A low energy theory for superfluid and crystalline matter

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Summary

1. Some systems in nature are remarkable in that they are crystalline as well as superfluid
2. Discuss the low energy theory for such systems
3. Relation of the low energy coefficients (LECs) of the lagrangian to thermodynamic derivatives
4. *(Cirigliano, Reddy, Sharma. arXiv:1102.5379)*
5. Application to the crystalline superfluids (LOFF phases) *(preliminary)*
6. Neutron star crust

Low energy fields

1. One Goldstone mode is associated with the phase modulation of the condensate $\langle\psi_1\psi_2\rangle \propto |\Delta| e^{-2i\phi(x)}$
2. The second set of Goldstone modes is associated with translations and are the lattice phonons $\xi^a(x)$
3. Symmetries require invariance under constant shifts
   - $\phi(x) \to \phi(x) + \theta$
   - $\xi^a(x) \to \xi^a(x) + b^a$

The effective action

1.
$$L_{eff} = \frac{f_\phi^2}{2}(\partial_0\phi)^2 - \frac{v_\phi^2 f_\phi^2}{2}(\partial_i\phi)^2 + \frac{\rho}{2}\partial_0\xi^a\partial_0\xi^a - \frac{1}{4}\mu(\xi^{ab}\xi^{ab}) - \frac{K}{2}(\partial_a\xi^a)(\partial_b\xi^b) + g_{\rm mix} f_\phi\sqrt{\rho}\partial_0\phi\partial_a\xi^a + \cdots$$

2. $\xi^{ab} = (\partial_a\xi^b + \partial_b\xi^a - \frac{2}{3}\partial_c\xi^c\delta^{ab})$ is the traceless part of the strain tensor
3. An interesting feature is the mixing between the ϕ and the longitudinal lattice mode
4. The LECs can be related to derivatives of the free energy Ω with respect to external fields (for eg. the chemical potential μ). We call this thermodynamic matching

Thermodynamic matching for pure superfluid

1. *(Son, Wingate (2006))*
2. $\mathcal{L} = \mathcal{L}_v[\psi] + (\mu + m)\psi^\dagger \psi$
3. Slightly more general form $\mathcal{L} = \mathcal{L}_v[\psi] + A_\mu(x) j^\mu$
4. For constant $A_\mu(x) = \bar{A}_\mu = (\mu + m, 0)$ we get back the standard grand canonical picture, but in the intermediate stages we keep the external field general
5. Action invariant under gauge transformations $\psi \to \psi \exp(i\theta(x))$, $A_\mu \to A_\mu - \partial_\mu \theta$

Thermodynamic matching

1. $Z[A(x)] = \int [d\psi] e^{i \int d^4x \mathcal{L}}$
2. Doing the path integral in two steps. First integrating out the high energy fields, and obtain an effective lagrangian for ϕ
3. Then $Z[A(x)] = \int [d\phi] e^{i\mathcal{L}_{\text{eff}}[\phi, A]}$
4. The combination $D_\mu \phi = \partial_\mu \phi + A_\mu$ is invariant under gauge transformations
5. The invariant building block $X = D_\mu \phi D^\mu \phi$
6. Write the lagrangian as $\mathcal{L}_{\text{eff}}(\phi, A_\mu) = f(Y) + ...$ where $Y = \sqrt{X} - m$, and ... represents terms with more derivatives than fields
7. For constant $A^\mu(x) = \bar{A}^\mu$, 1. gives $Z[\bar{A}] = e^{-i\Omega VT}$

Thermodynamic matching

1. Alternatively, we can expand the effective action about the classical solution $\phi_0 = 0$. $\phi = \phi_0 + \varphi(x)$
2. $e^{if(\mu)VT} \int d[\varphi] e^{i \int d^4x d^4x' \varphi(x)\varphi(x') \frac{\partial^2 \mathcal{L}_{\text{eff}}}{\partial\phi(x)\partial\phi(x')} + \dots}$
3. The loop corrections are zero for constant external fields because there is always a derivative acting on the external field
4. Therefore $f(Y) = -\Omega(\mu = Y) = P(\mu = Y)$
5. *for eg. (Son, Wingate (2006))*

