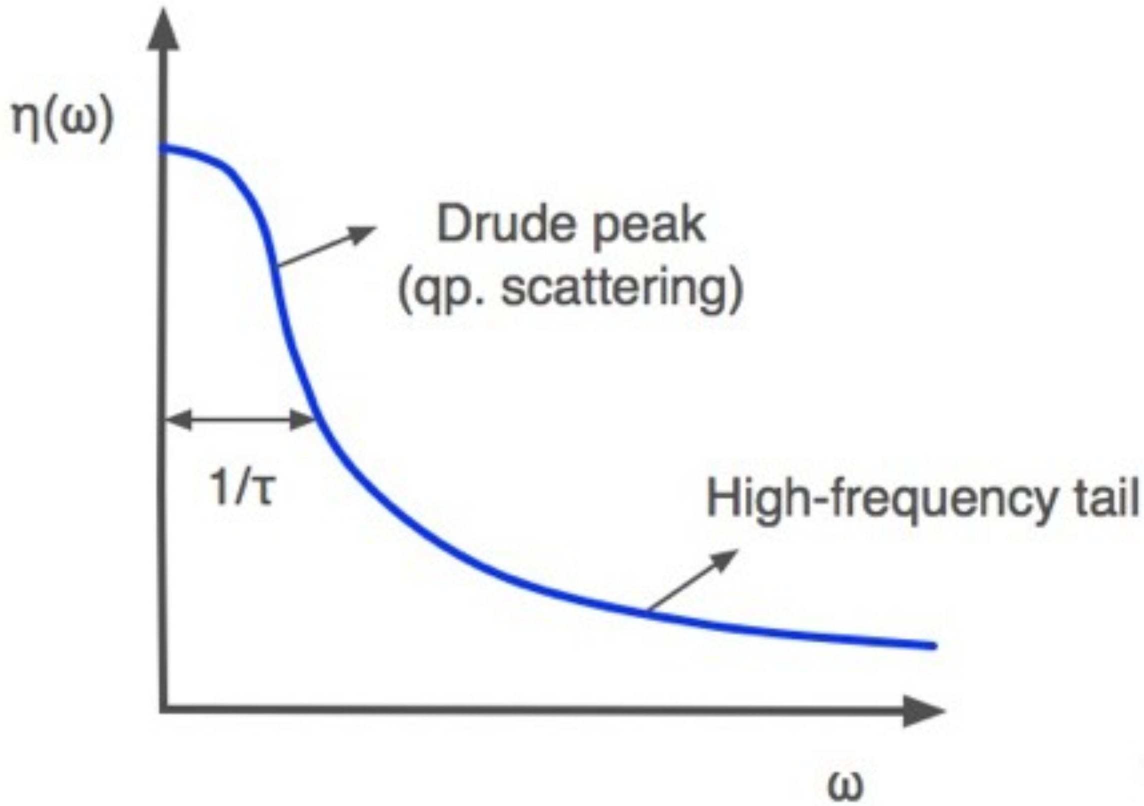
$$\sigma(\omega) = \frac{\sigma_{\mathrm{DC}}}{1 + (\omega\tau)^2}$$

$$\sigma_{\mathrm{DC}}(T) = \tau \left(\int_{-\Omega_c}^{\Omega_c} d\omega \sigma(\omega)/\pi \right) = \frac{ne^2}{m^*(T)} \tau(T)$$

