Tkachenko modes and correlation functions for rapidly rotating bosons

Gora Shlyapnikov

LPTMS, Orsay, France
University of Amsterdam

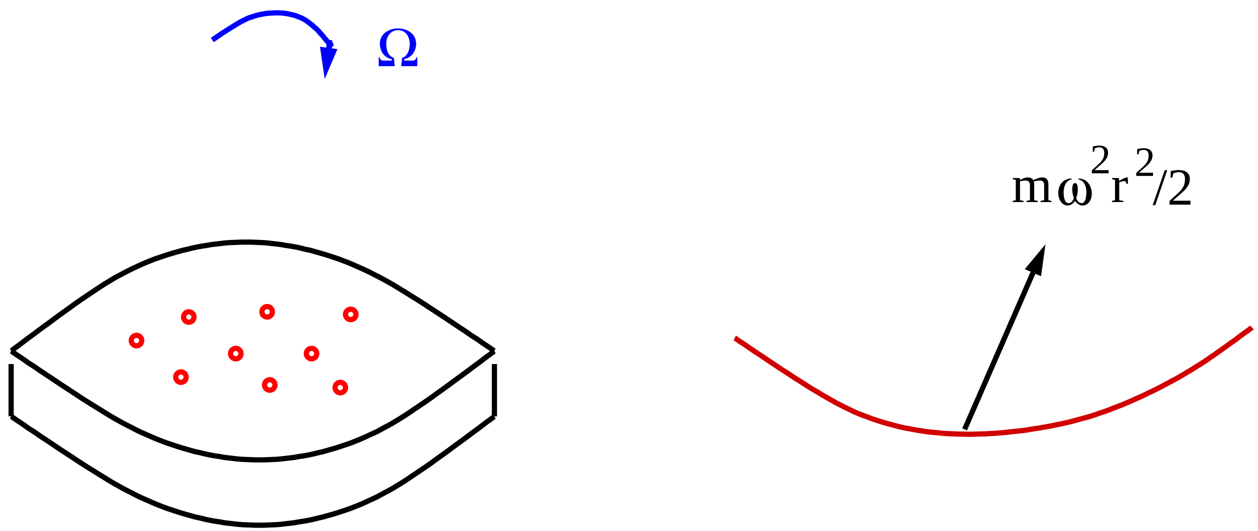
Outline

- Introduction
- Excitation spectrum
- Damping rates
- One-bode density matrix
- Conclusions and outlook

Collaborations: S.I. Matveenko (Landau Institute, Chernogolovka)

Seattle, April 29, 2011

Rapidly rotating bosons. Experiment and theory



$\Omega < \omega$, but $|\Omega - \omega| \ll \Omega \Rightarrow$ Lowest Landau Level (LLL)

Vortex lattice. Mean-field Quantum Hall regime.

Number of vortices is much smaller than the number of particles

JILA experiment (E. Cornell group) and ENS experiment (J. Dalibard group)

Theoretical studies based on the GP equation and hydrodynamic approach

Last 10 years Ho, Baym, Fetter, Cooper, Aftalion, Dalibard, Sonin, etc.

GP approach

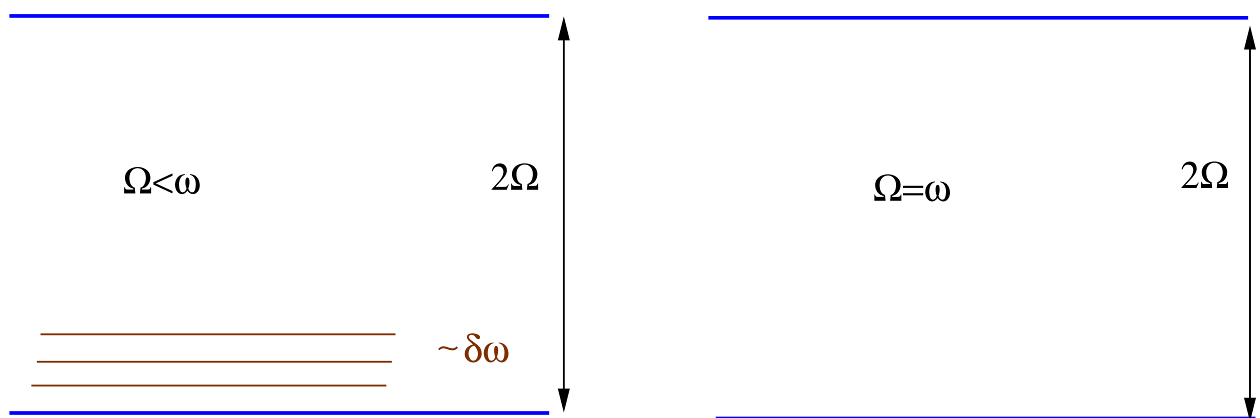
$$\hat{H} = \int d^2\mathbf{r} \left[\hat{\psi}^\dagger \frac{\hat{\mathbf{p}}^2}{2m} \hat{\psi} + \frac{g}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} + V(\mathbf{r}) \hat{\psi}^\dagger \hat{\psi} - \Omega \hat{\psi}^\dagger \hat{L} \hat{\psi} \right]$$