Hydrodynamics of the Goldstone mode

1. For a slowly varying field φ we can expand in $\partial\varphi$ about the equilibrium point
2. Taking the non-relativistic limit

$$\mathcal{L}_{\text{eff}}(\varphi) = P(\mu_n) + \frac{dP}{d\mu}(\partial_t\varphi) + \frac{1}{2}\frac{d^2P}{d\mu^2}(\partial_t\varphi)^2 - \frac{dP}{d\mu}\frac{(\partial_i\varphi)^2}{2m} + \ldots$$
$$\sim \frac{f_\phi^2}{2}\left[(\partial_t\varphi)^2 - c_\phi^2(\partial_i\varphi)^2\right]$$

3. $f_\phi^2 = \frac{\partial^2 P}{\partial\mu^2} = \frac{dn}{d\mu}$, $c_\phi^2 = \frac{n}{mf_\phi^2}$

The procedure

1. Identify the conserved current for the spontaneously broken global symmetry
2. Couple an external field to the conserved current, and promote the global symmetry to a local symmetry
3. Write a low energy lagrangian for the fields invariant under the local symmetry
4. For constant external fields this coincides with $-\Omega$
5. Perform a gradient expansion

Matching for superfluid and crystal

1. The conserved charge associated with translations is the stress tensor
2. The external fields are the spatial components of the external metric g^{ab}
3. To make invariant combinations it is useful to introduce the body fixed coordinates $z^a = x^a - \xi^a(x)$ *(Leutwyler 1997, Son 2002)*
4. There are three invariant combinations
   - $Y = \sqrt{D_\mu \phi D^\mu \phi} - m$
   - $W^a = \partial_\mu z^a D^\mu \phi$
   - $H^{ab} = \partial_\mu z^a \partial^\mu z^b$

Matching for superfluid and crystal

1. $\mathcal{L}_{\text{eff}}(\phi, \xi^a, A_\mu, g_{\mu\nu}) = f(Y, W^a, H^{ab}) + \ldots$
2. There are also terms that change by a total derivative on making gauge transformations

$$\epsilon^{\mu\nu\sigma\lambda}\epsilon^{abc}\Big(C_1\, A_\mu + C_2\, \partial_\mu\phi\Big)(\partial_\nu z^a \partial_\sigma z^b \partial_\lambda z^c)$$

3. For constant $g^{ab}(x) = \bar{g}^{ab}$ and $A_\mu(x) = \tilde{A}_\mu = (\mu + m, \mathbf{A})$ the action at the classical solution at $\phi = 0$, and $\xi^a = 0$ is the free energy
4. A new feature is that we need to allow for $\mathbf{A} \neq \mathbf{0}$
5. For constant external fields, the variables give $Y_0 = \mu$, $W_0^a = \mathbf{A}^\mathbf{a}$, $H_0^{ab} = \bar{g}^{ab}$
6. $f(\tilde{A}_0, \mathbf{A}, \bar{g}_{ab}) = -\Omega(\tilde{A}_\mu, \bar{g}_{ab}) = -\mathcal{E}(\tilde{A}_\mu, \bar{g}_{ab}) + \tilde{A}_\mu j^\mu$

Quadratic lagrangian

1. Expanding near the equilibrium, $Y = \mu$, $W^a = 0$, $H^{ab} = -\delta^{ab}$ and keeping only the quadratic terms

$$
\begin{aligned}
\mathcal{L}_0 = & \frac{1}{2}\left[\frac{\partial^2 f}{\partial Y^2}\right](\partial_0 \phi)^2 - \frac{1}{2}\left[\frac{1}{m}\frac{\partial f}{\partial Y} - \frac{\partial^2 f}{3\partial W^c \partial W^c}\right](\partial_i \phi)^2 \\
& + \frac{1}{2}\left[\frac{2}{3}\frac{\partial f}{\partial H^{cc}} + m^2 \frac{\partial^2 f}{3\partial W^c \partial W^c}\right]\dot{\xi}^a \dot{\xi}^a \\
& + \left[\frac{2}{3}\frac{\partial^2 f}{\partial H^{cc}\partial Y} + m\frac{\partial^2 f}{3\partial W^c \partial W^c}\right](\partial_c \xi^c)(\partial_0 \phi) \\
& - \frac{1}{4}[\mu]\, \xi^{ab}\, \xi^{ab} - \frac{1}{2}[K](\partial_c \xi^c)^2
\end{aligned}
$$

