INT Symposium May 15-20, 2011

Energy of a trapped Fermi gas with large scattering length

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Summary

- Contact
- Some previous results involving the contact
- A few new results in an external potential

Contact Interaction

$$T \lesssim 1\,\mu\mathrm{K}, \quad n \lesssim 10^{-6} n_{\mathrm{air}}$$

$$\lambda_{dB} \gtrsim 1\,\mu\mathrm{m}, \quad n^{-1/3} \gtrsim 1\,\mu\mathrm{m}$$

Range of interaction: $r_0 \sim 1\,\mathrm{nm}$

Atoms in the same spin state: noninteracting

Atoms in different spin states: interact with a

$$\lambda_{dB}, n^{-1/3}, |a| \gg r_0$$

$$\Rightarrow \; r_0 = 0$$

Contact

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \left(\frac{1}{r_{ij}} - \frac{1}{a}\right) A\left(\frac{\mathbf{r}_i + \mathbf{r}_j}{2}; \mathbf{R}'\right) + O(r_{ij})$$

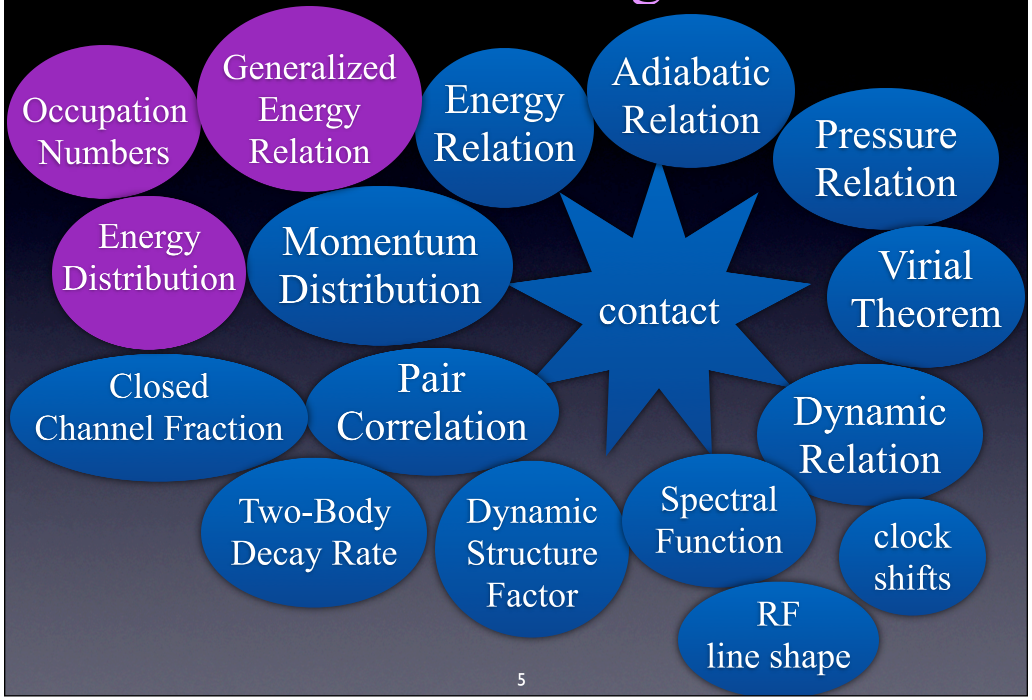
$$r_{ij} \equiv |\mathbf{r}_i - \mathbf{r}_j| \to 0 \quad (i, j : \textbf{different spin states})$$

Pair correlation: $$\langle n_\uparrow n_\downarrow \rangle = \frac{C(\mathbf{x})}{(4\pi r)^2}, \quad r \to 0$$

$$C(\mathbf{x}) \propto \int |A(\mathbf{x}, \mathbf{R}')|^2 d^{3N-6} R' : \ \text{Contact Density}$$

$$\mathcal{I} \equiv \int C(\mathbf{x}) d^3 x : \ \text{Contact}$$

Some results involving the contact



Momentum Distribution

$$n_{\mathbf{k}\sigma} = \frac{C}{k^4}, \quad k \to \infty$$

$C = \mathcal{I}/\mathrm{volume}$

Closed channel fraction

$$N_{\text{closed}} = s\mathcal{I}$$

s : from atomic physics

Two-body decay rate

$$\frac{dN_\sigma}{dt} = -v\mathcal{I}$$

if two atoms can scatter into other spin states with much lower energy

v : from atomic physics

Energy Relation

$$E = E_K + E_U + E_V$$

(kinetic+interaction+trapping)

$$E_K = \sum_{\mathbf{k}\sigma} \frac{\hbar^2 k^2}{2m} n_{\mathbf{k}\sigma} \sim +\infty \mathcal{I} \qquad E_U \sim -\infty \mathcal{I}$$

