Effects of retardation in the renormalization group for fermions

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Thanks: UC-Labs Program, NSF

INT - New applications of RG, February 2010

Outline

- system considered: interacting fermions
- RG for interacting fermions
- frequency dependent interactions
- simple example: circular Fermi surface in 2D
- Eliashberg theory
- Large-N expansion
- another simple example: 1D Holstein-Hubbard model
- 2D square lattice at half-filling
- Conclusions

Interacting fermions:

Examples: electrons, quasiparticles, fermionic atoms such as ^{6}Li, ^{40}K.

Generic action:

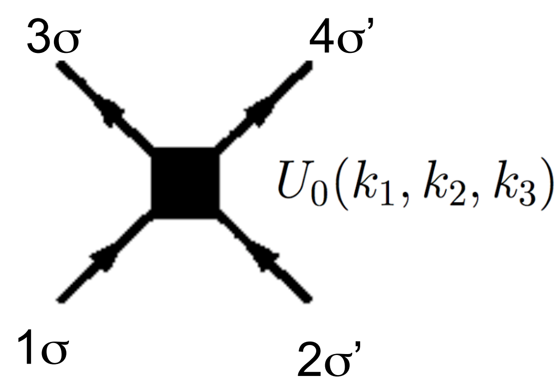
$$S = T \sum_{k,\sigma} \bar{\psi}_{\sigma k}(i\omega_n - \xi_{\mathbf{k}})\psi_{\sigma k} + T^3 \sum_{k_1,k_2,k_3} \sum_{\sigma,\sigma'} U_0(k_1,k_2,k_3)\bar{\psi}_{\sigma k_3}\bar{\psi}_{\sigma' k_4}\psi_{\sigma' k_2}\psi_{\sigma k_1}$$

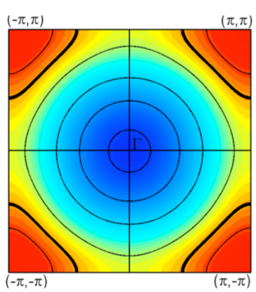
where:

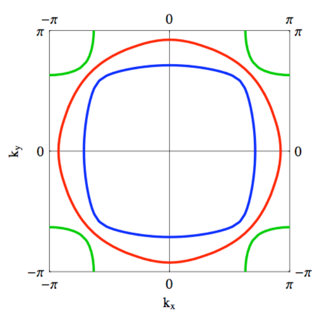
$$k \equiv (\omega_n, \mathbf{k})$$

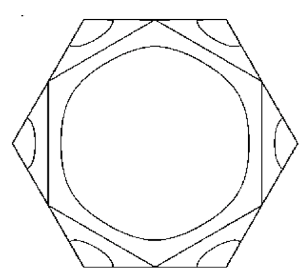
$$k_1 + k_2 = k_3 + k_4$$

$$\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$$









Frequency-dependent interactions

$U_0(k_1, k_2, k_3)$

Examples:

Dynamical Coulomb screening

Phonon-mediated interactions

Interactions mediated by BEC fluctuations

Interactions mediated by any boson-exchange coupling

Retardation effects important when $v_B \lesssim v_F$

$$\lambda = 2N(0)g^2/\omega_E$$

$$U_0(k_1, k_2, k_3) = u_0(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) - 2g(k_1, k_3)g(k_2, k_4)D(k_1 - k_3)$$

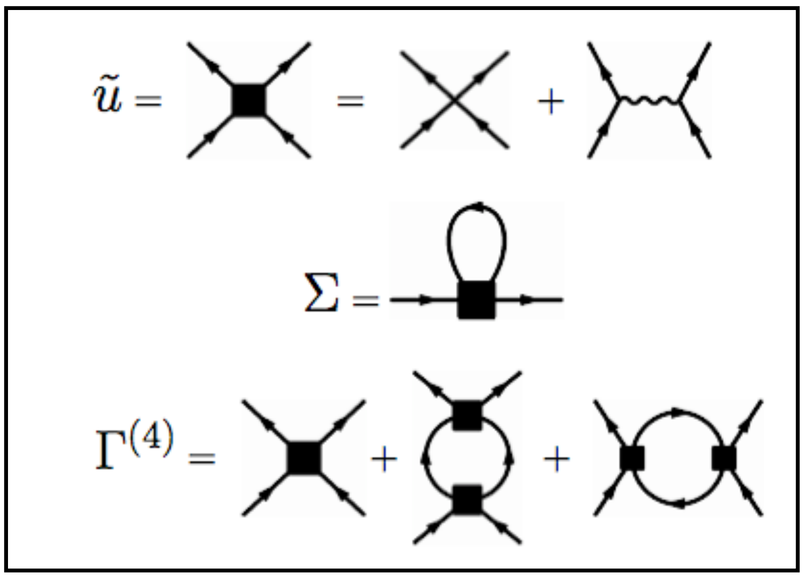
Circular Fermi surface, isotropic interaction, BCS channel:

s-wave:

$$\tilde{v}(\omega_1,\omega_3)=N(0)\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_3}{2\pi}\tilde{u}(-k_3,k_3,-k_1,k_1)$$

RG for the couplings:

$$\frac{d}{d\ell}\tilde{v}(\omega_1,\omega_3,\ell)=-\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{\Lambda_\ell\ \tilde{v}(\omega_1,\omega,\ell)\tilde{v}(\omega,\omega_3,\ell)}{\Lambda_\ell^2+Z_\ell^2(\omega)\omega^2}$$

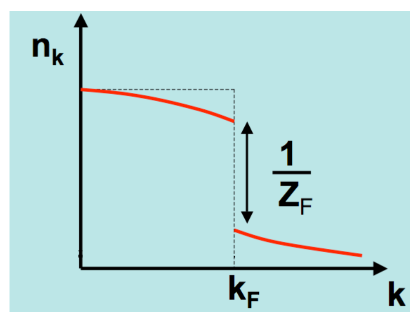


Initial condition:

$$\tilde{v}(\omega_1,\omega_3,\ell=0)=u_0-\lambda\omega_E D(\omega_1-\omega_3)$$

Self-energy correction:

$$\Sigma(\omega,\mathbf{k})\approx\Sigma_0+i(1-Z(\omega,\mathbf{k}))\omega$$



$$\frac{d}{d\ell}\tilde{v}(\omega_1,\omega_3,\ell) = -\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{\Lambda_\ell\ \tilde{v}(\omega_1,\omega,\ell)\tilde{v}(\omega,\omega_3,\ell)}{\Lambda_\ell^2 + Z_\ell^2(\omega)\omega^2}$$

matrix equation: $U_{ij} = v(\omega_i, \omega_j)$

$$\frac{d\mathbf{U}}{d\ell} = -\mathbf{U}\cdot\mathbf{M}\cdot\mathbf{U}$$

Exact solution:

$$\begin{aligned}\mathbf{U}(\ell) &= [1 + \mathbf{U}(0)\cdot\mathbf{P}(\ell)]^{-1}\,\mathbf{U}(0)\\ \mathbf{P}(\ell) &= \int_0^\ell d\ell'\mathbf{M}(\ell')\,.\end{aligned}$$