The mixing parameter and entrainment

1. $n_b = m\frac{\partial^2 f}{3\partial W^c \partial W^c}$ is the density of the superfluid that is entrained on the lattice
2. Related to the change in energy associated with relative motion between the superfluid and the lattice. To see that, note that in the non-relativistic limit $W^a \sim m_n(-\frac{1}{m_n}\partial_a\phi - \partial_0\xi^a + \frac{1}{m}\partial_i\phi\partial_i\xi^a)$

H^{ab}

1. $H^{ab} \equiv \eta^{ab} - (\partial^a \xi^b + \partial^b \xi^a) + \partial_\mu \xi^a \partial^\mu \xi^b$ is related to the deformations of the crystal
2. In the case where one conserved species (p) forms the lattice and the second species (n) is superfluid, $\delta H^{cc} = -\frac{1}{n_p} \delta n_p$ and therefore $g_{\mathrm{mix}} = \frac{1}{f_\phi \sqrt{\rho}} [n_b - n_p \frac{\partial n_n}{\partial n_p}]$

Relating g^{ab} to deformations

1. The $g^{ab} = H^{ab} \equiv \eta^{ab} - (\partial^a \xi^b + \partial^b \xi^a) + \partial_\mu \xi^a \partial^\mu \xi^b$, relates the external metric to the deformation
2. The elastic constants are given by,

$$K = \bar{K} + \frac{1}{3} P$$
$$\mu = \bar{\mu} - P$$

   where, $P = -\frac{1}{3}\langle T^a_a \rangle$ is the trace of the stress tensor
3. $\bar{K} = (\frac{10}{9}\delta_{abcd} - \frac{2}{3}\delta_{ab}\delta_{cd} - \frac{4}{9}\delta_{ac}\delta_{bd})\frac{\partial^2 \sqrt{-g} f}{\partial g^{ab} \partial g^{cd}}$
4. $\bar{\mu} = (\frac{2}{3}\delta_{abcd} - \frac{2}{3}\delta_{ac}\delta_{bd})\frac{\partial^2 \sqrt{-g} f}{\partial g^{ab} \partial g^{cd}}$

Application to LOFF phases

1. Asymmetric Fermi gases for $(\mu_1 - \mu_2) = 2\delta\mu \neq 0$
2. Weak coupling, mean field analysis to get qualitative understanding
3. $\Delta(x) = \Delta \sum_{\{\mathbf{q}^a\}} e^{i2\mathbf{q}^a\cdot\mathbf{r}}$
4. LOFF phases *(Larkin, Ovchinnikov; Fulde, Ferrell)* are possible ground states for $\delta\mu \sim [0.707, 0.754]\Delta_0$, where Δ_0 is the gap in the symmetric phase
5. For simple lattice structures there is a second order phase transition from the normal phase to the LOFF phase at $\delta\mu = 0.754\Delta_0$

Application to LOFF phases

1. We do the calculation for a $\cos(2qz)$ condensate
2. A Ginzburg-Landau expansion in Δ can be used near the second order transition
3. $|\mathbf{q}^a|$ is chosen to minimize the free energy. $|\mathbf{q}^a| v_f = \eta \delta\mu$ with $\eta \sim 1.2$

LECs in the LOFF phase

1. $\mathcal{L}_\psi =$

$$\begin{pmatrix} \psi_1^\dagger & \psi_2 \end{pmatrix} \left[\begin{array}{cc} i\partial_t - \frac{p^2}{2m} + (\mu + \delta\mu) & \Delta(x) \\ \Delta^*(x) & i\partial_t + \frac{p^2}{2m} - (\mu - \delta\mu) \end{array} \right] \begin{pmatrix} \psi_1 \\ \psi_2^\dagger \end{pmatrix}$$

2. In this case more convenient to compute the free energy in the deformed state: $\mathbf{r} \to \mathbf{r} + \xi(r)$

LECs in the LOFF phase

1. For the cos condensate, one lattice phonon ξ^z
2. Expand to the second order in ξ since we want the quadratic lagrangian

$$2\Delta\cos\big(2q(z-\xi)\big) \sim 2\Delta[\cos(2qz) - 2q\xi\sin(2qz) \\ -4q^2\xi^2\cos(2qz)]$$