Ground state $\quad \frac{\hat{\mathbf{p}}^2}{2m} \Psi_0 + g|\Psi_0|^2 \Psi_0 + V(\mathbf{r}) \Psi_0 - \Omega \hat{L} \Psi_0 = \mu \Psi_0$

Single-particle Hamiltonian $\; H = (\hat{\mathbf{p}} - \mathbf{A})^2/2 + m(\omega^2 - \Omega^2) r^2/2; \;\; \mathbf{A} = m[\boldsymbol{\Omega} \times \mathbf{r}]$

$\delta\omega = (\omega - \Omega) \ll \Omega$ and $ng \ll 2\hbar\Omega \Rightarrow$ LLL $\Rightarrow \Psi_0(\mathbf{r}) = \sqrt{n} f_0(z) \exp(-|z|^2/2)$

$$z, \bar{z} = (x \pm y)/l; \quad l = (\hbar/m\Omega)^{1/2}$$



Expand Ψ_0 in $\psi_n = (z^n/\sqrt{\pi n!}) \exp(-|z|^2/2)$

Projected GP equation

Alternative $\Rightarrow$ projected GP equation

$$\hat{P}F(z,\bar{z}) = \frac{1}{\pi}\int dw d\bar{w}\, \exp[-|w|^2 + z\bar{w}]F(w,\bar{w}) \;\Rightarrow \text{LLL}$$

$\Omega = \omega \;\Rightarrow$ geometry of an infinite plane

$$\frac{Ng}{\pi}\int dw d\bar{w} \mathrm{e}^{-2w\bar{w}+z\bar{w}}|f_0(w)|^2 f_0(w) = \tilde{\mu} f_0(z);\; \tilde{\mu} = \mu - \hbar\Omega$$

Triangular vortex lattice $f_0(z) = (2v)^{1/4}\vartheta_1(\sqrt{\pi v}z, q)\,\mathrm{e}^{z^2/2}$

$$q = \exp(i\pi\tau),\ \tau = u + iv,\ v = \sqrt{3}/2,\ u = -1/2$$

$$\vartheta(z,q) = 2\sum_{n=1}^{\infty}(-1)^{n+1}q^{(n-1/2)^2}\sin(2n-1)z$$

$$\tilde{\mu} = \alpha n g; \quad \alpha = 0.1596$$

Alternative mean field approach

$$i\hbar\frac{\partial\hat{\psi}}{\partial t} = \frac{\hat{\mathbf{p}}^2}{2m}\hat{\psi} + g\hat{\psi}^\dagger\hat{\psi}\hat{\psi} + V(\mathbf{r})\hat{\psi} - \Omega\hat{L}\hat{\psi}$$

$$\hat{\psi} = \exp i\hat{\Phi}\sqrt{\hat{n}}; \quad \hat{\psi}^\dagger = \sqrt{\hat{n}}\exp -i\hat{\Phi}; \quad [\hat{n}(\mathbf{r}), \hat{\Phi}(\mathbf{r}')] = i\delta(\mathbf{r}-\mathbf{r}')$$

Small fluctuations of the density $n = n_0(\mathbf{r}) + \delta\hat{n}; \quad \hat{\Phi} = \Phi_0(\mathbf{r}) + \delta\hat{\Phi}$