Coupling diverges at $\ell = \ell_c$, where:

$$\det\left[1 + \mathbf{U}(0)\cdot\mathbf{P}(\ell_c)\right] = 0$$

Weak-intermediate coupling
McMillan, '68

$$T^* \approx 1.13\,\omega_E \exp\{-(1+\lambda)/(\lambda-\mu^*(1+\lambda))\}$$

Strong Coupling
Allen-Dynes, '75

$$T^* \approx 0.16\sqrt{\lambda}\omega_E$$

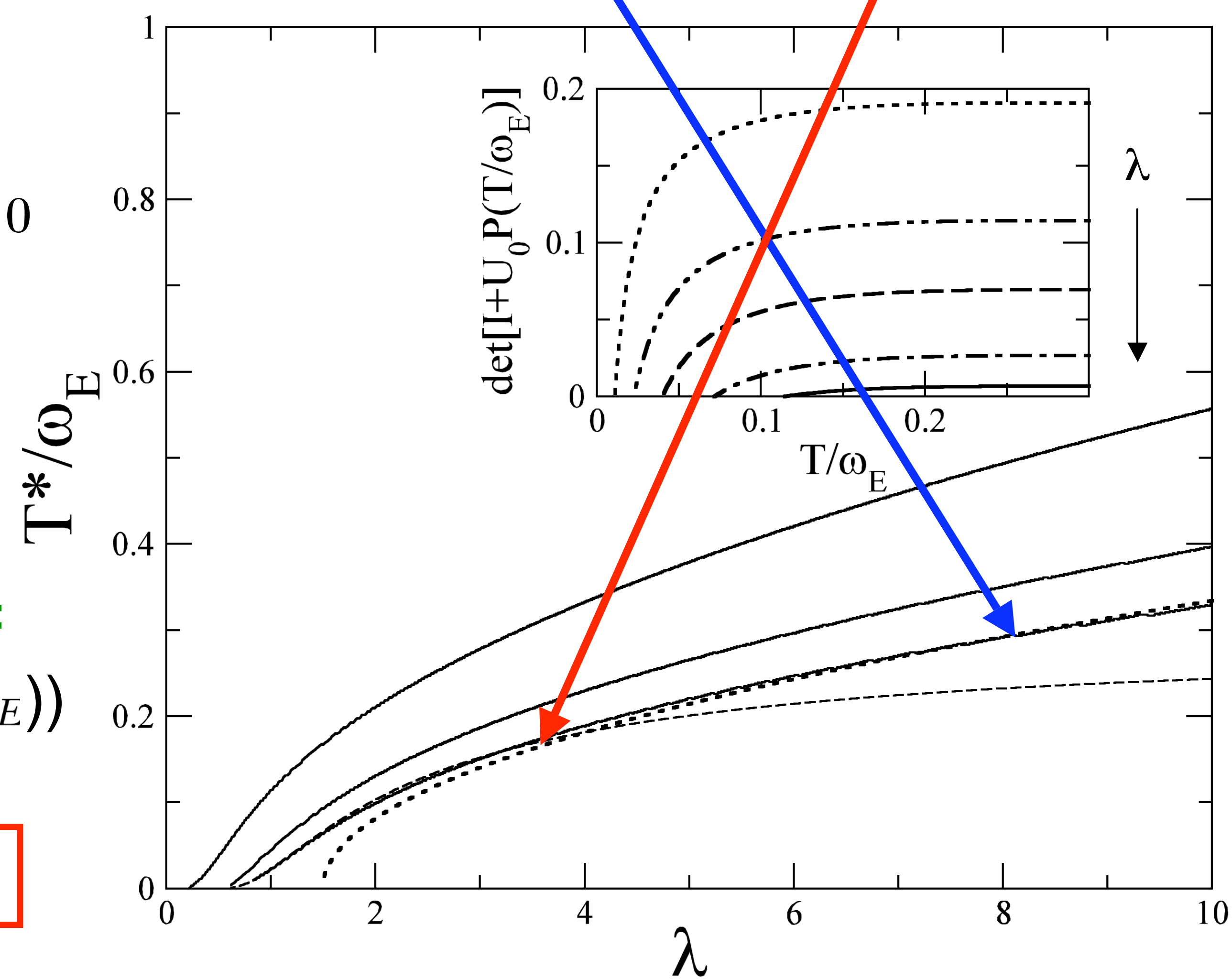
T* calculated from:

det [**1** + **U**(0). **P**(T^*/ω_E)] = 0

$$\lambda = 2N(0)g^2/\omega_E$$

Effective e-e interaction:

$$\mu^* = u_0/(1+u_0 \ln(\Lambda_0/\omega_E))$$



SWT, AH Castro Neto, R Shankar, DK Campbell, PRB **72**, 054531(2005) matrix equation: $U_{ij} = v(\omega_i, \omega_j)$

$$\frac{d\mathbf{U}}{d\ell} = -\mathbf{U}\cdot\mathbf{M}\cdot\mathbf{U}$$

Exact solution:

$$\begin{aligned}\mathbf{U}(\ell) &= [1+\mathbf{U}(0)\cdot\mathbf{P}(\ell)]^{-1}\,\mathbf{U}(0)\\ \mathbf{P}(\ell) &= \int_0^{\ell} d\ell'\mathbf{M}(\ell')\,.\end{aligned}$$

Coupling diverges at $\ell = \ell_c$, where: $$\det[1+\mathbf{U}(0)\cdot\mathbf{P}(\ell_c)] = 0$$

which is equivalent to: $$[1+\mathbf{U}(0)\cdot\mathbf{P}(\ell_c)]\cdot\mathbf{f} = 0$$

$$f(\omega) = -\frac{1}{\pi}\int_{\omega'}\int_{\Lambda_c}^{\infty}\frac{[u_0-\lambda\omega_E D(\omega-\omega')]}{Z_\Lambda^2(\omega')\,\omega'^2+\Lambda^2}f(\omega')$$

→ **gives ELIASHBERG's equations at T=T_c**

Finite temperature: find a temperature T *
above which Fermi liquid is stable

$$\omega_n = \pi T^*(2n+1) \qquad \Lambda_c \to 0$$

Define: $$\boxed{\phi(\omega_n) = f(\omega_n)/Z(\omega_n)}$$

Interaction:

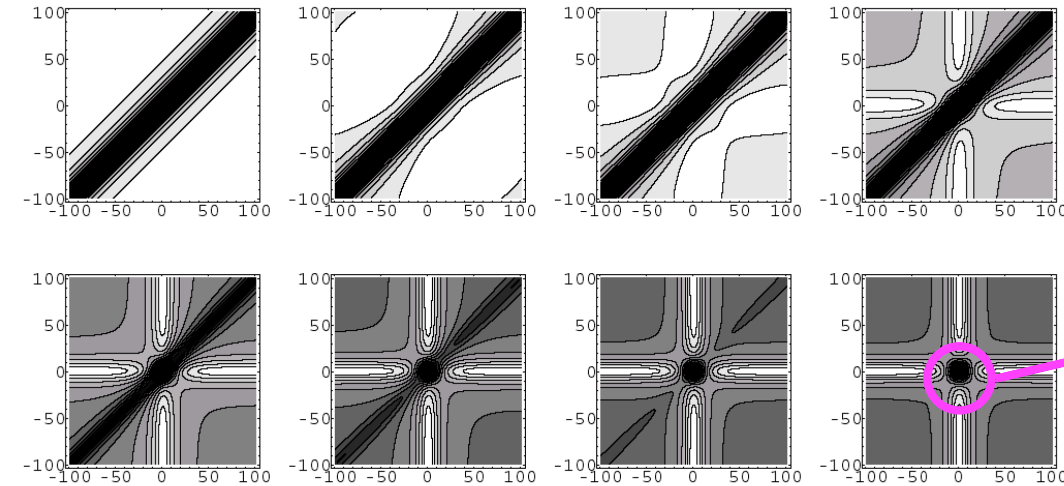
$$\boxed{Z(\omega_n)\phi(\omega_n) = -\pi T^* \sum_m [u_0 - \lambda\omega_E D(\omega_n - \omega_m)] \frac{\phi(\omega_m)}{|\omega_m|}}$$

Self-energy:

$$\boxed{Z(\omega_n) = 1 + \lambda\omega_E \frac{\pi T^*}{\omega_n} \sum_m \operatorname{sgn}(\omega_m) D(\omega_n - \omega_m)}$$

Eliashberg's equations

RG evolution of the couplings in the BCS channel (λ = 0.3, 4.0)

