Exact relations for few-body and many-body problems with short-range interactions

Félix Werner
UMass (Amherst)

Yvan Castin
Ecole Normale Supérieure (Paris)

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2-body scattering state: $\Phi_{\mathbf{k}}(\mathbf{r}) \underset{r\to\infty}{=} e^{i\mathbf{k}\cdot\mathbf{r}} + f_{\mathbf{k}} \frac{e^{ikr}}{r} + \ldots$

$$f_k \underset{k\to 0}{=} \frac{-1}{\frac{1}{a} + ik - \frac{1}{2}k^2 r_e + \ldots}$$

$$-\frac{1}{a}\Phi_{\mathbf{k}\to 0}(\mathbf{r}) \underset{r\to\infty}{\simeq} -\frac{1}{a} + \frac{1}{r}$$

A. FERMIONS Spin 1/2 $N_\uparrow + N_\downarrow = N$

$$\psi(\underbrace{\mathbf{r}_1, \ldots, \mathbf{r}_{N_\uparrow}}_{\text{spin } \uparrow}, \underbrace{\mathbf{r}_{N_\uparrow+1}, \ldots, \mathbf{r}_N}_{\text{spin } \downarrow})$$

Zero-Range Model:

$$\begin{cases} \psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) \underset{r_{ij}\to 0}{=} \left(\frac{1}{r_{ij}} - \frac{1}{a}\right) A_{ij}\left(\mathbf{R}_{ij}, (\mathbf{r}_k)_{k\neq i,j}\right) + O(r_{ij}) \\ \sum_{i=1}^{N} \left[-\frac{\hbar^2}{2m}\Delta_{\mathbf{r}_i} + U(\mathbf{r}_i)\right]\psi = E\,\psi \end{cases}$$

$$\mathbf{R}_{ij} \equiv \frac{\mathbf{r}_i + \mathbf{r}_j}{2}$$

Exact relations [Tan 2008]: for any eigenstate:

① $$\frac{dE}{d(-1/a)} = \frac{4\pi\hbar^2}{m}(A, A)$$ where

$$(A^{(1)}, A^{(2)}) \equiv \sum_{i<j} \int \Big(\prod_{k \neq i,j} d^d r_k \Big) \int d^d R_{ij} (A_{ij}^{(1)} A_{ij}^{(2)})(\mathbf{R}_{ij}, (\mathbf{r}_k)_{k \neq i,j})$$

$$\Uparrow$$

Lemma:

$$\begin{cases} \psi_1 : \ a_1, \ A^{(1)} \\ \psi_2 : \ a_2, \ A^{(2)} \end{cases}$$

$$\langle \psi_1, H\psi_2 \rangle - \langle H\psi_1, \psi_2 \rangle = \frac{4\pi\hbar^2}{m} \left(\frac{1}{a_1} - \frac{1}{a_2} \right) \ (A^{(1)}, A^{(2)})$$

[$\Leftarrow$ Ostrogradsky's Thm.]

② $$C \equiv \lim_{k\to\infty} k^4\, n_\sigma(\mathbf{k}) = (4\pi)^2\,(A,A)$$

$$\int \frac{d^3k}{(2\pi)^3}\, n_\sigma(\mathbf{k}) = N_\sigma.$$

Derivation à la Olshanii-Dunjko [2003, in 1D]:

$$n_\uparrow(\mathbf{k}) = \sum_{i=1}^{N_\uparrow} \int \Big(\prod_{l\neq i} d\mathbf{r}_l\Big) \left| \int d\mathbf{r}_i e^{-i\mathbf{k}\cdot\mathbf{r}_i}\, \underbrace{\psi(\mathbf{r}_1,\ldots,\mathbf{r}_N)} \right|^2$$

$k \to \infty$

$$\sum_{j,j\neq i} \frac{1}{r_{ij}} A_{ij}(\mathbf{r}_j, (\mathbf{r}_l)_{l\neq i,j}).$$

$$g^{(2)}_{\uparrow\downarrow}(\mathbf{r}_1,\mathbf{r}_2) \equiv \langle \hat{n}_\uparrow(\mathbf{r}_1)\hat{n}_\downarrow(\mathbf{r}_2)\rangle$$