$$\Omega_c \lesssim \text{bandwidth} \;\; \& \;\; \tau(T) \sim 1/T \Rightarrow$$

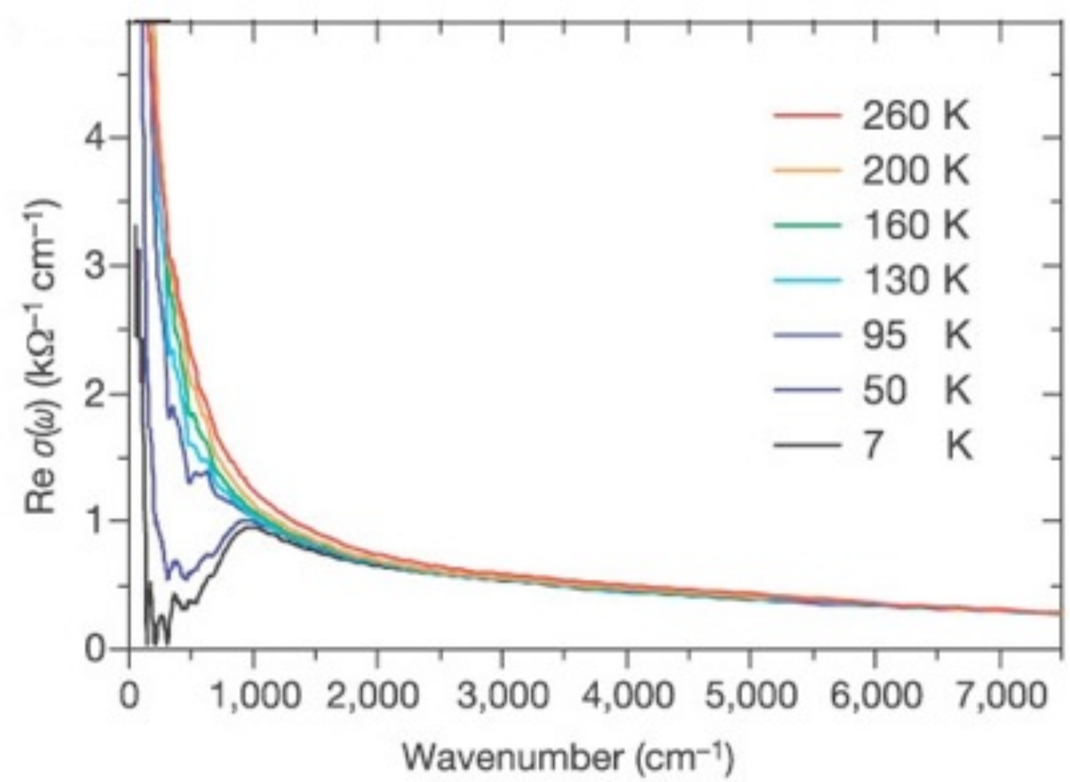
$$\rho(T) \equiv \frac{1}{\sigma_{\mathrm{DC}}(T)} \sim \left(\frac{m^*}{ne^2} \right) T$$

Spectral phenomenology above T_c



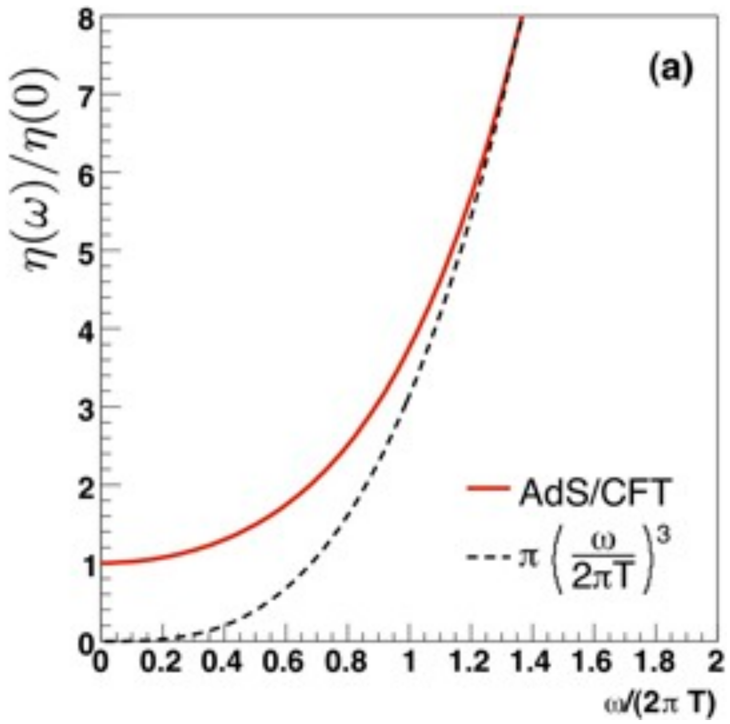
$$\sigma(\omega) = \frac{\sigma_{\rm DC}}{1 + (\omega\tau)^2}$$

$$\rho(T) \equiv \frac{1}{\sigma_{\rm DC}(T)} \sim \left(\frac{m^*}{ne^2}\right) T$$