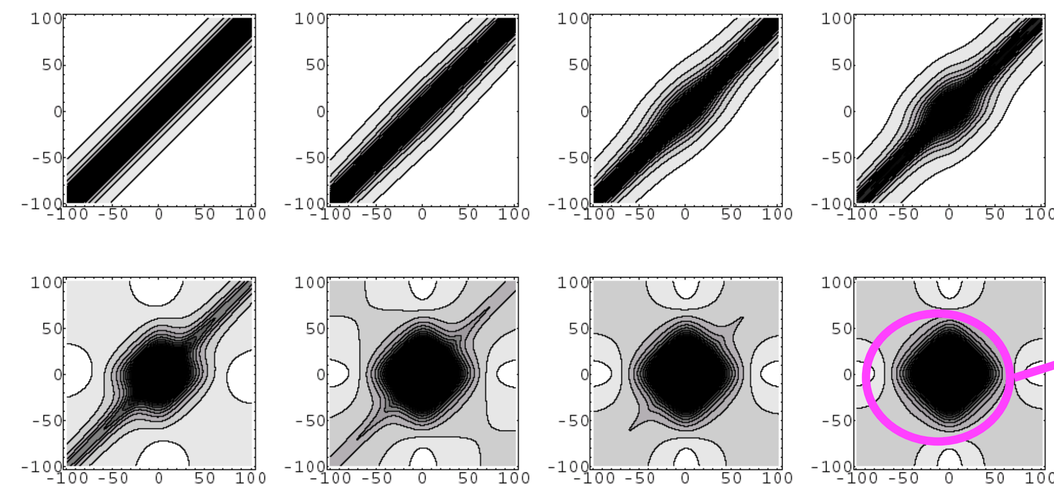


“weak-coupling”
λ < 1

~ ω_E

“two-step” RG is OK

Figure 1: Plots of the NxN matrix U at different RG scales ℓ. Here the number of frequency divisions $N=200$, and the value of the parameters used are $\lambda=0.3$, $\Lambda_0=100$, $\omega_E=10$, $u_0=0.1$. Panels correspond to ℓ = 0, 2.5, 3, 5, 6.5, 6.9, 7.1, and 7.19.



“strong-coupling”
λ > 1

~ T_c

SWT *et al.*, Philos. Mag. **86**, 2631 (2006)

Figure 2: Plots of the NxN matrix U at different RG scales ℓ. Here the number of frequency divisions $N=200$, and the value of the other parameters are $\lambda=4$, $\Lambda_0=100$, $\omega_E=10$, and $u_0=0.1$. Panels correspond to ℓ = 0, 1, 2, 2.5, 3, 3.13, 3.157, 3.172. The scale $2W_c \approx 40$ distinguishes the high and low frequencies close to ℓ_c.

Circular Fermi surface, but anisotropic boson-exchange couplings

$\lambda = 0.4$ $\lambda = 4.0$

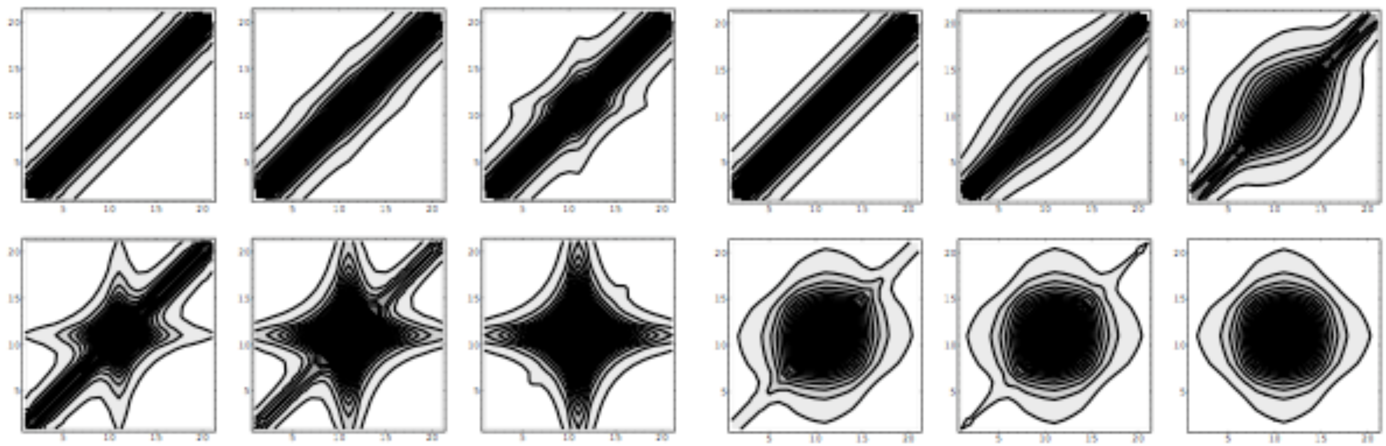


FIG. 2: Same as Fig.[1] but for the $d_{x^2-y^2}$-channel, $\gamma = 2$. The six panels on the left side are the evolution of the matrix elements at weak coupling, $\lambda = 0.4$ and the panels on the right side are the same but for the strong coupling regime, $\lambda = 4.0$

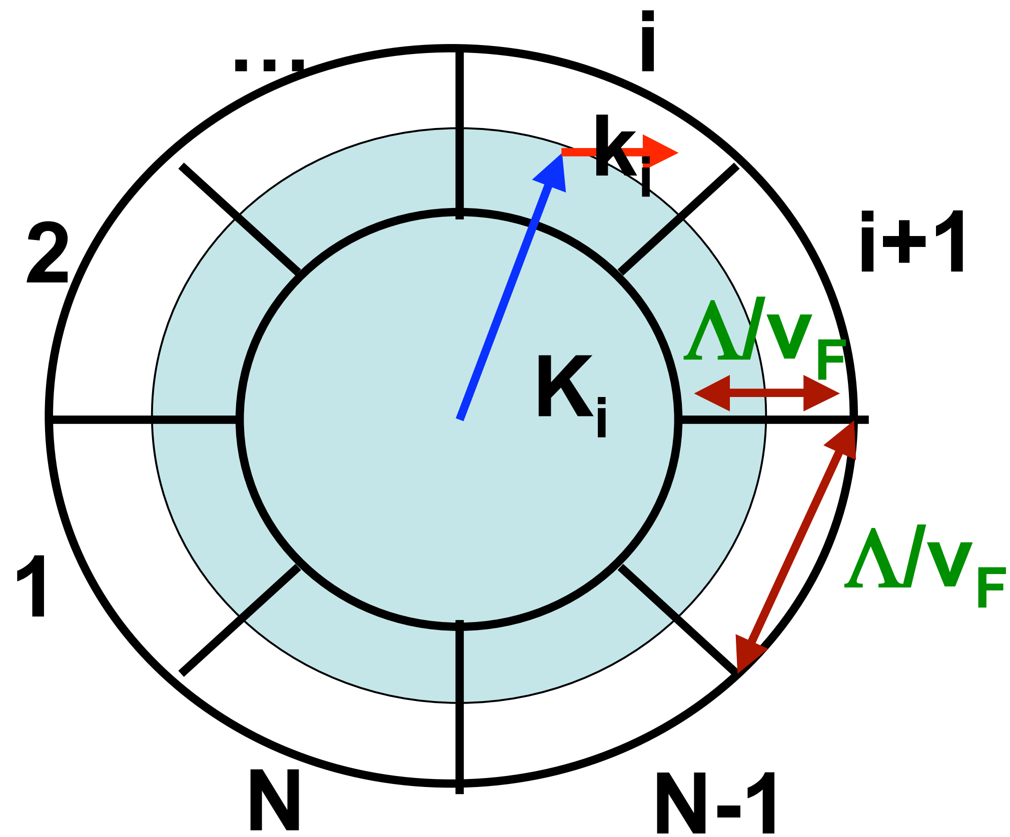
Generalized Eliashberg equations.