Internal energy = linear functional of momentum distribution

$$E_{\text{internal}} \equiv E_K + E_U = \sum_{\mathbf{k}\sigma} \frac{\hbar^2 k^2}{2m} \left(n_{\mathbf{k}\sigma} - \frac{C}{k^4} \right) + \frac{\hbar^2 \mathcal{I}}{4\pi m a} \quad \text{(finite)}$$

$$C = \mathcal{I}/\text{volume}$$

$$E_V = \int n(\mathbf{x}) V(\mathbf{x}) d^3 x$$

S. Tan, Ann. Phys. 2008

Braaten and Platter, PRL 2008

Adiabatic Relation

$$\frac{dE}{d(-1/a)}\Big|_{\text{adiabatic}} = \frac{\hbar^2 \mathcal{I}}{4\pi m}$$

$\mathcal{I}$ is a thermodynamic quantity

$\mathcal{I}$ → Energy, Entropy, Pressure, ...

Dynamic Relation

$$\frac{dE}{dt} = \frac{\hbar^2 \mathcal{I}(t)}{4\pi m} \frac{d[-a^{-1}(t)]}{dt}$$

S. Tan, Ann. Phys. 2008

Braaten and Platter, PRL 2008

Pressure Relation

Uniform System: $$P = \frac{2}{3}\rho_E + \frac{\hbar^2}{12\pi m a}C$$

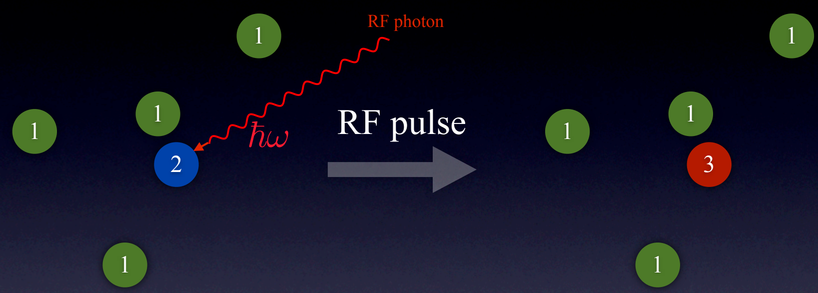
Generalized Virial Theorem

Harmonically Trapped System: $$E - 2E_V = -\frac{\hbar^2 \mathcal{I}}{8\pi m a}$$

S. Tan, Ann. Phys. 2008

Braaten and Platter, PRL 2008

Clock Shifts



$$\overline{\omega} - \omega_0 = \frac{\hbar}{4\pi m N_2}\left(\frac{1}{a_{12}} - \frac{1}{a_{13}}\right)\mathcal{I}$$

Dynamic Relation $\Delta E_{\mathrm{abrupt}} = \frac{\hbar^2 \mathcal{I}}{4\pi m}\Delta(-a^{-1})$

Punk and Zwerger, PRL 2007

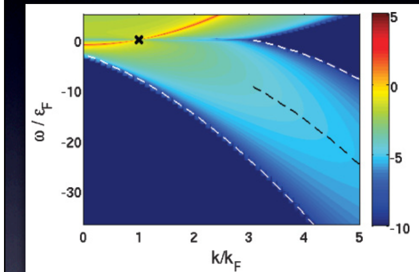
Baym, Pethick, Yu, and Zwierlein, PRL 2007

Schneider, Shenoy, and Randeria, arXiv:0903.3006

Zhang and Leggett, PRA 2009

Spectral Function

$\Psi \propto 1/r_{\uparrow\downarrow}, \quad r_{\uparrow\downarrow} \to 0$ → High-k atoms are paired:
$\{\mathbf{k}\uparrow, \mathbf{k}'\downarrow\} \quad (\mathbf{k}' \approx -\mathbf{k})$



Schneider and Randeria, PRA 2010

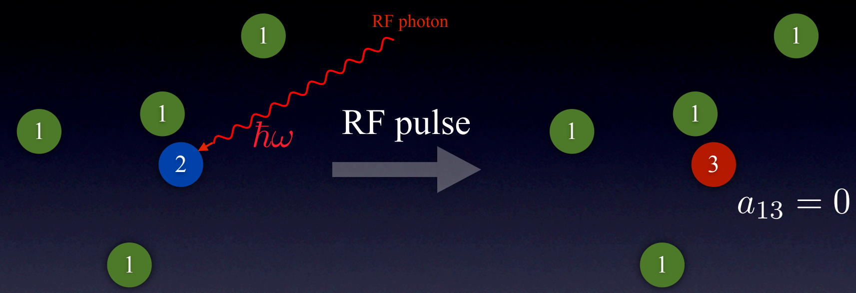
$$A(k,\omega) \approx \frac{C}{k^4}\delta\Big(\omega + \frac{\hbar k^2}{2m}\Big), \;\; \omega < 0, \;\; k \gg k_F$$

Bending back of the dispersion relation even in the normal state

Bending back seen in Photoemission Spectroscopy
Stewart, Gaebler, and Jin, Nature 2008.