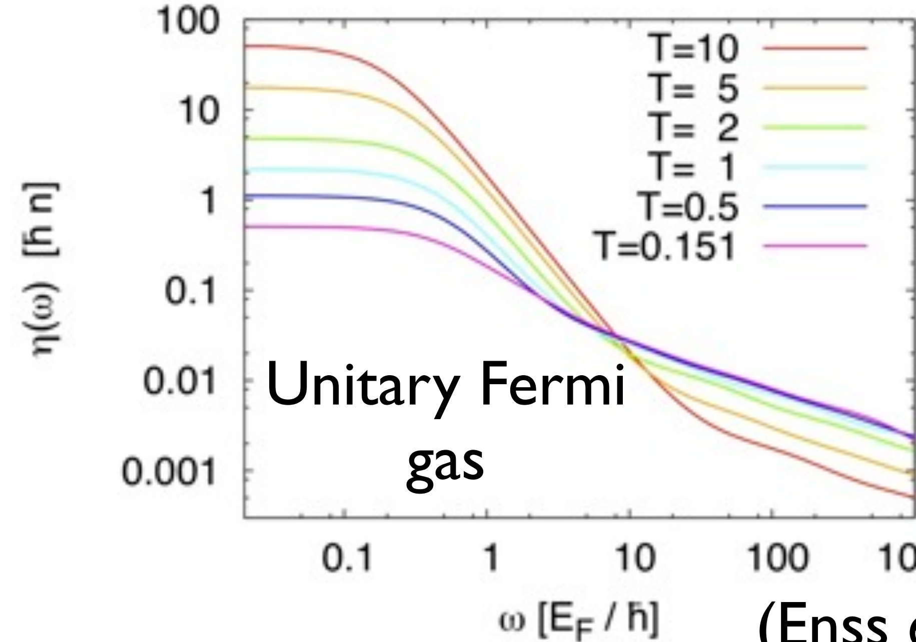


Optical conductivity at optimal doping, (Van Der Marel *et al*, Nature '03)

Sum rule estimate of viscosity



(Teaney, PRD '06)



(Enss *et al*, '11)