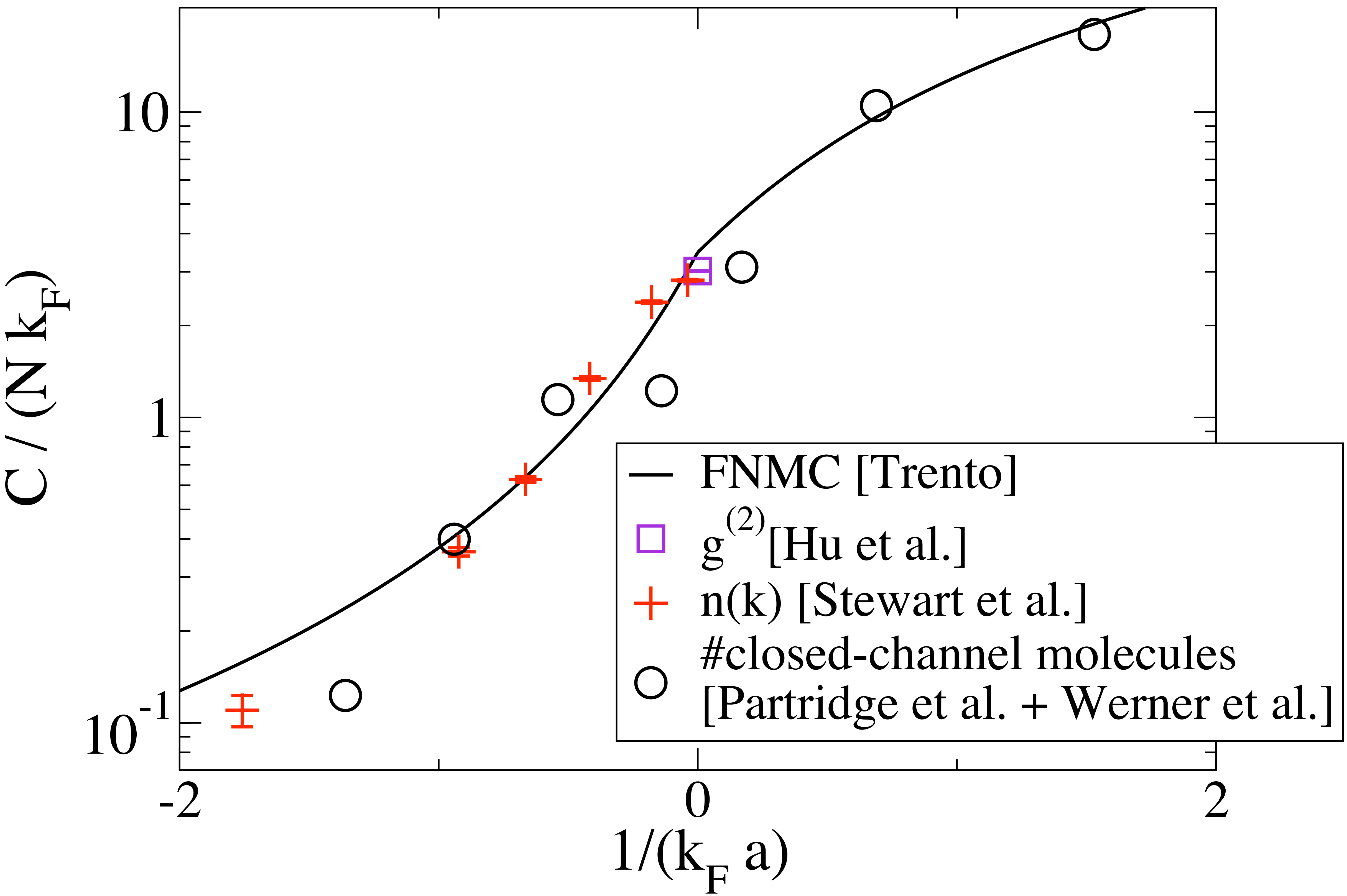
③
$$\int d^3R\, g^{(2)}_{\uparrow\downarrow}\left(\mathbf{R}+\frac{\mathbf{r}}{2},\mathbf{R}-\frac{\mathbf{r}}{2}\right) \underset{r\to 0}{\sim} \frac{(A,A)}{r^2}$$