Combescot et al, PRA 2009
Schneider and Randeria, PRA 2010

High-Frequency Tail of the RF spectrum



RF Transition Rate $\propto \mathcal{I}(\omega - \omega_0)^{-3/2}$ at $\omega - \omega_0 \gg E_F/\hbar$

Perali, Pieri, and Strinati, PRL 2008

Schneider, Shenoy, and Randeria, arXiv:0903.3006

Schneider and Randeria, PRA 2010

Dynamic Structure Factor

Solving two-body problem at $\omega,\ \dfrac{\hbar q^2}{2m} \to \infty,$

$$S(q,\omega) = \frac{C}{\pi}\sqrt{\frac{\hbar}{m\omega^3}}\,f(x), \quad 0 < x \equiv \frac{\hbar q^2}{2m\omega} < 2, \quad x \neq 1$$

$$S(\mathbf{q},\omega) \equiv \sum_{\sigma\sigma'}\int d^3r dt\, e^{i\omega t - i\mathbf{q}\cdot\mathbf{r}}\langle n_\sigma(\mathbf{r},t)n_{\sigma'}(0,0)\rangle$$

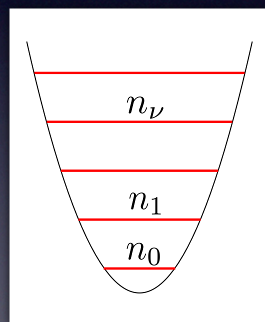
$$f(x) = \frac{\sqrt{1-x/2}}{(1-x)^2} + \frac{1}{\sqrt{2x}}\ln\frac{1+\sqrt{2x-x^2}}{|1-x|}$$
$$+\frac{1}{x\sqrt{1-x/2}}\left[\pi^2\theta(x-1) - \ln^2\frac{1+\sqrt{2x-x^2}}{|1-x|}\right]$$

Son and Thompson, PRA 2010

Taylor and Randeria, PRA 2010

Previous: $E = \sum_{\mathbf{k}\sigma} \eta(\mathbf{k}) \frac{\hbar^2 k^2}{2m} n_{\mathbf{k}\sigma} + \int V(\mathbf{x}) n(\mathbf{x}) d^3x$

Alhassid's Question



In a harmonic trap,
is *E* a functional of occupation numbers of energy levels?

$$E \stackrel{?}{=} E[n_\nu]$$

Yoram Alhassid
(*Yale Univ*)

The Answer: YES, $E = E[n_\nu]$

Moreover, this is valid for any smooth potential $V(\mathbf{x})$ having a lower bound, not just harmonic traps!

Generalized Energy Relation:

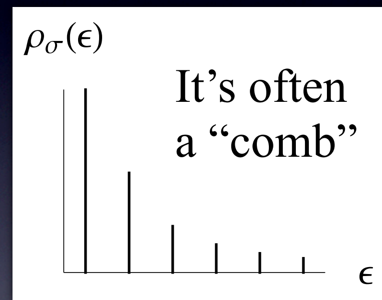
$$E = \frac{\hbar^2 \mathcal{I}}{4\pi m a} + \lim_{\epsilon_{\max}\to\infty} \left(\sum_{\epsilon_\nu < \epsilon_{\max}} \epsilon_\nu n_\nu - \frac{\hbar \mathcal{I}}{\pi^2} \sqrt{\frac{\epsilon_{\max}}{2m}} \right)$$

ϵ_ν : ν-th energy level in the potential

$n_\nu = \sum_\sigma n_{\nu\sigma}$

Energy Distribution Function

$$\rho_\sigma(\epsilon) \equiv \sum_\nu n_{\nu\sigma}\delta(\epsilon - \epsilon_\nu)$$



$$\rho_\sigma(\epsilon)\big|_{\text{coarse-grained}} = \frac{\hbar \mathcal{I}}{4\pi^2\sqrt{2m}}\epsilon^{-3/2} + O(\epsilon^{-5/2}), \quad \epsilon \to \infty$$

Occupation Numbers

$$n_{\nu\sigma} = \frac{1}{k_\nu^4} \int C(\mathbf{x}) \left| \phi_\nu(\mathbf{x}) \right|^2 d^3x, \quad \epsilon_\nu \to \infty$$

$$k_\nu = \frac{\sqrt{2m\epsilon_\nu}}{\hbar}$$

$\phi_\nu(\mathbf{x})$: wave function of the ν-th energy level (normalized)

A remark about the derivation

These 3 relations can be derived from the expansion of

$$\langle \psi^{\dagger}_{\sigma}(\mathbf{x}) \psi_{\sigma}(\mathbf{x}+\mathbf{r}) \rangle$$

at $r \to 0$, which was found previously

Possible applications of the 3 relations

- Provide a robust benchmark for theories of trapped Fermi gases
- Provide a robust benchmark for numerical simulations of trapped Fermi gases
- Determine the relative importance of high-lying energy levels
- Influence the theory of Fermi gases in optical lattices