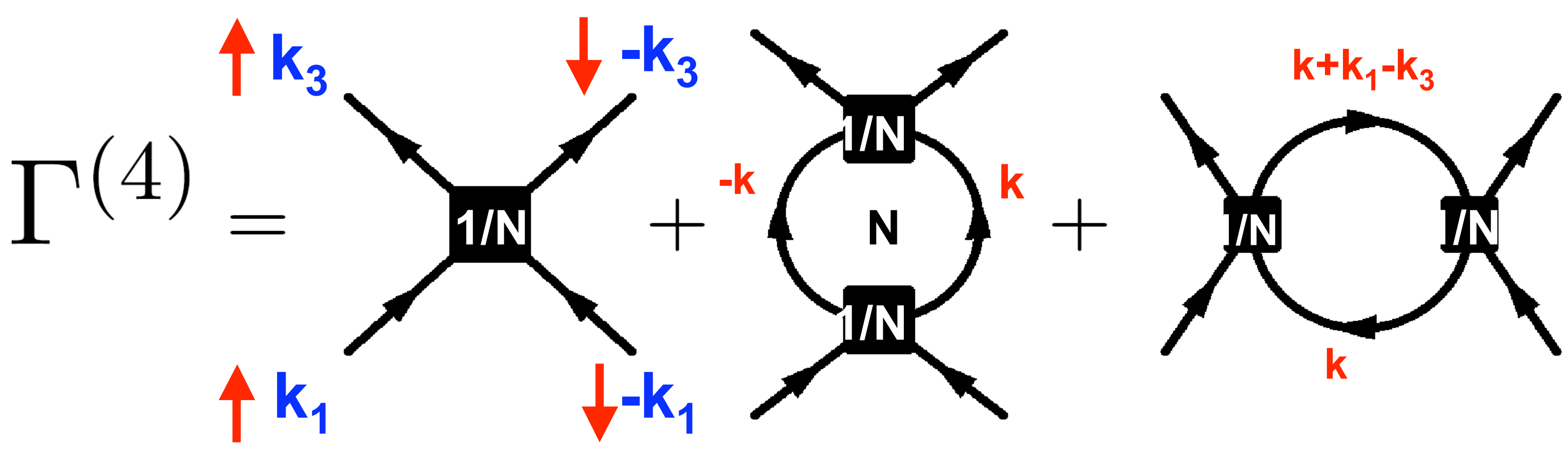
R. Roldan, SWT, M. P. Lopez-Sancho, PRB 2009

Large-N analysis:

$N = E_F / \Lambda$

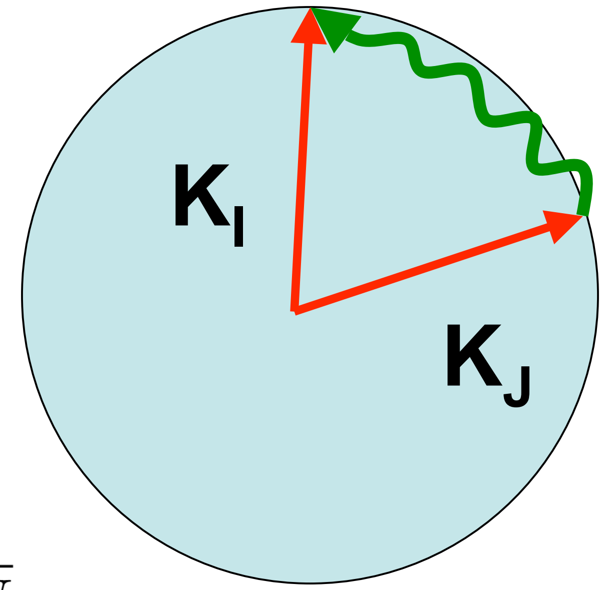
R. Shankar, Rev. Mod. Phys. **68**, 129 (1994)



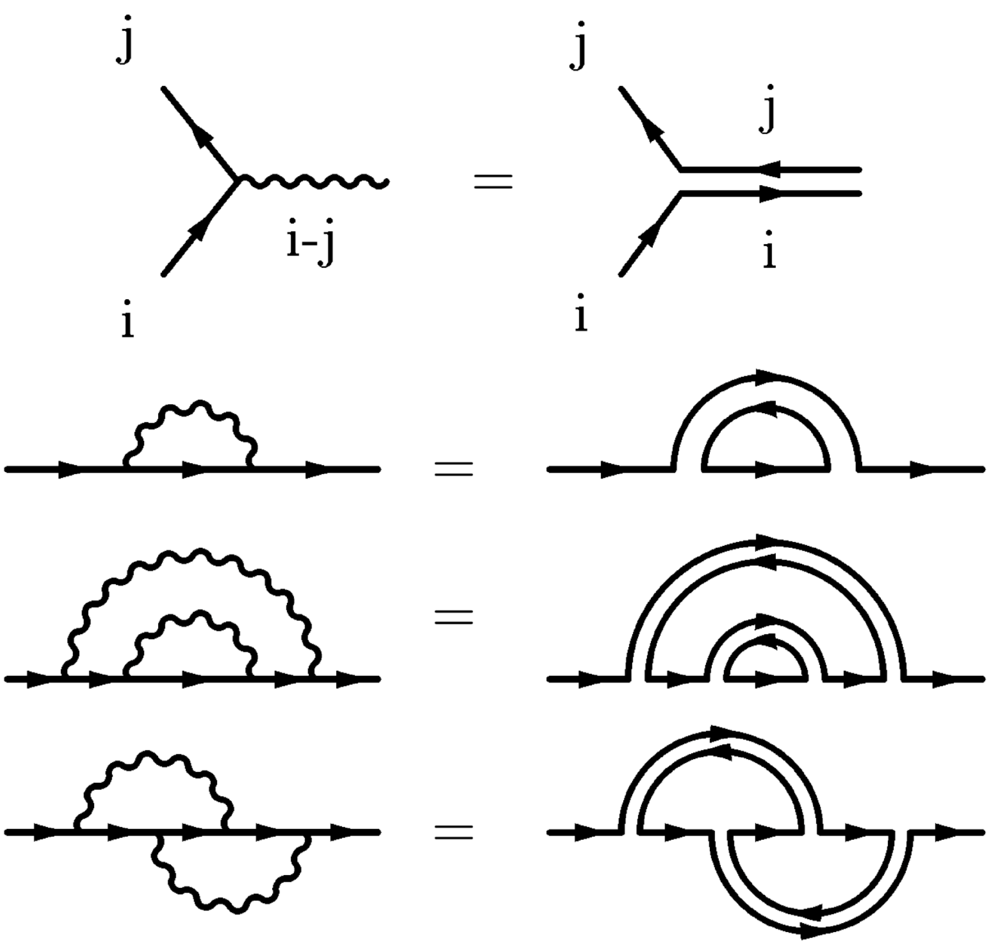


Migdal's theorem ('58)

$$u \sim g^2 \sim 1/N$$



t'Hooft 1974



$\propto \; g/\sqrt{N}$

$\propto \; \frac{g^2}{N} \; N$

$\propto \; g^4$

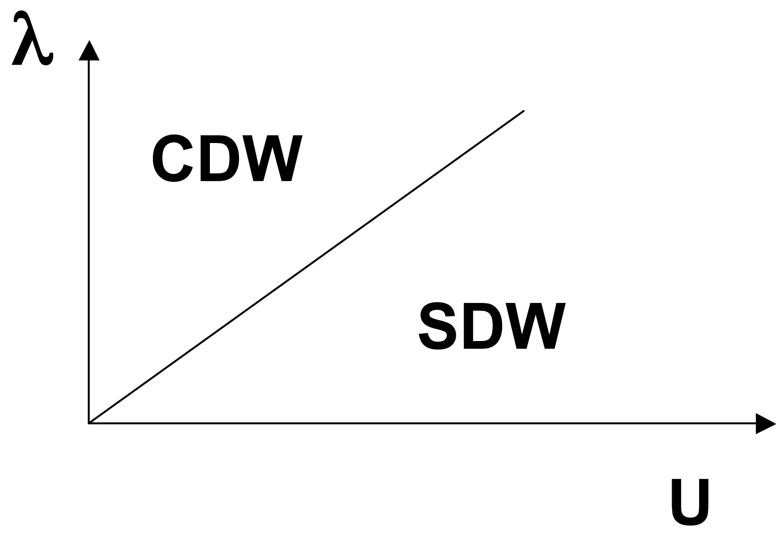
$\propto \; g^4/N^2$

Self-energy

No e-ph vertex corrections

1D Holstein-Hubbard model:

$$H = -t\sum_{i,\sigma}(c^\dagger_{i+1,\sigma}c_{i,\sigma} + H.c.) + U\sum_i n_{i,\uparrow}n_{i,\downarrow}$$
$$+g_{ep}\sum_{i,\sigma}(a^\dagger_i + a_i)n_{i,\sigma} + \omega_0\sum_i b^\dagger_i b_i,$$

