Fisher zeros and singularities of the gap equation for nonlinear O(N) sigma models at finite volume.

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Introduction

In 2D nonlinear $O(N)$ sigma models, partition functions have the same property as $SU(2N)$ gauge theories under the transformation $g_0^2 \to -g_0^2$.

Zeros of partition functions in the complex β plane (Fisher's zeros) for 2D nonlinear O(N) sigma model have similar features as 4D lattice gauge theory. They will stay away from the physical (i.e. real) domain, this indicates the absence of phase transition. (The confinement for lattice gauge theory)

Both $O(N)$ sigma models with $N \geq 3$ and $SU(N)$ gauge with $N \geq 2$ has the property of asymptotic freedom.

Introduction

The lattice sites are denoted $\mathbf{x}$ and the scalar fields $\vec{\phi}_{\mathbf{x}}$ are N-dimensional unit vectors. The partition function reads:

$$Z = C \int \prod_{\mathbf{x}} d^N \phi_{\mathbf{x}} \delta(\vec{\phi}_{\mathbf{x}}.\vec{\phi}_{\mathbf{x}} - 1) \mathrm{e}^{-(1/g_0^2) E[\{\phi\}]} \ , \quad (1)$$

with

$$E[\{\phi\}] = -\sum_{\mathbf{x},\mathbf{e}} (\vec{\phi}_{\mathbf{x}}.\vec{\phi}_{\mathbf{x}+e} - 1) \ , \quad (2)$$

with $\mathbf{e}$ running over the D positively oriented unit lattice vectors.

Introduction

We insert the δ function as

$$\delta(\vec{\phi}_{\mathbf{x}}.\vec{\phi}_{\mathbf{x}} - 1) = \int_{K-i\infty}^{K+i\infty} dM_x^2 \frac{e^{\frac{M_x^2}{2g_0^2}\left(1-\overrightarrow{\phi_x}^2\right)}}{4\pi i g_0^2} \tag{3}$$

Keeping only the zero mode of M_x^2, we get a Berlin-Kac model with ($\sum_x \overrightarrow{\phi}_x . \overrightarrow{\phi}_x = V = L^2$) and rescaling,

$$M^2 = M_x^2/\beta, \beta = \frac{1}{g_0^2}, b = \frac{1}{N g_0^2}, b \equiv 1/\lambda^t \tag{4}$$

we get partition functions as:

$$Z \propto \int_{K-i\infty}^{K+i\infty} dM^2 \frac{e^{\frac{VN}{2}[bM^2 - \mathfrak{L}\left(M^2\right)]}}{b^{(\frac{VN}{2}-1)}} \tag{5}$$

with,

$$\mathfrak{L}\left(M^2\right) = \frac{1}{V}\sum_{k_i} \ln[2\sum_{i=1}^{D}\left(1-\cos\left(k_i\right)\right) + M^2] \tag{6}$$

where $k_i = \frac{2\pi}{L} n_i (n_i = 0, 1, ..., L-1)$

$Z(b)$ is normalize by $Z(0) = 1$

Gap equations

In the large-N limit, it possible to calculate the partition function in the saddle point approximation , we obtain:

$$b = \mathfrak{B}(M^2) \ , \tag{7}$$

with

$$\mathfrak{B}(M^2) \equiv \frac{1}{V} \sum_{k_i} \frac{1}{2(\sum_{j=1}^{D}(1 - \cos(k_j)) + M^2} \ . \tag{8}$$

M^2 is the saddle point value of the suitably rescaled Lagrange multiplier and can be interpreted as the mass gap or as the renormalized mass in cutoff units.

Gap equations

The saddle point equation is invariant under the simultaneous changes:

$$\begin{aligned} \lambda^t &\rightarrow -\lambda^t \\ M^2 &\rightarrow -M^2 - 4D \ . \end{aligned} \tag{9}$$

This can be seen by changing variables $k_j \rightarrow k_j + \pi$ for all j. Note that this change of variable sends the zero-momentum mode into the fastest oscillating one (that changes sign at every lattice site).