Linearize NLSE with respect to $\delta\hat{n}$ and $\nabla\delta\hat{\Phi}$

Zero order $\Rightarrow$ GP equation for $\Psi_0(\mathbf{r}) = \sqrt{n_0(\mathbf{r})}\exp[i\Phi_0(\mathbf{r})]$

Linear order

$$\hbar\frac{\partial}{\partial t}\frac{\delta\hat{n}}{2\sqrt{n_0}} = -\frac{\hbar^2}{2m}\left(\frac{\nabla[n_0\nabla\hat{\Phi}]}{\sqrt{n_0}} + \nabla\Phi_0\nabla\frac{\delta\hat{n}}{\sqrt{n_0}} + \nabla^2\hat{\Phi}_0\frac{\delta\hat{n}}{\sqrt{2n_0}}\right) + \hbar\Omega\frac{\partial}{\partial\phi}\frac{\delta\hat{n}}{2\sqrt{n_0}}$$

$$-\hbar\sqrt{n_0}\frac{\partial\delta\hat{\Phi}}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + 3n_0 g + V(\mathbf{r}) - \mu - \hbar\Omega\frac{\partial}{\partial\phi}\right)\frac{\delta\hat{n}}{2\sqrt{n_0}} - \hbar\Omega\sqrt{n_0}\frac{\partial\delta\hat{\Phi}}{\partial\phi}$$

$$\delta\hat{n} = \sqrt{n_0}\mathrm{e}^{-|z|^2/2}\sum_{\mathbf{k}}[u_{\mathbf{k}}\exp[-i\Phi_0] - \tilde{v}^*_{\mathbf{k}}\exp[i\Phi_0]]\exp[-i\epsilon_{\mathbf{k}}t]\hat{a}_{\mathbf{k}} + \text{h.c.}$$

$$\delta\hat{\Phi} = \frac{-i\mathrm{e}^{-|z|^2/2}}{2\sqrt{n_0}}\sum_{\mathbf{k}}[u_{\mathbf{k}}\exp[-i\Phi_0] + \tilde{v}^*_{\mathbf{k}}\exp[i\Phi_0]]\exp[-i\epsilon_{\mathbf{k}}t]\hat{a}_{\mathbf{k}} + \text{h.c.}$$

Solution of projected BdG equations

$u_{\mathbf{k}},\ \tilde{v}_{\mathbf{k}} \rightarrow$ solutions of projected BdG equations

$$2g\hat{P}(|\Psi_0|^2 u_{\mathbf{k}}) - g\hat{P}(\Psi_0^2 \tilde{v}_{\mathbf{k}}^*) = (\tilde{\mu} + \epsilon_{\mathbf{k}}) u_{\mathbf{k}}$$

$$2g\hat{P}(|\Psi_0|^2 \tilde{v}_{\mathbf{k}}) - g\hat{P}(\Psi_0^2 u_{\mathbf{k}}^*) = (\tilde{\mu} - \epsilon_{\mathbf{k}}) \tilde{v}_{\mathbf{k}}$$

$$u_{\mathbf{k}} = \frac{c_{1\mathbf{k}}}{\sqrt{S}} f_0 \left(z + \frac{ik_+}{2} \right) \mathrm{e}^{ik_- z/2} \mathrm{e}^{-k^2/4} = c_{1\mathbf{k}} P(f_0 \mathrm{e}^{i\mathbf{kr}})$$

$$\tilde{v}_{\mathbf{k}} = \frac{c_{2\mathbf{k}}}{\sqrt{S}} f_0 \left(z - \frac{ik_+}{2} \right) \mathrm{e}^{-ik_- z/2} \mathrm{e}^{-k^2/4} = c_{2\mathbf{k}} P(f_0 \mathrm{e}^{-i\mathbf{kr}})$$

$$k_\pm = k_x \pm k_y; \quad c_{1\mathbf{k}} = \left[\frac{\tilde{K}(\mathbf{k}) + \epsilon_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}} \right]^{1/2} \mathrm{e}^{k^2/8}; \quad c_{2\mathbf{k}} = \left[\frac{\tilde{K}(\mathbf{k}) - \epsilon_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}} \right]^{1/2} \frac{|K_2(\mathbf{k})|}{K_2(\mathbf{k})} \mathrm{e}^{k^2/8}$$