④
$$E - E_{\text{trap}} = \frac{\hbar^2 C}{4\pi m a} + \sum_\sigma \int \frac{d^3k}{(2\pi)^3}\frac{\hbar^2 k^2}{2m}\left[n_\sigma(\mathbf{k}) - \frac{C}{k^4}\right]$$

$$E_{\text{trap}} \equiv \left\langle \sum_{i=1}^{N} U(\mathbf{r}_i)\right\rangle$$

Numerical verification of exact relations for 4 fermions [Daily&Blume 2009]

Contact measurements for balanced Fermi gas at low T in a trap



C also appears in radiofrequency spectra

Exp: [Stewart, Gaebler, Drake & Jin 2010]

Theo:
[Punk&Zwerger 2007]
[Baym, Pethick, Yu&Zwierlein 2007]
[Pieri, Perali&Strinati 2009]
[Schneider, Shenoy&Randeria 2009]
[Haussmann, Punk&Zwerger 2009]
[Braaten, Kang&Platter 2010]
[Schneider&Randeria 2010]

Some new relations

① $$\frac{1}{2}\frac{d^2E_n}{d(-1/a)^2} = \left(\frac{4\pi\hbar^2}{m}\right)^2 \sum_{n',E_{n'}\neq E_n} \frac{|(A^{(n')},A^{(n)})|^2}{E_n - E_{n'}}$$

$$\Downarrow$$

② $$\frac{\partial C}{\partial(1/a)} > 0 \quad \text{at constant } T \text{ or } S$$

③

$$\sum_{\sigma=\uparrow,\downarrow} \frac{1}{4\pi} \int d\hat{r} \int d^d R \left\langle \hat{\psi}^\dagger_\sigma \left(\mathbf{R} - \frac{\mathbf{r}}{2}\right) \hat{\psi}_\sigma \left(\mathbf{R} + \frac{\mathbf{r}}{2}\right) \right\rangle \underset{r\to 0}{=} N - \frac{C}{4\pi} r - \frac{m}{3\hbar^2} \left(E - E_{\mathrm{trap}} - \frac{\hbar^2 C}{4\pi m a} \right) r^2 + \dots$$

④

$$\left(\frac{\partial E}{\partial r_e}\right)_a = 2\pi \sum_{i<j} \int d^3R \int \Big(\prod_{k\neq i,j} d^3 r_k \Big) A_{ij}(\mathbf{R}, (\mathbf{r}_k)_{k\neq i,j}) \cdot \left[E + \frac{\hbar^2}{4m}\Delta_{\mathbf{R}} + \frac{\hbar^2}{2m} \sum_{k\neq i,j} \Delta_{\mathbf{r}_k} - \sum_{l=1}^{N} U(\mathbf{r}_l) \right] A_{ij}(\mathbf{R}, (\mathbf{r}_k)_{k\neq i,j})$$

generalises [Efimov 1993]

[$\Leftarrow$ Ostrogradsky's Thm. + effective range model]

generalisations to 2D
and to finite-range interactions:
see arXiv

B. BOSONS Spinless Efimov effect

→ additional boundary condition in Zero-Range Model:

$$\psi(\mathbf{r}_1,\ldots,\mathbf{r}_N) \underset{R\to 0}{\sim} \frac{1}{R^2} \sin\left[|s_0| \ln \frac{R}{R_t}\right] \Phi(\mathbf{\Omega})\, B(\mathbf{C},\mathbf{r}_4,\ldots,\mathbf{r}_N)$$

$$R \equiv \sqrt{\left({r_{12}}^2 + {r_{23}}^2 + {r_{13}}^2\right)/3}$$

$R_t =$ 3-body parameter
(directly related to binding energy of Efimov trimers

$s_0 = i \cdot 1.00624\ldots$ is imaginary solution of

$$-s\cos\left(s\frac{\pi}{2}\right) + \frac{8}{\sqrt{3}}\sin\left(s\frac{\pi}{6}\right) = 0$$

$\Phi(\mathbf{\Omega}) =$ hyperangular part of
wavefunction of Efimov trimer

① $$\left(\frac{dE}{d(-1/a)}\right)_{R_t} = \frac{4\pi\hbar^2}{m}(A,A)$$

② $$\int d^3R\, g^{(2)}_{\uparrow\downarrow}\left(\mathbf{R}+\frac{\mathbf{r}}{2},\mathbf{R}-\frac{\mathbf{r}}{2}\right) \underset{r\to 0}{\sim} \frac{(A,A)}{r^2}$$