Gap equations in infinite volume

$$\mathfrak{B}(M^2) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{2(\sum_{j=1}^{D}(1-\cos(p_j)) + M^2} \tag{10}$$

Let M^2 go to zero. Correspondingly $\mathfrak{B}(0)$ go to some criticcal value with

$$\mathfrak{B}(0) = \frac{1}{2}\int_0^\infty d\alpha e^{-\alpha d} I_0(\alpha)^d \tag{11}$$

$d \to 2$,$\mathfrak{B}(0) \to \infty$. More precisely:

$$2d\mathfrak{B}(0) \sim \frac{2}{\pi(d-2)} \tag{12}$$

Gap equations in infinite volume

And the gap equation is then,

$$\mathfrak{B}(M^2) = \frac{1}{4\pi}\int_0^8 du \frac{F(1/2, 1/2; 1; u(8-u)/16)}{u + M^2} \quad (13)$$

$d = 3$:

$$\mathfrak{B}(0) = \frac{2\sqrt{6}}{3\pi^2}\Gamma(\frac{1}{24})\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{11}{24}) = 0.2527... \quad (14)$$

$d \to \infty$ (mean field prediction):

$$2d\mathfrak{B}(0) = 1 + \frac{1}{2d} + 3(\frac{1}{2d})^2 + 12(\frac{1}{2d})^3 + ... \quad (15)$$

Map between M^2 plane and λ plane

We calculate the 2D (and 3D) gap equation for $L = 2$ exactly by solving $\lambda \equiv \frac{1}{b} = P(M^2)/Q(M^2)$. The singular points (where $\partial b/\partial M^2 = 0$) appear to be the roots of discriminant equation $(\Delta = 0)$ for a 3-order (4-order) polynonial equation.

the map requires a Riemann surface with $q + 1$ sheets and $2q$ cuts in the λ^t plane, where q is an integer of order L^D. By connecting the sheets in a specific way, we construct one circle at infinity that maps into the circle at infinity in the M^2 plane and q others that maps into the cut on the real axis $[-4D, 0]$.