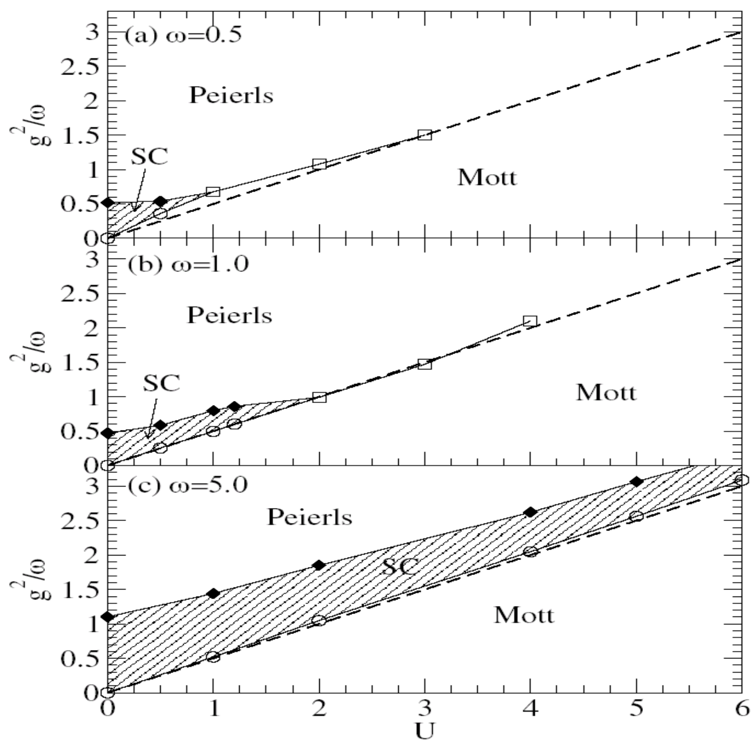


$\lambda = 2g_{ep}^2 / \omega_0$ $\qquad$ $U_{eff} \equiv U - \lambda$

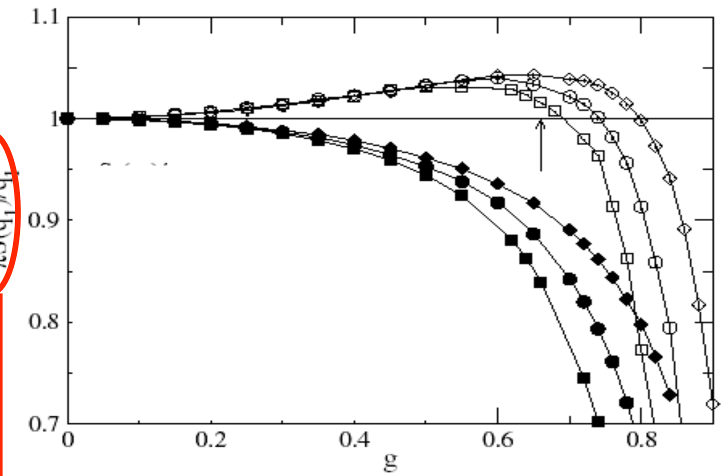
J. E. Hirsch and E. Fradkin, PRB 27, 4302 (1983)

also: H. Fehske, *et al.,* PRB 69, 165115 (2004);
I. P. Bindloss, PRB 71, 205113 (2005)

More recently a third phase has been proposed:



R. T. Clay and R. P. Hardikar, PRL 95, 096401 (2005)



Kρ

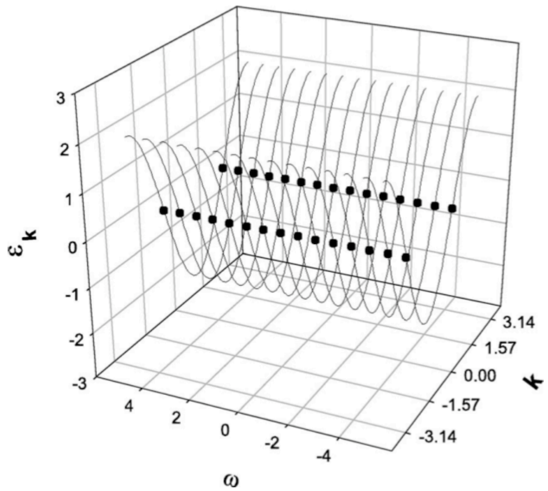
From Tomonaga-Luttinger liquid theory:

$O^{CDW} \sim x^{-K\rho}$ $O^{SC} \sim x^{-1/K\rho}$

also: C. Wu, *et al.*, PRB 52, 15683 (1995)
E. Jeckelmann, *et al.*, PRB 60, 7950 (1999)
Y. Takada and A. Chatterjee, PRB 67, 0811102 (2003)
Y. Takada, J. Phys. Soc. Jpn., 65, 1544 (1996)

Functional RG analysis:

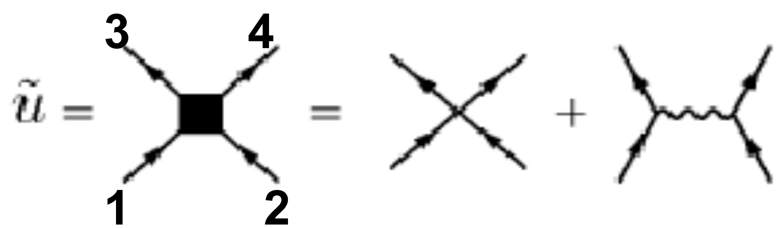
K.-M. Tam *et al.*, PRB **75**, 161103 (2007)



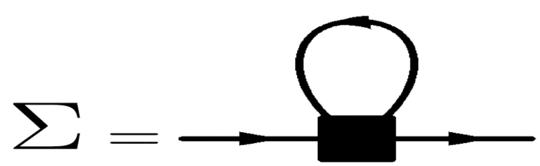
$$g_1 \rightarrow g_1(\omega_1, \omega_2, \omega_3, \omega_4)$$
$$g_2 \rightarrow g_2(\omega_1, \omega_2, \omega_3, \omega_4)$$
$$g_3 \rightarrow g_3(\omega_1, \omega_2, \omega_3, \omega_4)$$
$$g_4 \rightarrow g_4(\omega_1, \omega_2, \omega_3, \omega_4)$$

$$\omega_1 + \omega_2 = \omega_3 + \omega_4$$

Initial conditions:

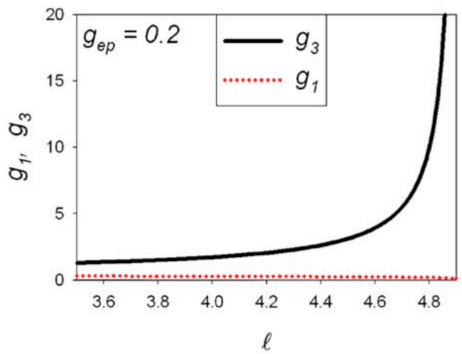


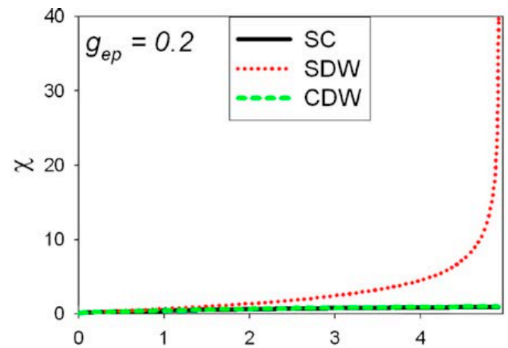
$$g_i(\omega_1, \omega_2, \omega_3, \omega_4) = U - \frac{2g_{ep}^2}{\omega_0}\left(\frac{\omega_0^2}{\omega_0^2 + (\omega_1 - \omega_3)^2}\right)$$



RG flows of susceptibilities and couplings ($\omega_0 = 1$):

$U_{eff} > 0$



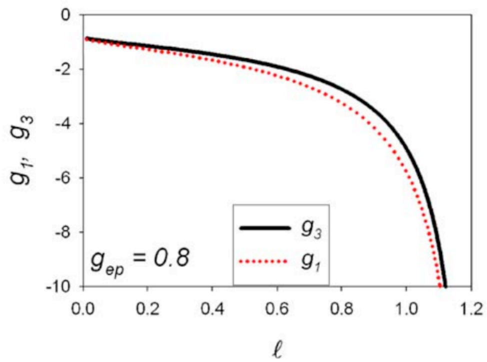


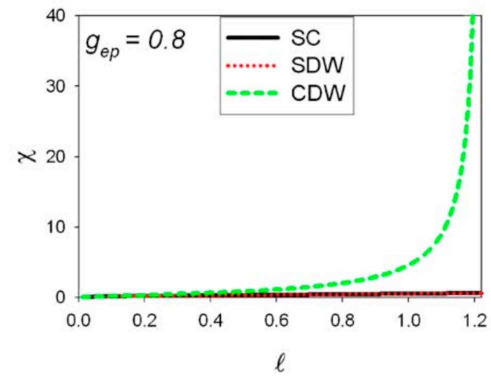
SDW

$\Delta_C \neq 0$

$\Delta_S = 0$

$U_{eff} < 0$





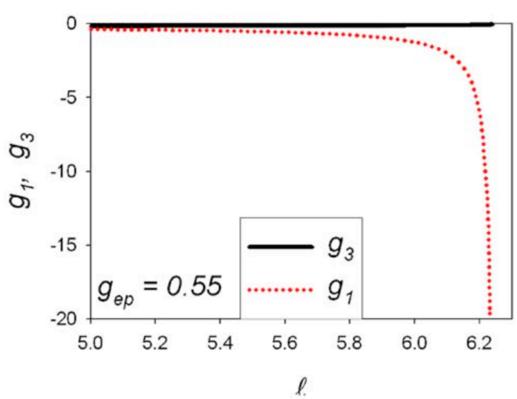
CDW

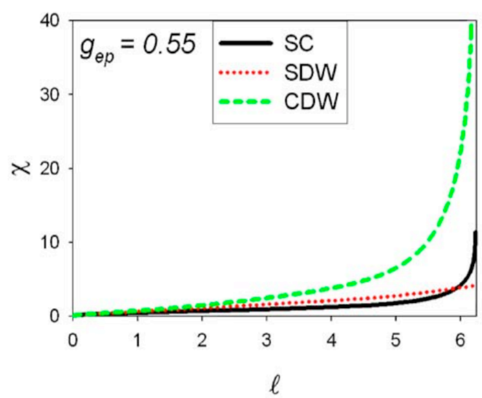
$\Delta_C \neq 0$

$\Delta_S \neq 0$

"Intermediate" region:

$U_{eff} = -\delta$





CDW

$\Delta_C = ?$

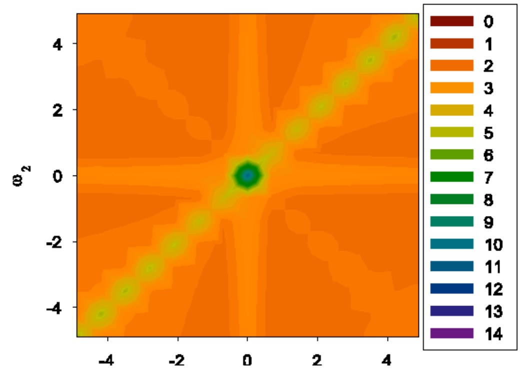
$\Delta_S \neq 0$

Frequency structure of $g_3(\omega_1,\omega_2,\omega_1,\omega_2)$

K.-M. Tam *et al.*, PRB **75**, 161103 (2007)

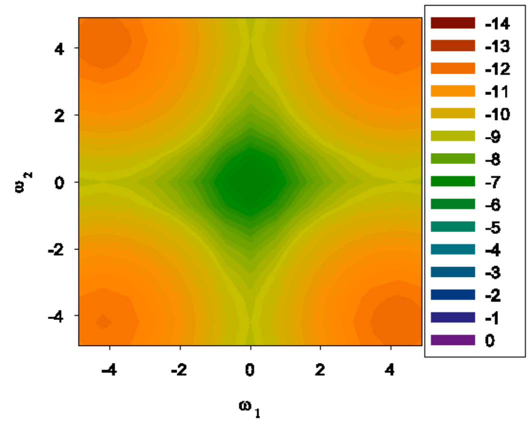
g_{ep} = 0.2 (SDW)

$U_{eff} < 0$



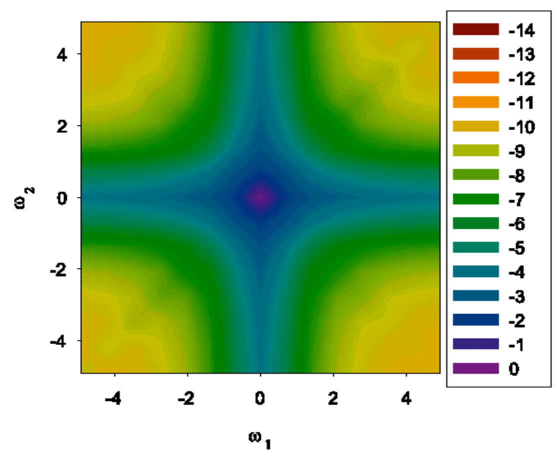
g_{ep} = 0.8 (CDW)

$U_{eff} > 0$



g_{ep} = 0.55 (CDW)

$U_{eff} \approx 0$



"dynamical umklapp"

How to conciliate with Kρ > 1:

- Kρ > 1 does not mean SC is dominant!