③ $$\left(\frac{\partial E}{\partial \ln(R_t)}\right)_a = \frac{\hbar^2}{m}\frac{\sqrt{3}}{32}|s_0|^2 N(N-1)(N-2) \cdot \int d\mathbf{C} \int d\mathbf{r}_4 \ldots d\mathbf{r}_N \left|B(\mathbf{C},\mathbf{r}_4,\ldots,\mathbf{r}_N)\right|^2$$

[$\Leftarrow$ Ostrogradsky's Thm.]

④ <u>Virial Thm.</u> [Werner 2008]:

$$2(E-E_{\mathrm{trap}}) = \sum_{i=1}^{N}\left\langle \vec{r}_i \cdot \vec{\nabla} U(\vec{r}_i)\right\rangle + \frac{1}{a}\left(\frac{\partial E}{\partial(1/a)}\right)_{R_t} - \left(\frac{\partial E}{\partial \ln(R_t)}\right)_a$$

For Efimov trimer:

$$n(k) \underset{k\to\infty}{=} \frac{C}{k^4} + \frac{D}{k^5}\cos\left[2|s_0|\ln(k\sqrt{3}/\kappa_0) + \varphi\right] + \ldots$$

$$E_{\text{trimer}} = -\frac{\hbar^2{\kappa_0}^2}{m}$$

$$\Rightarrow \int \frac{d^3k}{(2\pi)^3}\frac{\hbar^2 k^2}{2m}\left[n(\mathbf{k}) - \frac{C}{k^4}\right] \quad \text{diverges}$$

$$\text{conjecture} : D \propto \left(\frac{\partial E}{\partial \ln(R_t)}\right)_a$$

3 particles $\quad a=\infty$

$$-s\cos\left(s\frac{\pi}{2}\right)+\eta\frac{4}{\sqrt{3}}\sin\left(s\frac{\pi}{6}\right)=0 \quad \text{where} \quad \eta=2 \text{ for bosons}, \ -1 \text{ for fermions}$$

Efimov trimers: wavefunction known [Efimov 1970] $\Rightarrow$

$$\left(\frac{\partial E}{\partial(-1/a)}\right)_{R_t}=\sqrt{-E\frac{\hbar^2}{m}}\cdot\frac{\pi\tan(s\pi)\sin\left(s\frac{\pi}{2}\right)}{\cos\left(s\frac{\pi}{2}\right)-s\frac{\pi}{2}\sin\left(s\frac{\pi}{2}\right)-\frac{4\pi}{3\sqrt{3}}\cos\left(s\frac{\pi}{6}\right)}$$