Map between M^2 plane and λ plane

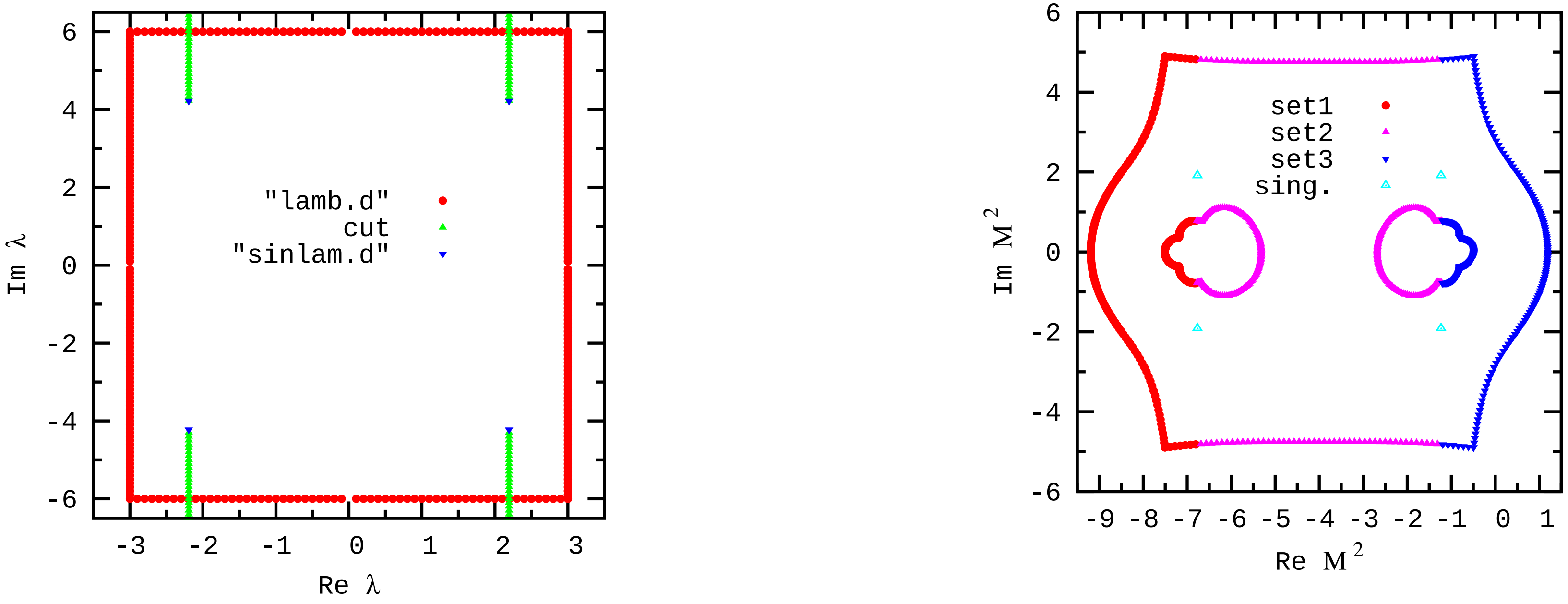


Figure 1: loop in λ plane and loops in M^2 plane

Map between M^2 plane and λ plane

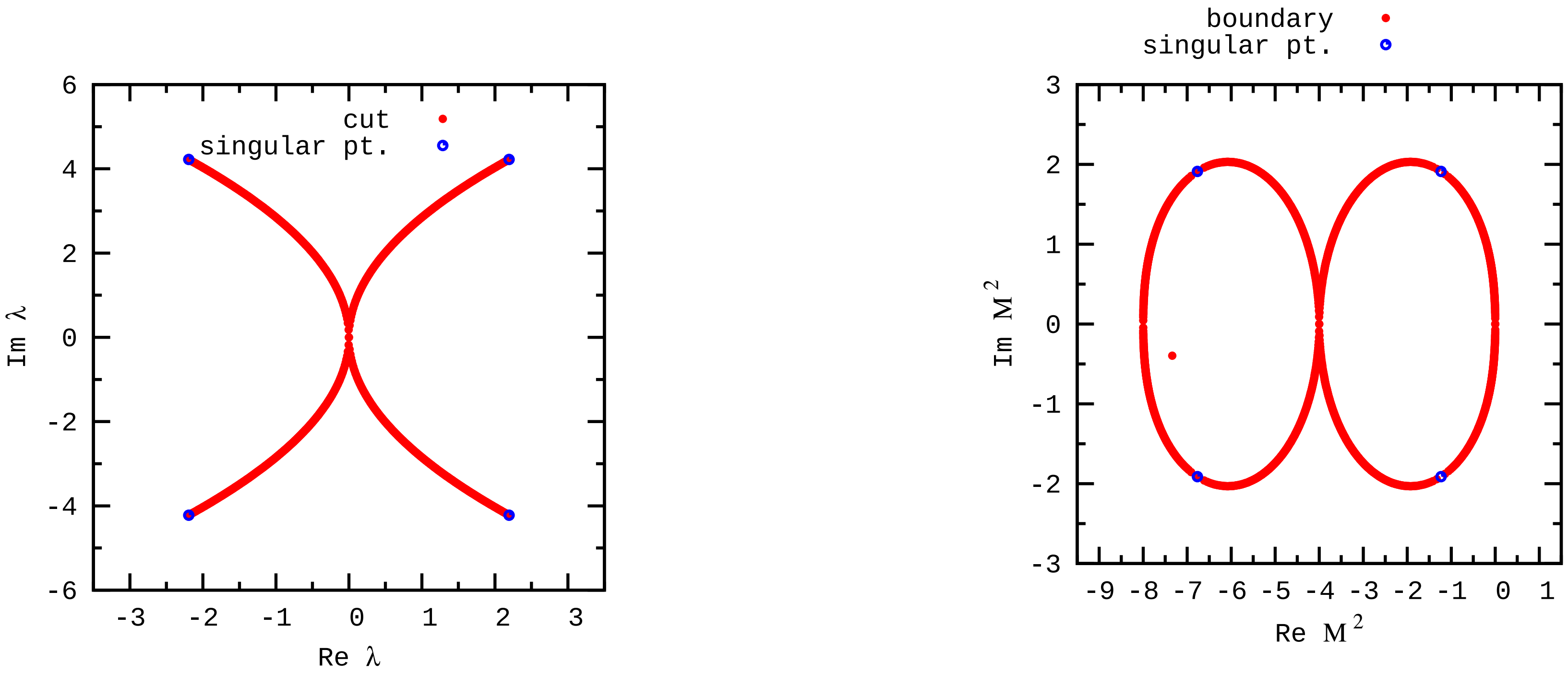