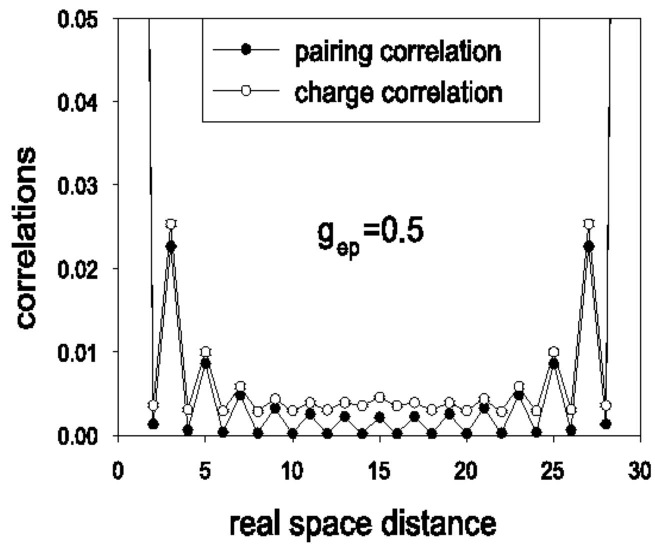
$$O^{CDW}(x) \propto x^{-\alpha K_\rho} \equiv x^{-K_{CDW}}$$

$$O^{SC}(x) \propto x^{-\beta/K_\rho} \equiv x^{-K_{SC}}$$

D. Loss and T. Martin, PRB 50, 12160 (1994)
M. Tezuka, *et al.*, PRL 96, 226401 (2005)

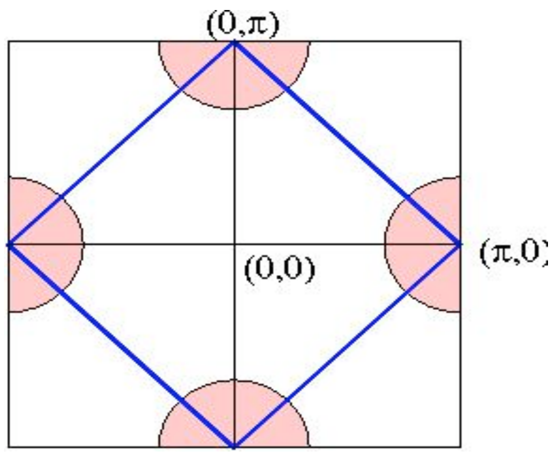
Ladder systems:
K.-M. Tam *et al.*, PRB 75, 195119 (2007)

Direct calculation of susceptibilities (Determinantal Quantum Monte-Carlo):

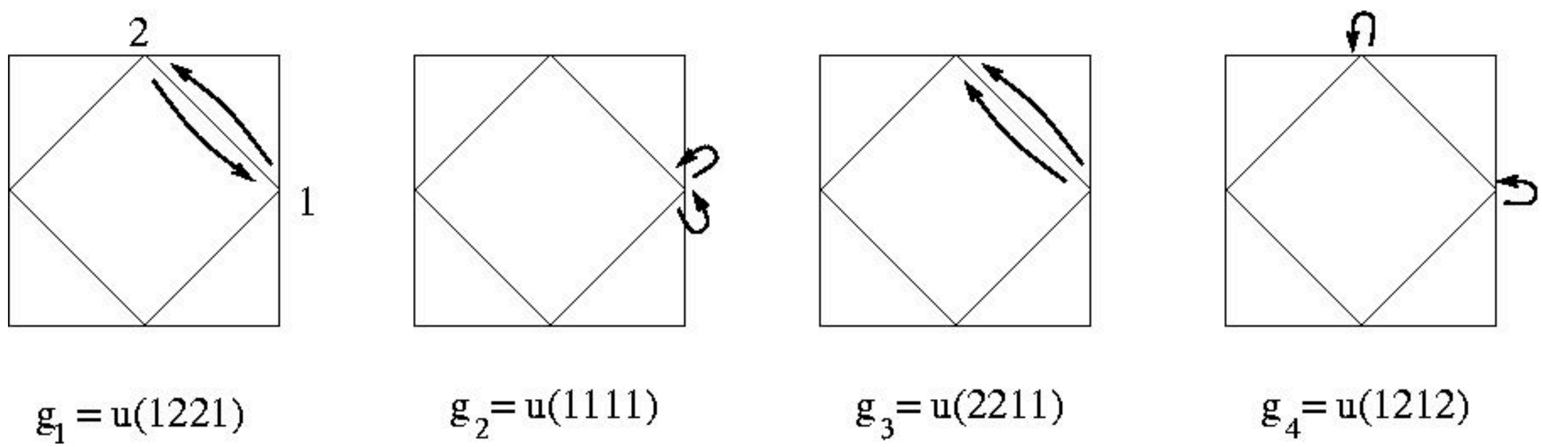


K.-M. Tam *et al.* (unpublished)

Two-patch model for van Hove problem:



H. Schulz, Europhys. Lett. (1987)



phonon coupling: $\lambda = 2N(0)g^2/\omega_E$

Allow for anisotropic phonons, calculate flow of susceptibilities:

$u_0 = 0.5,\ \omega_E = 1.0$

$\lambda_0 < \lambda_\pi$ $\lambda_0 > \lambda_\pi$

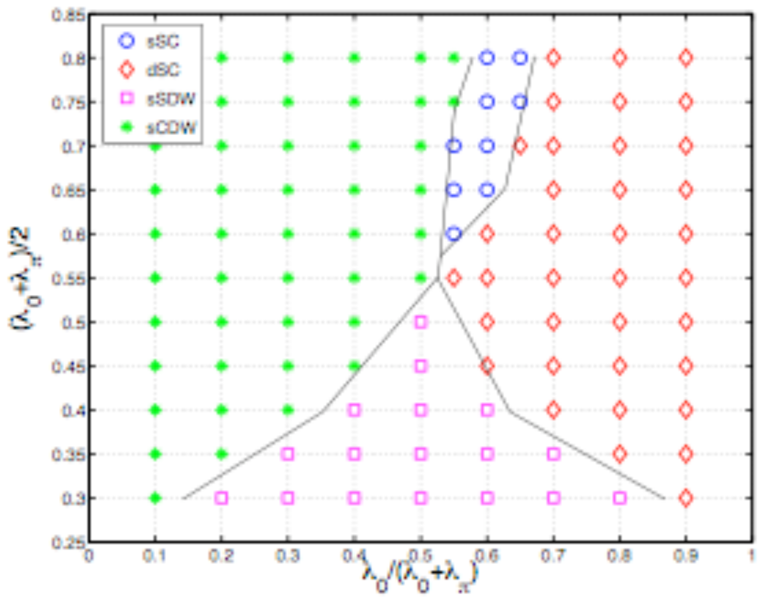


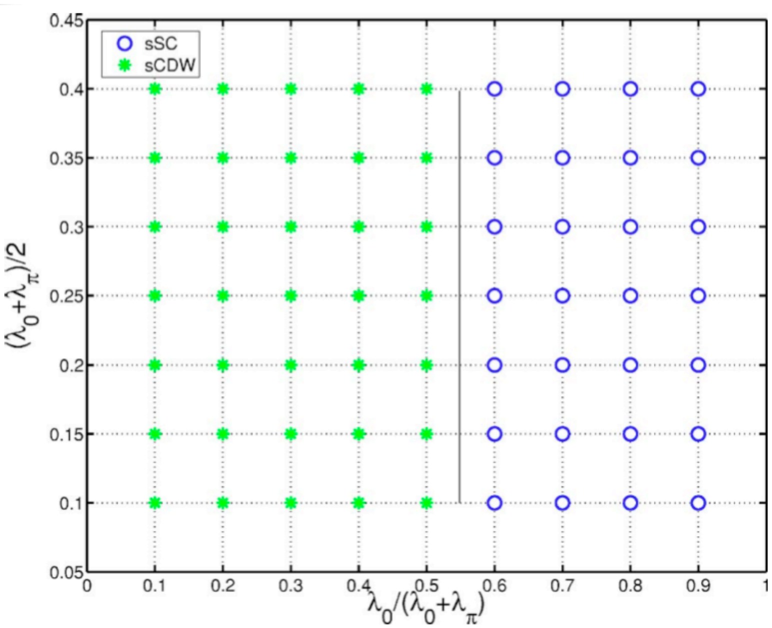
FIG. 1: (Color online) Phase diagram for Einstein phonons of frequency $\omega_E = 1.0$. Four phases involving antiferromagnetism (sSDW) (purple squares), charge density wave (sCDW) (green stars), and s-wave (sSC) (blue circles) and d-wave (dSC) (red rhombs) superconductivity compete in the vicinity where the average phononic strength $\bar{\lambda}$ approaches the bare on-site repulsion $u_0 = 0.5$. The lines distinguishing the different domains are guides to the eye.