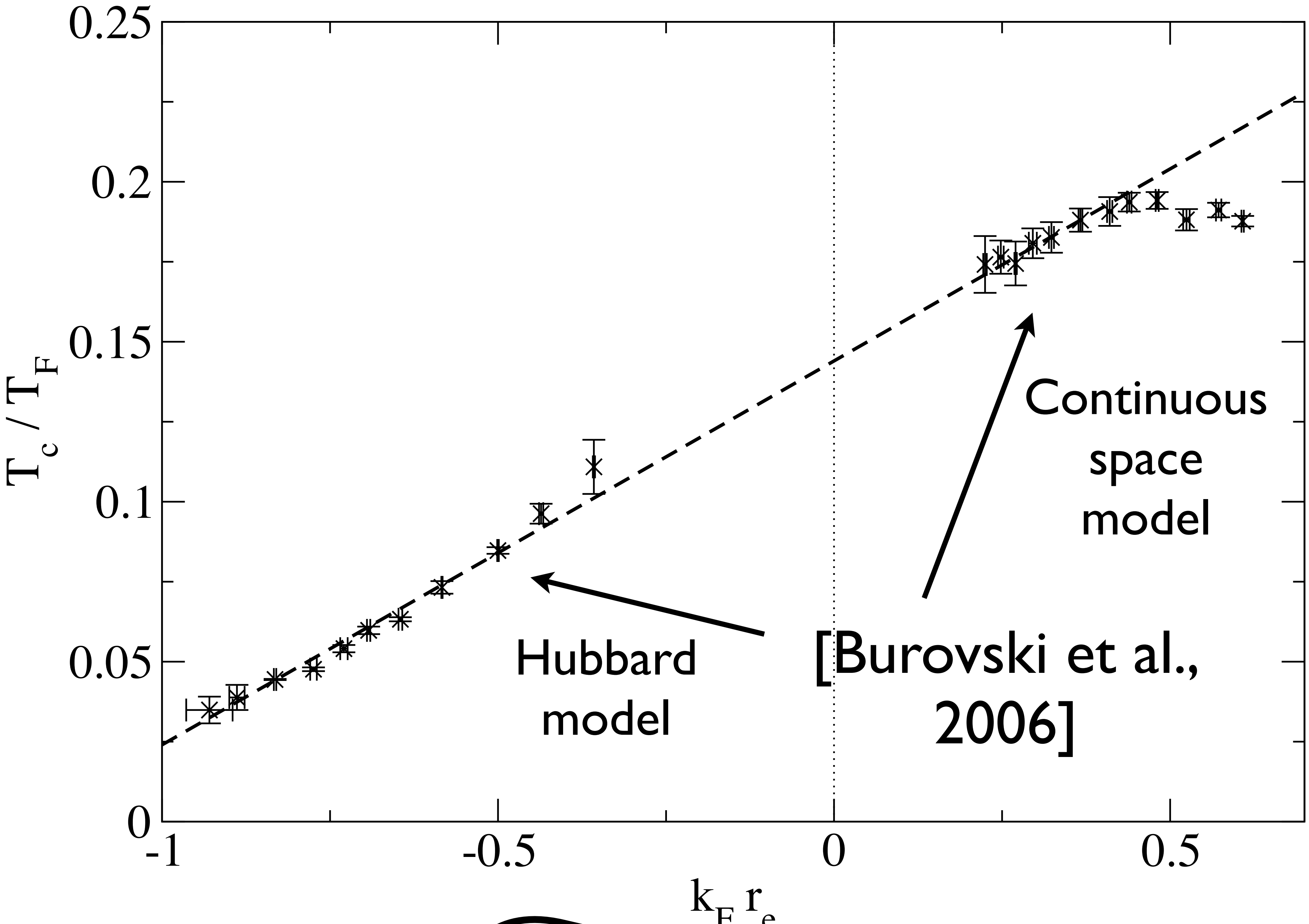
In isotropic harmonic trap: wavefunctions known
[Jonsell, Heiselberg&Pethick 2002; Tan 2004; Werner&Castin 2005]
(universal states; $l=0$; $q=0$) $\Rightarrow$

$$\frac{\partial E}{\partial(-1/a)}=\sqrt{\frac{2\hbar^3\omega}{m}}\cdot\frac{\Gamma(s+\frac{1}{2})\,s\sin\left(s\frac{\pi}{2}\right)}{\Gamma(s+1)\left[\cos\left(s\frac{\pi}{2}\right)-s\frac{\pi}{2}\sin\left(s\frac{\pi}{2}\right)-\eta\frac{2\pi}{3\sqrt{3}}\cos\left(s\frac{\pi}{6}\right)\right]}$$

$$\left(\frac{\partial E}{\partial r_e}\right)_a=\sqrt{\frac{\hbar^3\omega}{8m}}\cdot\frac{\Gamma(s-\frac{1}{2})\,s(s^2-\frac{1}{2})\sin\left(s\frac{\pi}{2}\right)}{\Gamma(s+1)\left[-\cos\left(s\frac{\pi}{2}\right)+s\frac{\pi}{2}\sin\left(s\frac{\pi}{2}\right)+\eta\frac{2\pi}{3\sqrt{3}}\cos\left(s\frac{\pi}{6}\right)\right]}$$

Agreement with numerical results of: Braaten & Hammer; von Stecher, Greene & Blume; Werner & Castin

Critical Temperature of Unitary Gas



$$\frac{T_c}{T_F} \simeq 0.15 + 0.12\, k_F\, r_e$$

$\sim$ 2% for ^{6}Li
$\sim$ 6% for ^{40}K (TBC)