Figure 2: cuts in λ plane and sheet boundaries in M^2 plane

Singular points (Finite V and Infinite V)

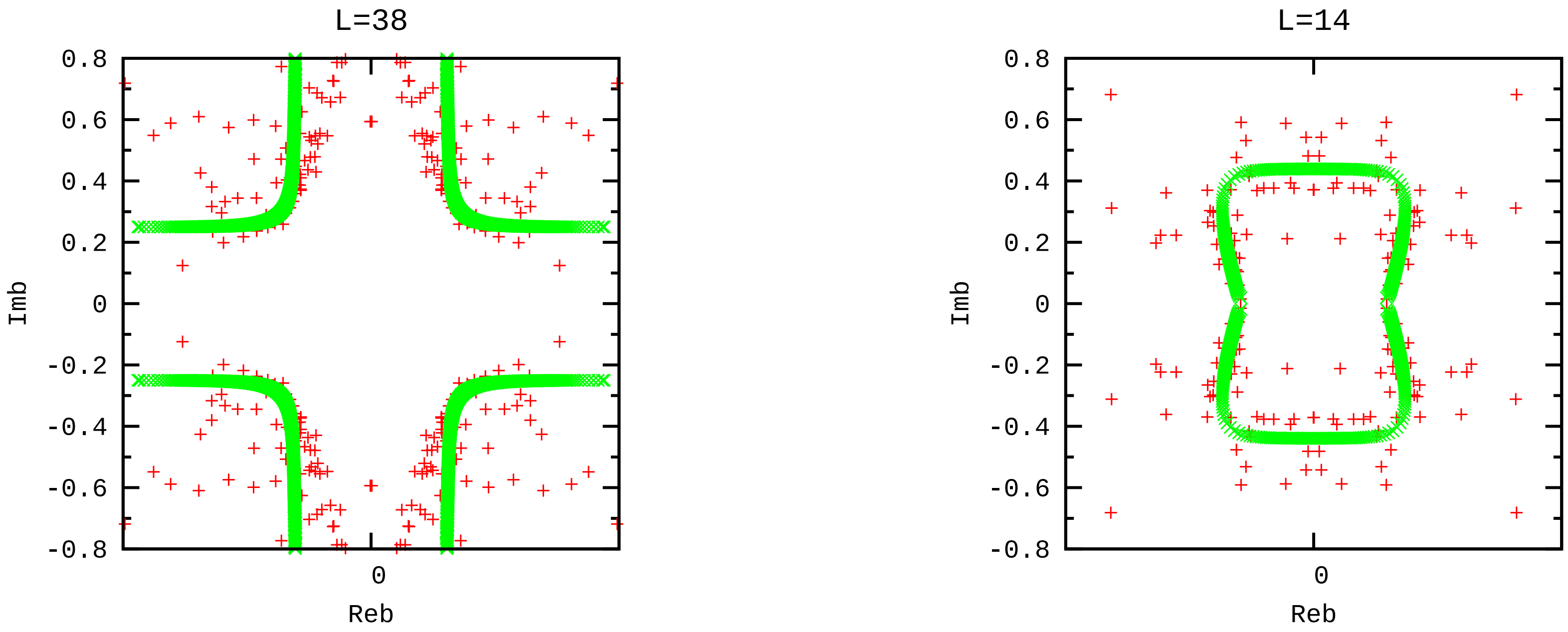


Figure 3: singular points and infinite volume asymptotic mass gap in b plane for $L = 38$ in 2D and $L = 14$ in 3D.