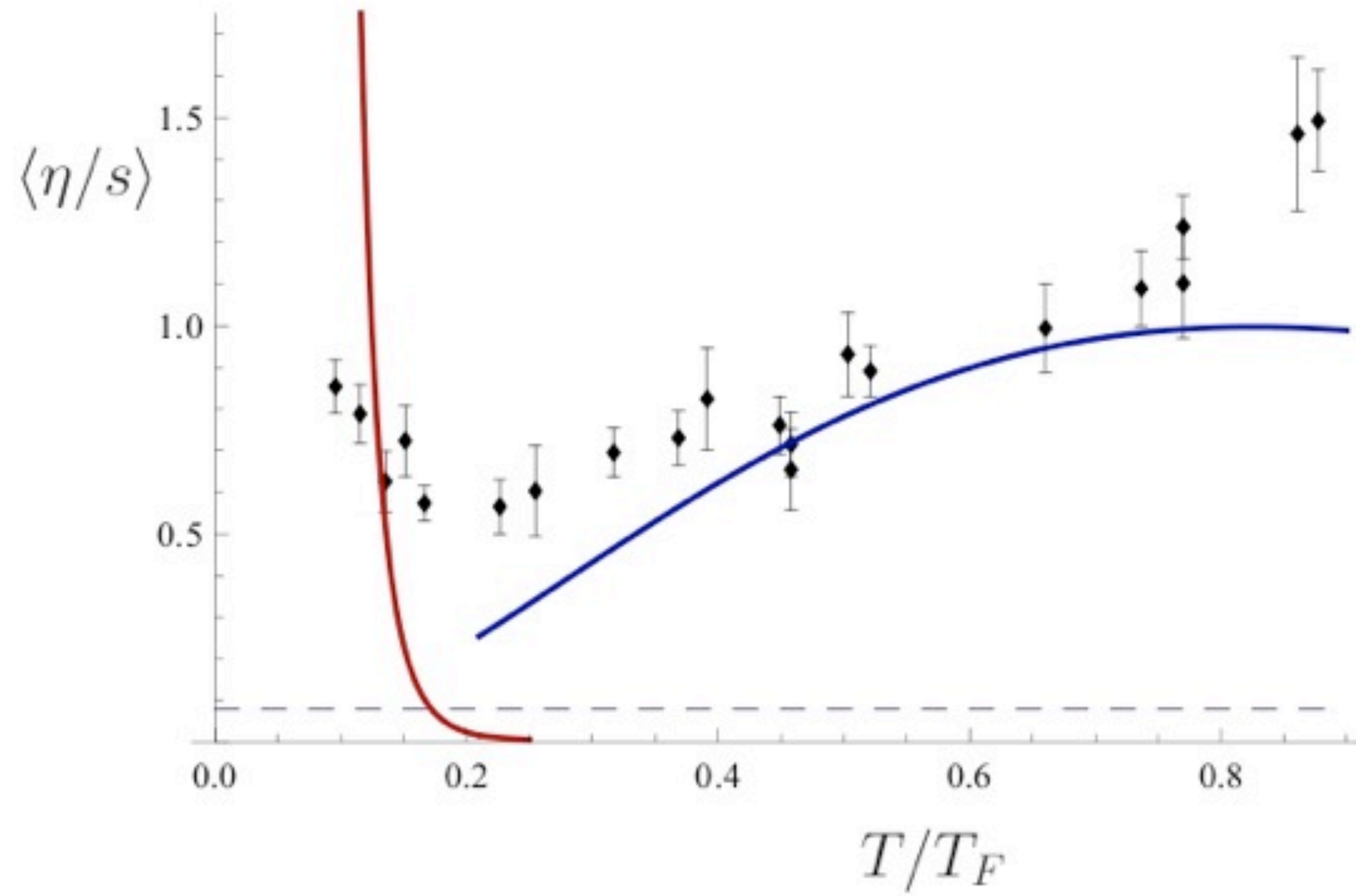
- N=4 SSYM and unitary Fermi gas: area under "Drude peak" = energy density ε:

$$\int_0^\infty [\eta(\omega) - A\omega^3]/\pi = \varepsilon/5 \qquad \int_0^\infty d\omega\, [\eta(\omega)/\pi - CF(\omega)/\sqrt{m\omega}] = \varepsilon/3$$