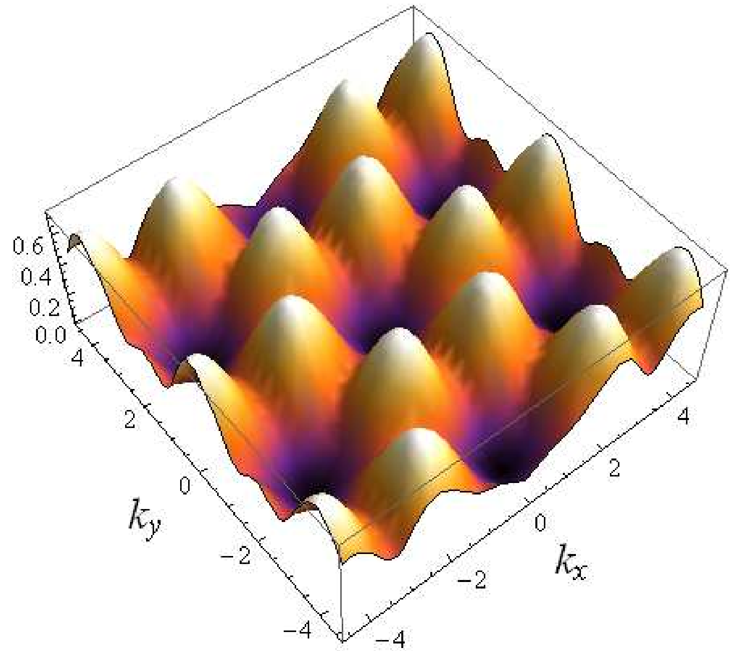
$$\tilde{K}(\mathbf{k}) = 2K_1(\mathbf{k}) - K_1(0; \quad K_1(0) = K_2(0) = \tilde{K}(0) = \alpha \simeq 1.1596$$

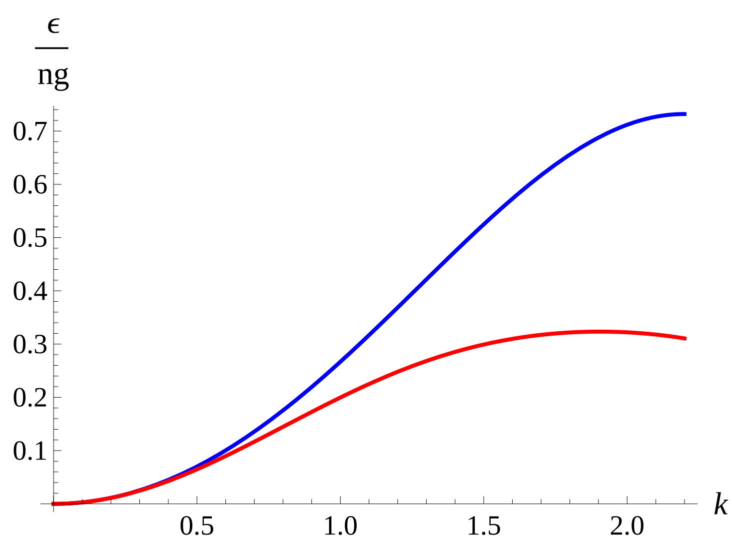
$$K_1(\mathbf{k}) = \sqrt{v} \sum_{n,m=-\infty}^{\infty} (-1)^{nm} \mathrm{e}^{-\pi v (n^2+m^2)} \mathrm{e}^{-\sqrt{\pi v} k_x n + i\sqrt{\pi v} k_y m} \mathrm{e}^{-k_x^2/4}$$

$$K_2(\mathbf{k}) = \sqrt{v} \sum_{n,m=-\infty}^{\infty} (-1)^{nm} \mathrm{e}^{-\pi v (n^2+m^2)} \mathrm{e}^{-\sqrt{\pi v}(k_x - ik_y)(n+m)} \mathrm{e}^{-k_x^2/2 + ik_x k_y/2}$$

Excitation spectrum

$$\epsilon_{\mathbf{k}}^2 = |2K_1(\mathbf{k}) - K_0|^2 - |K_2(\mathbf{k})|^2$$





Low-energy excitations

$$kl \ll 1 \;\Rightarrow\; K_1 = \alpha\left[1 - \frac{k^2}{8} + \frac{(\eta+1)k^4}{64}\right];\; K_2 = \alpha\left(1 - \frac{k^2}{4} + \frac{k^4}{32}\right);\; \eta = 0.8219$$

$$\epsilon = \frac{\alpha\sqrt{\eta}}{4}\, ng(kl)^2 \simeq 0.2628\, ng\,(kl)^2$$

Exactly coincides with Sonin (2005)