Exact form of partition functions $Z(b)$

Using theorem of residues, we get the general fomula of partition functions,

$$Z(b) = \sum_i P(1/b) e^{\frac{VNbx_i}{2}} \tag{16}$$

where $P(1/b)$ are polynomials of $1/b$ and x_i are the roots of M^2 of the equations $2\sum_{i=1}^{D}(1-\cos(k_i)) + M^2 = 0$

For example, for L=2,N=2,

$$Z(b) = \frac{3}{32}\left(\frac{1}{128b^3} - \frac{e^{-32b}}{128b^3} - \frac{e^{-16b}}{4b^2}\right) \tag{17}$$

Density of states

By doing inverse Laplace transform, the density of states is then

$$n(E) = \frac{N}{2\pi i} \int_{K-i\infty}^{K+i\infty} dbe^{bNE} Z(b) \tag{18}$$

n(E) is piecewise function with the form

$$n(E) = \sum_i P(E - E_i)(2\theta(E - E_i) - 1) \tag{19}$$

which is a quasi-gaussian distribution at large volume and N.

For example,Fig 4 show the density of states at L=2, N=4.

Density of states

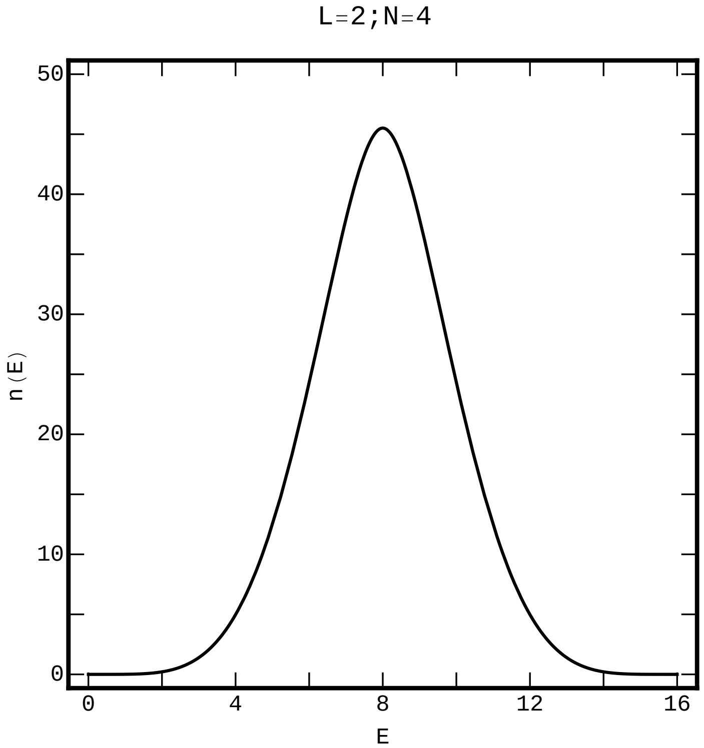


Figure 4: Density of states at L=2, N=4

Zeros of the partition functions (2D)

The average energy per unit of volume V is

$$\mathcal{E} = -(1/(VN))\partial \ln Z/\partial b \ . \tag{20}$$

In the saddle point approximation, we obtain

$$\mathcal{E} = (1/2)(\lambda^t - M^2) \ . \tag{21}$$

If we now integrate over a contour C which close a zero of Z,we have,

$$\oint_C db(dZ/db)/Z = i2\pi \ \times \text{number of zeros in C} \tag{22}$$

We change variable to write

$$\oint_C db(dZ/db)/Z = VN \oint_{C'} dM^2(db/dM^2)(1/2)(1/b - M^2) \qquad (23)$$

where C' is the contour corresponding to C in the M^2 plane.

Consequently, if the contour C' in the M^2 plane does not cross the cut, then there are no zeros of the partition function inside the corresponding C in the b-plane. We conclude that in the large-N limit,there is no Fisher's zero below the image of singular points line in b-plane. (Phys.Rev.D80:054020,2009. Y.Meurice)

Here is a easy way to understand this conclusion in infinite volume: We choose the contour C just alone the 4 hyperbolas, all the points on the contour have the property that $db/dM^2 = 0$. From Equ(23), we can also get the conclusion.