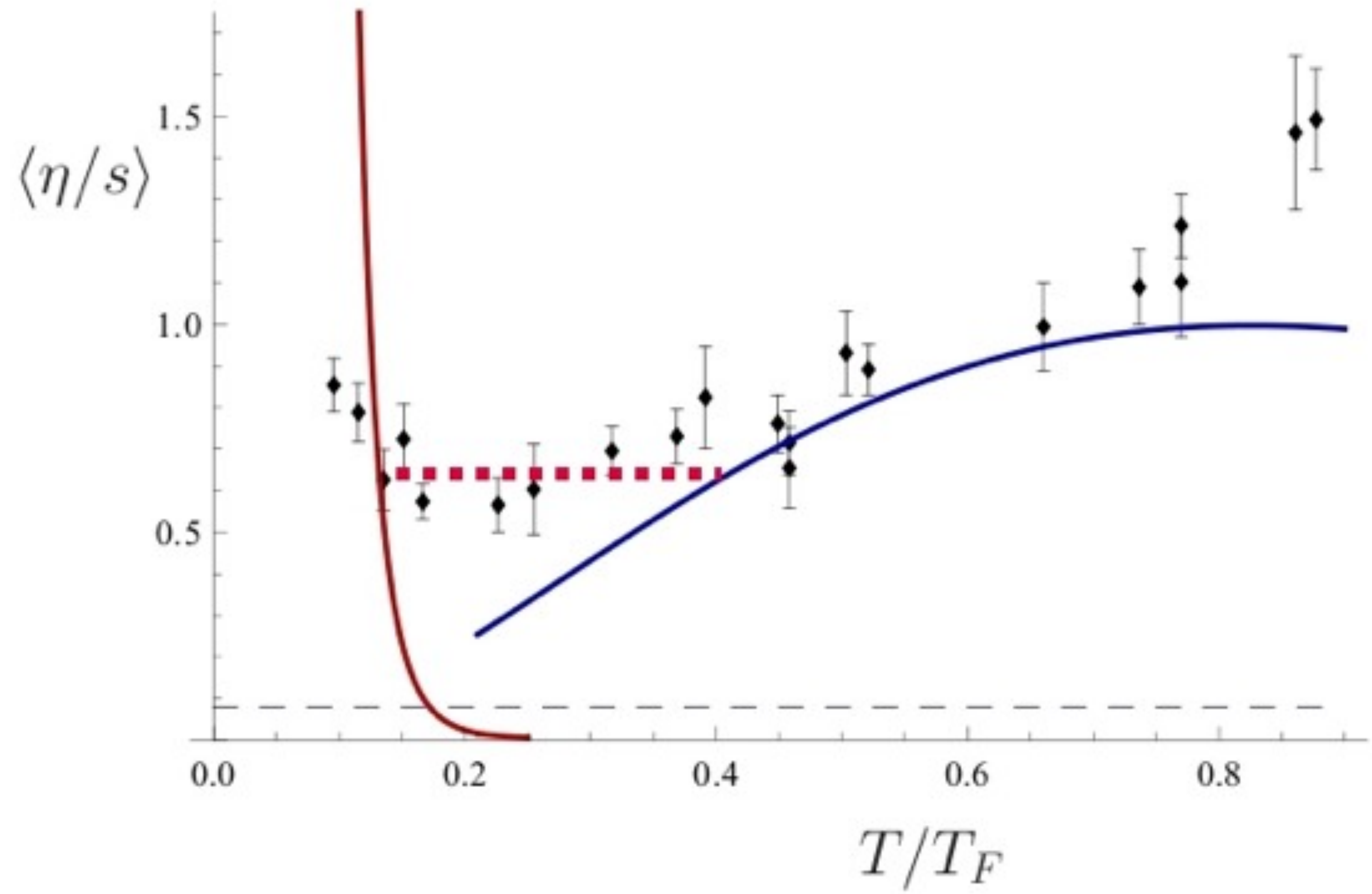
- Width of peak ~ *T*. ⇒ η(0) ~ ε/T ~ s.

- *η ~ ε/T in strongly correlated systems.* (Including unitary Fermi gas.) *Analogue of linear resistivity in strongly correlated electronic systems.*

Is η(T) ∝ s(T) in a unitary Fermi gas?



Is η(T) ∝ s(T) in a unitary Fermi gas?



- Maybe. Relevant *T* range: $0.2 < T/T_F < 0.5$.

- Would be interesting to compare η with ε/T instead of s.

Summary: Exact results from spectral functions & sum rules

- Vanishing normal fluid density ⇒ $\eta=0$.
- Exact sum rules for the shear η and bulk ζ viscosities.
- High frequency tails for $\eta(\omega)$ and $S(q,\omega)$.
- $\zeta(\omega) = 0$ at all ω and T at unitarity.
- Sum rules suggest signature of maximally incoherent quasiparticles: $\eta \sim \varepsilon(T)/T$.