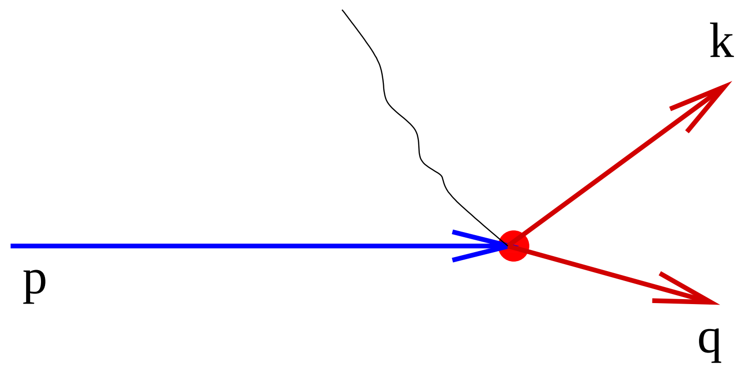
Tight confinement in one direction $\Rightarrow g = 2\sqrt{2\pi}\hbar^2 a/ml_0;\; l_0 = \sqrt{\hbar/m\omega_0}$

Rb^{87} $\Omega = 100$ Hz $\omega_0 = 300$ Hz $\Rightarrow ng/\hbar\Omega \simeq 0.1$ at $n \simeq 3\times 10^8$ cm^{-2} (LLL!)

Low-energy excitations $\Rightarrow \epsilon < 1$ Hz

$\nu = \pi n l^2 \gg 1 \;\rightarrow$ mean-field regime

Damping rate



Beliaev damping

$$\hat{\psi} = \left(\sqrt{n} f_0(z) + \sum_{\mathbf{k}} [u_{\mathbf{k}} \hat{a}_{\mathbf{k}} \exp(-i\epsilon_{\mathbf{k}} t) - \tilde{v}_{\mathbf{k}} \hat{a}^{\dagger}_{\mathbf{k}} \exp(i\epsilon_{\mathbf{k}} t)] \right) \exp(-i\mu t - |z|^2/2)$$

$$\hat{V} = g\sqrt{n} \sum_{\mathbf{k},\mathbf{q}} \int dx\, dy\, [f_0 u_{\mathbf{p}} u^*_{\mathbf{k}} u^*_{\mathbf{q}} + 2 f_0 \tilde{v}^*_{\mathbf{p}} u^*_{\mathbf{q}} \tilde{v}_{\mathbf{k}} - f^*_0 \tilde{v}^*_{\mathbf{p}} \tilde{v}_{\mathbf{k}} \tilde{v}_{\mathbf{q}} - 2 f^*_0 u_{\mathbf{p}} u^*_{\mathbf{k}} \tilde{v}_{\mathbf{q}}]\, \mathrm{e}^{-2|z|^2} \hat{a}^{\dagger}_{\mathbf{q}} \hat{a}^{\dagger}_{\mathbf{k}} \hat{a}_{\mathbf{p}} + \text{h.c.}$$

$$\Gamma_{\mathbf{p}} = \frac{2\pi}{\hbar} \sum_{\mathbf{k},\mathbf{q}} |\langle \mathbf{k}, \mathbf{q} | \hat{V} | \mathbf{p} \rangle|^2 (1 + N_{\mathbf{k}} + N_{\mathbf{q}}) \delta(\epsilon_{\mathbf{p}} - \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{q}})$$

$$u_{\mathbf{k}}(\mathbf{r}) \exp(-|z|^2/2) = c_{1\mathbf{k}} \frac{\Psi_0(z + ik_+ l/2)}{\sqrt{N}} \exp(-k^2 l^2/8) \exp(i\mathbf{kr}/2)$$

$$\tilde{v}_{\mathbf{k}}(\mathbf{r}) \exp(-|z|^2/2) = c_{2\mathbf{k}} \frac{\Psi_0(z - ik_+ l/2)}{\sqrt{N}} \exp(-k^2 l^2/8) \exp(-i\mathbf{kr}/2)$$