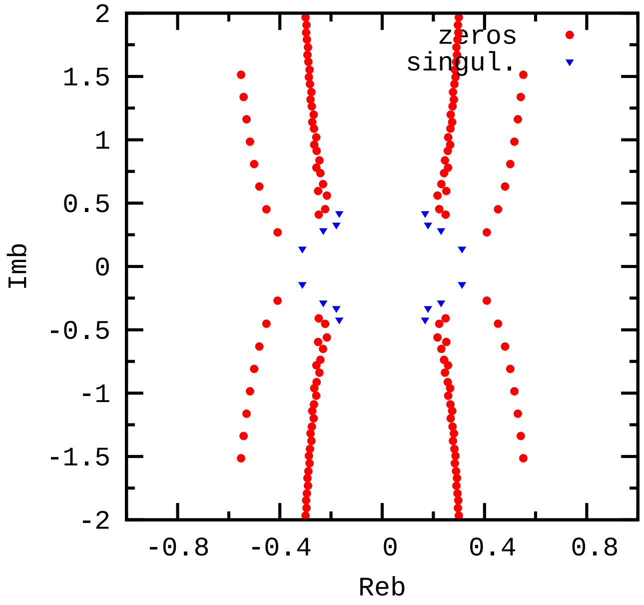
Zeros of the partition functions (2D)

The argument has been checked by numerical calculations. Fig 5 show the zeros in b and λ plane with $L = 6$,$N = 2$.

The pictures in the finite volume indicate that the singular points of the gap equation correspond to the end of lines of complex zeros at infinite volume. Hence, zeros always stay away from the physical(i.e. real)domain in 2D.

The density of zeros in complex b plane (number of zeros in a fix area at b plane) proportional to NV(Fig 6 are the log-log plots and the fit lines).

Zeros of the partition functions (2D)



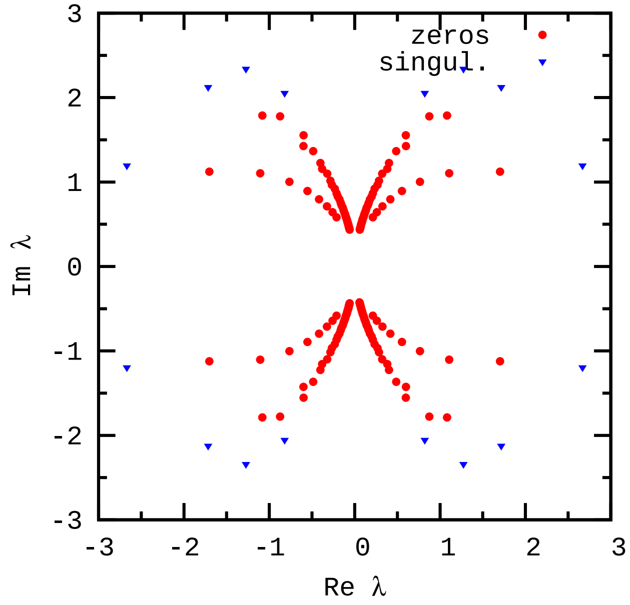
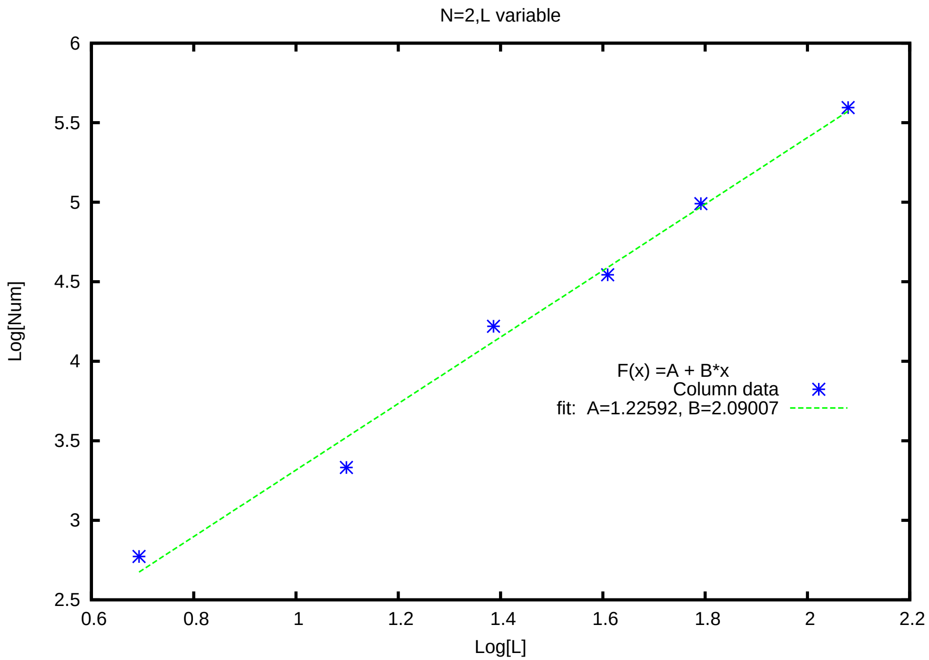