3. Integrating out the fermions still difficult beccause a space dependent condensate
4. Simplify further by making a Ginzburg Landau expansion in Δ, *(Mannarelli, Rajagopal, Sharma (2007))*

LECs in the LOFF phase

1. $\mathcal{L} = \frac{1}{2}[\frac{mk_f}{\pi^2}(\partial_0\phi)^2 - 2\alpha v_f^2(\partial_x\phi)^2 + 2\alpha q^2 v_f^2(\partial_0\xi^x)^2 - 2\alpha q^2 v_f^4(\partial_x\xi^x)^2] + [\alpha q v_f^2(\partial_0\phi\partial_x\xi^x)]$
2. $\alpha = \frac{2mk_f\Delta^2}{\pi^2\delta\mu^2(\eta^2-1)}$
3. $g_{mix} = \frac{\Delta}{\delta\mu}\frac{v_f}{\sqrt{(\eta^2-1)}}$
4. The mixing is parameterically small near the second order point, but may be important when Δ is larger
5. Requires a more careful consideration of gapless fermions

The neutron star inner crust

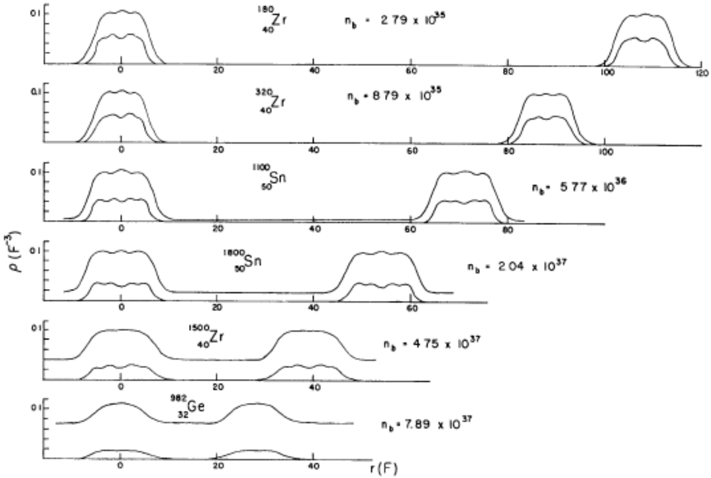


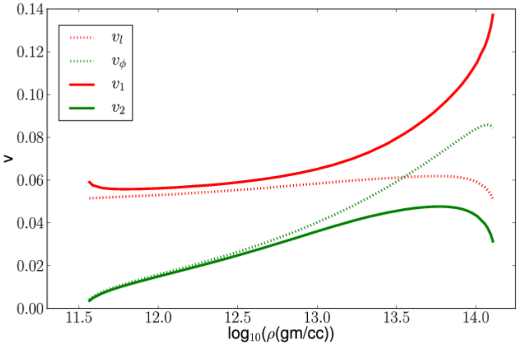
Fig. 3. Proton and neutron density distributions occurring along an axis joining the centers of two adjacent unit cells.

(Negele Vautherin (1973))

LECs in the neutron star crust

1. Model the system as clusters of protons localized on lattice sites, with some neutrons (n_b) bound or entrained on the sites, and the rest ($n_f = n_n - n_b$) unbound
2. $g_{\text{mix}} = \frac{1}{f_\phi \sqrt{\rho}}[n_b - n_p \frac{\partial n_n}{\partial n_p}]$
3. Use nuclear mass fomulae to obtain a rough estimate for n_n and n_p as a function of density
4. We take n_b as the density of bound neutrons in the Wigner-Seitz approximation
5. The second contribution is estimated by noting that $n_p \frac{\partial n_n}{\partial n_p} \sim n_p f_\phi^2 \tilde{V}_{np}$. For typical values of $\tilde{V}_{np}$, the first term dominates over the second term

Mixing in the neutron star crust



Conclusions and future work

1. The LECs can be calculated from the thermodynamic properties of the systems. Generalizations of "susceptibilities"
2. The existence of a phase with two modes with different dispersions that mix with each other could affect hydrodynamic oscillations
3. For future work, a more careful calculation of the elastic and mixing parameters