Damping of low-energy modes

$$\langle \mathbf{k},\mathbf{q}|\hat{V}|\mathbf{p}\rangle = \frac{\alpha^{5/2}}{S}\sqrt{\frac{Nng}{8\epsilon_p\epsilon_k\epsilon_q}}\left\{\frac{\epsilon_k}{\tilde{K}(k)}+\frac{\epsilon_q}{\tilde{K}(q)}-\frac{\epsilon_p}{\tilde{K}(p)}\right\} = \frac{\alpha}{4\sqrt{2}\,\eta^{1/4}}\sqrt{\frac{ng^2}{S}}\,\frac{(k^4+q^4-p^4)l}{kqp}$$

$$\Gamma_p = \frac{\alpha}{16\pi\eta}\,\frac{g}{\hbar}\,p^2\int_0^1 dx x^2\sqrt{1-x^2}\,\coth\left(\frac{\epsilon_p}{2T}\,x^2\right)$$

$$T=0 \;\Rightarrow\; \Gamma_{p0} = \frac{\alpha}{256\eta}\,\frac{g}{\hbar}\,p^2 \simeq 0.0055\frac{g}{\hbar}\,p^2$$

$$\frac{\hbar\Gamma_{p0}}{\epsilon_p} =\simeq \frac{0.065}{\nu}; \quad \nu=\pi n l^2 \gg 1 \;\Rightarrow\; \hbar\Gamma_{p0} \ll \epsilon_p$$

Finite $T \;\Rightarrow\; \Gamma_{pT} = \frac{\pi}{8\eta^{3/2}}\,\frac{T}{\hbar}\,\frac{1}{\nu} \simeq 0.53\frac{T}{\hbar}\,\frac{1}{\nu}; \quad \epsilon_p \ll T$

$$\epsilon_p \gg \frac{T}{\nu} \;\Rightarrow\; \hbar\Gamma_{pT} \ll \epsilon_p$$

Excitations with $\epsilon_p \lesssim \epsilon_c = \frac{T}{\nu}$ are overdamped

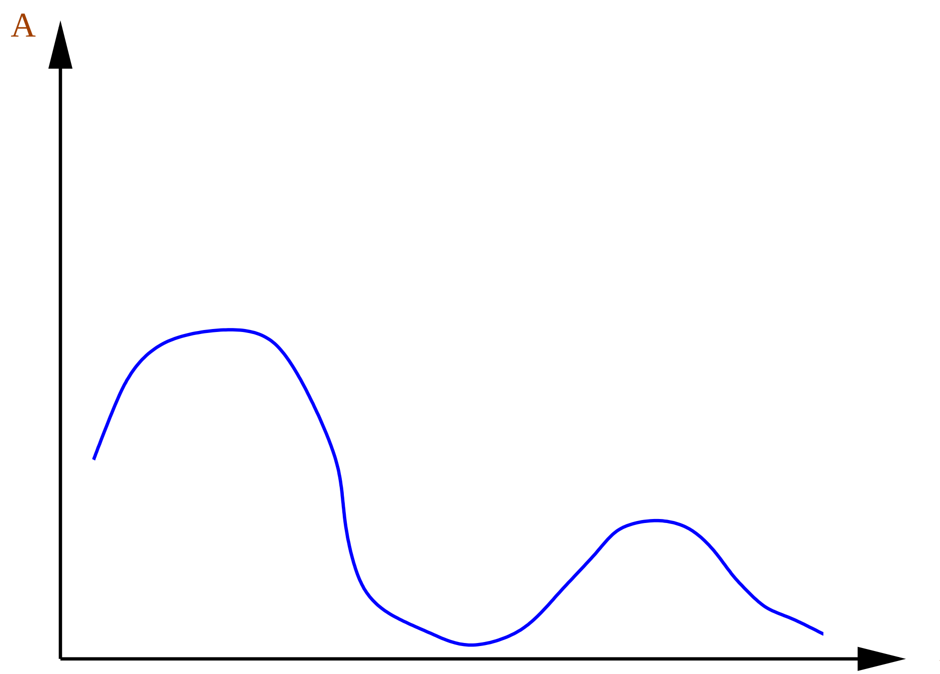
Damping of low-energy modes

Involved thermal excitations $\Rightarrow \epsilon \sim \epsilon_p$

Relaxation time $\Rightarrow \tau_R \sim \Gamma_p^{-1}$

$\epsilon_p \lesssim \epsilon_c \;\Rightarrow\; \tau_R \epsilon_p \lesssim 1 \;\Rightarrow$ hydrodynamic regime