Figure 5: Results of zeros and singular points in b plane and λ plane at L=6,N=2.

Density of zeros in b-plane



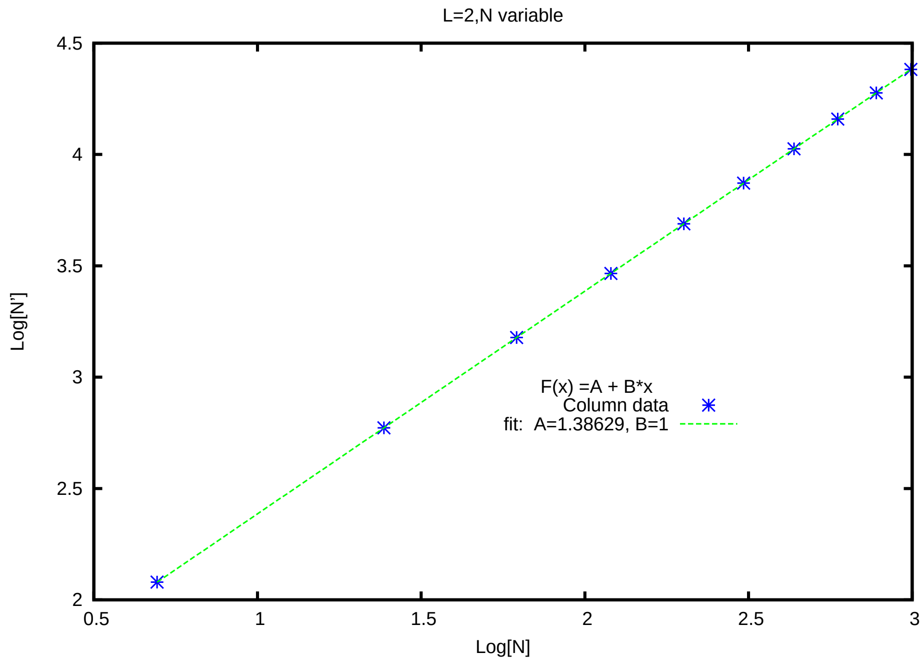


Figure 6: density of zeros in b-plane (N=2, L variable and L=2,N variable)

Conclusions

- we study the map between the mass gap M^2 and the 't Hooft coupling on finite size L and in the large-N limit. We show that the map requires a Riemann surface with $q+1$ sheets and 2q cuts in the 't Hooft coupling plane, where q is an integer of order L^D.
- From the partition functions in finite volume,we searched Fishers zeros. It has a similar features as 4D lattice gauge theory:Zeros stay away from the physical(i.e. real)domain. And At infinite volume, the singular points of the gap equation correspond to the end of lines of complex zeros.
- The number of zeros in unit area in b-plane is proportional to V N
- Density of states are piecewise functions.

Thank you!