$$g_{1,3}^{\ell=0}(\omega_1,\omega_2,\omega_3) = u_0 - \lambda_\pi \frac{\omega_E^2}{\omega_E^2 + (\omega_1 - \omega_3)^2}$$

$$g_{2,4}^{\ell=0}(\omega_1,\omega_2,\omega_3) = u_0 - \lambda_0 \frac{\omega_E^2}{\omega_E^2 + (\omega_1 - \omega_3)^2}$$

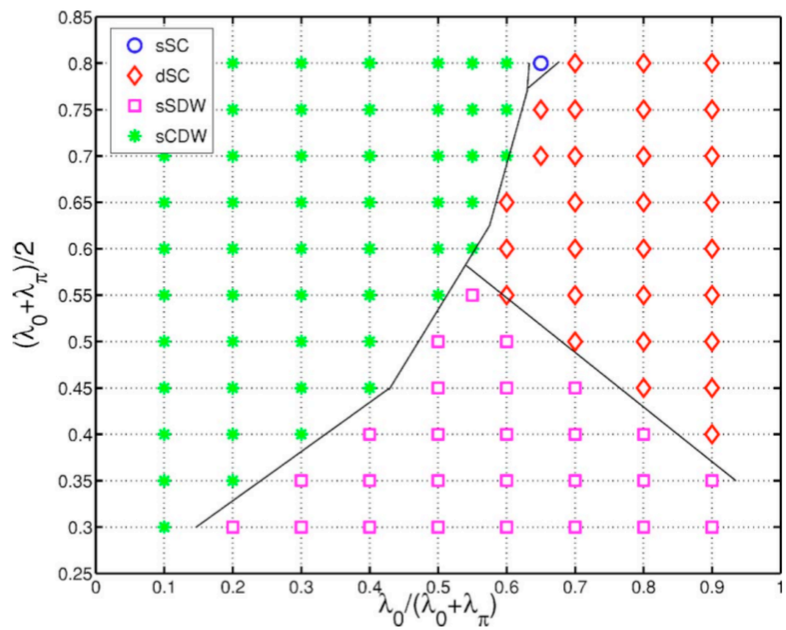
F. D. Klironomos and SWT, PRB **74**, 205109 (2006)

$u_0 = 0, \omega_E = 1.0$



Need repulsive component for d-wave SC to develop.

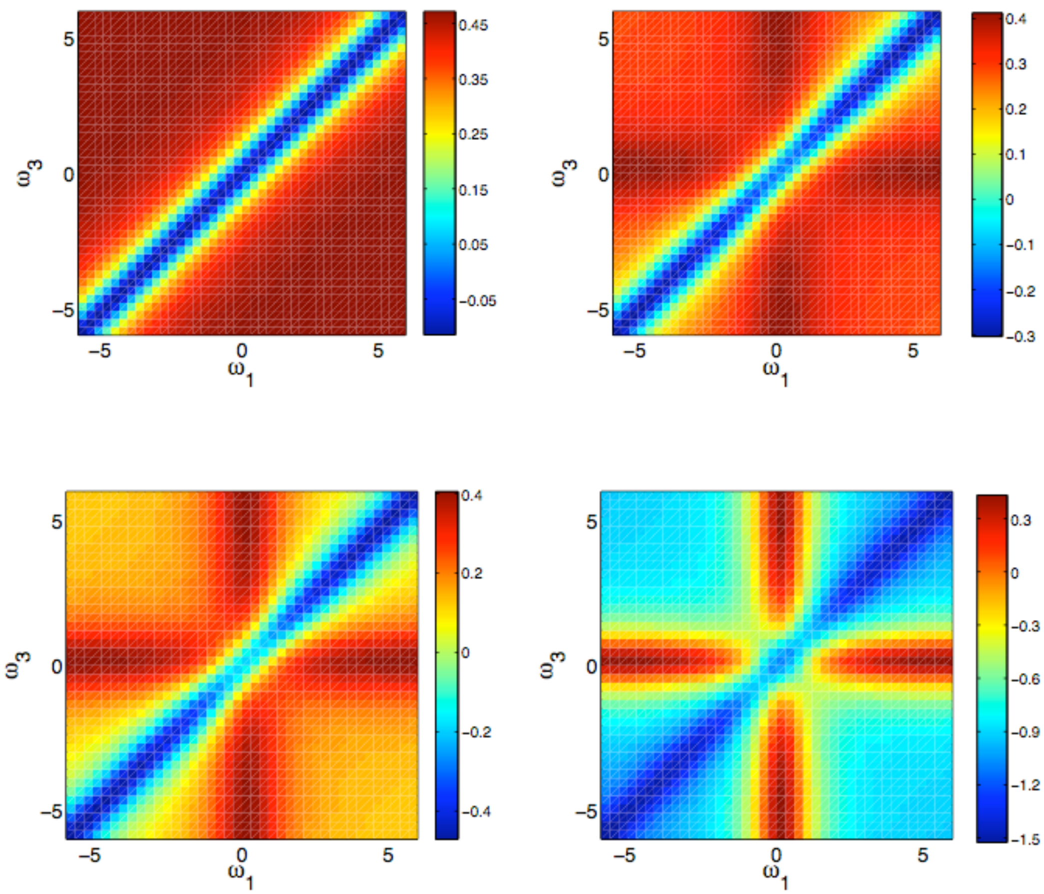
$u_0 = 0.5, \omega_E = 0.1$



Density-wave phases regions increase when ω_E is decreased.

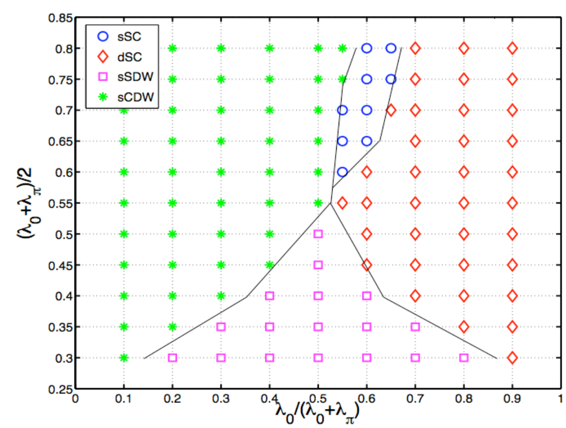
F. D. Klironomos and SWT, PRB **74**, 205109 (2006)

RG evolution of $g_2(\omega_1, -\omega_1, \omega_3, -\omega_3)$ for λ_0=0.6, λ_π=0.4, and ω_E=1.0.



$$g^{\ell=0}_{1,3}(\omega_1,\omega_2,\omega_3) = u_0 - \lambda_\pi \frac{\omega_E^2}{\omega_E^2 + (\omega_1-\omega_3)^2}$$

$$g^{\ell=0}_{2,4}(\omega_1,\omega_2,\omega_3) = u_0 - \lambda_0 \frac{\omega_E^2}{\omega_E^2 + (\omega_1-\omega_3)^2}$$



F. D. Klironomos and SWT, PRB **74**, 205109 (2006)

There can be dominant BCS pairings even at half-filling due to a separation of scales: a given coupling may have a different sign at low and high frequencies

Summary:

- Functional RG for interacting fermions with frequency-dependent interactions.
- Multiple energy scales.
- Applications:
- *2D Circular Fermi surface*, *Eliashberg equations, large-N expansion*
- *1D Holstein-Hubbard model*
- *2D square lattice at half-filling*

Fermion-Boson mixtures of cold atoms:

- *fermionic atoms + BEC of bosonic atoms*
- *on-site repulsion + long-range attraction + lattice geometry*
- *square lattice, triangular lattice*
- *L. Mathey et al., PRL 2006; PRB 2007, Klironomos et al., PRL 2007.*

Collaborators:

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Chuntai Shi *(UCR)*
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