$\Gamma \sim \Gamma_p(\epsilon_p \tau_R/\hbar) \Rightarrow$ approaches ϵ_p



Typical picture from the JILA experiment

One-body density matrix

$$g_1(\mathbf{r}) = \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(0)\rangle = \Psi_0^*(\mathbf{r})\Psi_0(0)\exp\left\{-\frac{1}{2}\langle(\delta\hat{\Phi}(\mathbf{r}) - \delta\hat{\Phi}(0))^2\rangle\right\}$$

$$\delta\hat{\Phi}(\mathbf{r}) = -\frac{i}{2}\sum_{\mathbf{k}}\frac{(c_{1\mathbf{k}} + c_{2\mathbf{k}})}{\sqrt{N}}\exp(i\mathbf{k}\mathbf{r}/2)\,\hat{a}_{\mathbf{k}} + \text{h.c.}$$

$$\langle(\delta\hat{\Phi}(\mathbf{r}) - \delta\hat{\Phi}(0))^2\rangle = \alpha g\int\frac{d^2k}{(2\pi)^2}\,\frac{(1+2N_k)}{\epsilon_k}[1 - J_0(kr/2)]$$

$$T = 0 \Rightarrow \quad \langle(\delta\hat{\Phi}(\mathbf{r}) - \delta\hat{\Phi}(0))^2\rangle_0 \simeq \frac{2}{\sqrt{\eta}}\,\frac{1}{\nu}\ln\left(\frac{e^C r}{2l}\right)$$

$$g_1(r) \propto \left(\frac{l}{r}\right)^{1/\sqrt{\eta}\pi n l^2}, \qquad r \gg l$$ Baym (2004)

One-body density matrix at finite T

Validity of the mean-field approach $\Rightarrow$ Small density fluctuations

$$\frac{\langle(\delta\hat{n}(\mathbf{r})-\delta\hat{n}(0))^2\rangle}{n^2}=\int_{0<k<l^{-1}}\frac{d^2k}{(2\pi)^2}\,\frac{g\alpha k^2 l^2}{2\epsilon_k}[1-J_0(kr/2)](1+2N_k)$$

$$\frac{\langle(\delta\hat{n}(\mathbf{r})-\delta\hat{n}(0))^2\rangle}{n^2}\simeq\frac{4T}{\alpha\eta n g\nu}\ln\left(\frac{r}{2\tilde{l}}\right)$$

$$t\gg ng \;\;\Rightarrow\;\; \tilde{l}=k_T^{-1}$$

The state is ordered in the vortex lattice at $r\simeq r_0=2\tilde{l}\exp(\alpha\eta n g\nu/4T)$

Low-momentum cut-off $\langle(\delta\hat{\Phi}(\mathbf{r})-\delta\hat{\Phi}(0))^2\rangle=\dfrac{8T}{\alpha\eta n g\nu}\dfrac{r^2}{l^2}\ln\left(\dfrac{r_0}{r}\right)$

$$g_1(r)\propto\exp\left[-\frac{4T}{\alpha\eta n g\nu}\frac{r^2}{l^2}\ln\left(\frac{r_0(T)}{r}\right)\right]$$

Outlook

$r_0 \Rightarrow$ very large $T \simeq ng;\ \nu \simeq 20 \ \Rightarrow \ r_0 \simeq 300l$

Is likely to exceed the size of realistic systems

$\nu \simeq 40 \ \Rightarrow \langle(\delta\hat{n}(\mathbf{r}) - \delta\hat{n}(0))^2\rangle \sim 0.3$ for $r \simeq 10l$ and $T \simeq ng \ \rightarrow$

"Irregularity" of the lattice

$\epsilon_c \simeq T\nu \ \sim 10$ Hz for $\nu \simeq 40$ even for $T \simeq 20$ nK

Damping of Tkachenko modes $\Rightarrow$ signature of the approach to the melting point