Towards a quantitative FRG approach for the BCS-BEC crossover

Michael M. Scherer

Theoretisch Physikalisches Institut, Jena University

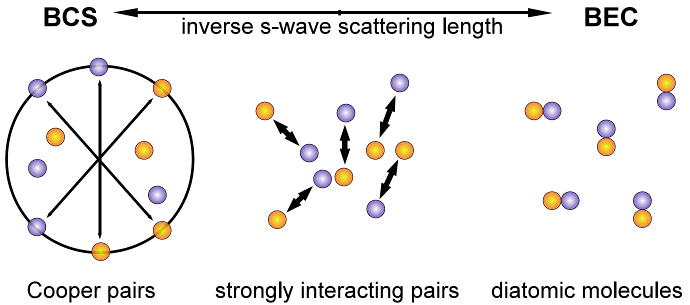
in collaboration with Sebastian Diehl, Stefan Flörchinger, Holger Gies, Jan Martin Pawlowski and Christof Wetterich

presented at INT-10-45W, Seattle, USA, Feb 26, 2010

Outline

The BCS-BEC Crossover

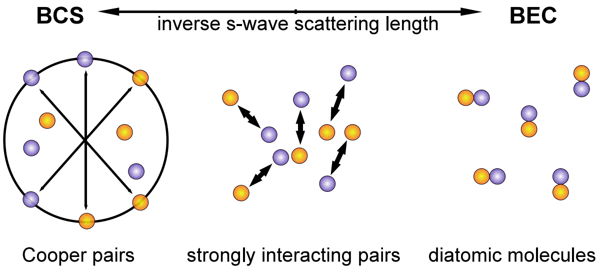
Ultracold gases of fermionic atoms near a Feshbach resonance show a crossover between BCS superfluidity and Bose-Einstein condensation (BEC) of molecules (→ talk by Ryan Kalas)



- different pairing mechanisms in fermionic systems, weak coupling/strong coupling/many body effects

Parametrization

- Crossover can be parametrized by the dimensionless inverse s-wave scattering length $(ak_F)^{-1}$
- Experimentally: phenomenon of Feshbach resonances in an external magnetic field



- $(ak_F)^{-1} < -1$: weakly attractive, Cooper pairing $\rightarrow$ below T_c: BCS superfluidity (perturbative)
- $(ak_F)^{-1} > 1$: two-body bound state, formation of molecules $\rightarrow$ below T_c: (interacting) BEC (see also talk by J.-P. Blaizot)
- $|(ak_F)^{-1}| < 1$: strongly correlated regime, Unitarity limit at $c^{-1} \rightarrow 0$ (non-perturbative regime)

Universality and Challenges

Universality:

- Limit of broad Feshbach resonances (experiments, e.g. with $^{6}\mathrm{Li}$ and $^{40}\mathrm{K}$)
- Thermodynamic quantities are independent of the microscopic details and can be expressed in terms of two dimensionless parameters:
  1. the concentration ak_F
  2. the temperature T/T_F
- Units are set by the density $n = k_F^3/(3\pi^2)$.

BCS-BEC crossover physics is a challenge for theoretical physics:

- BEC side: interacting QFT beyond perturbation theory (→ talk by J.-P. Blaizot)
- BCS side: Complex many-body physics beyond BCS theory leads to significant quantitative effects
- Unitarity limit: Perturbation theory fails, Non-perturbative techniques required, e.g. QMC

We will provide a method, that gives a unified description of the whole crossover. Quantitative measurements provide a testing ground for non-perturbative theoretical techniques in QFT.

Microscopic action and Parameters

We start with the microscopic action

$$S = \int_0^{1/T} d\tau \int_{\vec{x}} \psi^\dagger(\partial_\tau - \Delta - \mu)\psi + \phi^*(\partial_\tau - \frac{\Delta}{2} - 2\mu + \nu)\phi - h(\phi^* \psi_1 \psi_2 + h.c.)$$

- $\psi = (\psi_1, \psi_2)$ is a two component Grassmann field (fermions in two hyperfine states)
- μ is the chemical potential
- ϕ is a complex scalar field (molecules, Cooper pairs,...)
- $\nu = \mu(B - B_0)$ determines the detuning from the Feshbach resonance
- Yukawa coupling h couples the fermionic and bosonic fields, related to the width of the Feshbach resonance
- Nonrelativistic natural units with $\hbar = k_B = 2M = 1$

Hubbard-Stratonovich Transformation

Model is equivalent to a purely fermionic theory with an interaction term

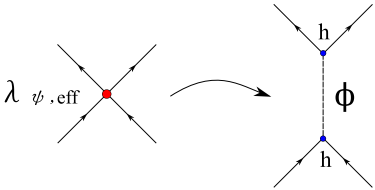
$$S_{\text{int}} = \int_{q_1,..,q_4} \left\{ \frac{-h^2}{P_\phi(q_1+q_3)} \right\} \psi_1^*(q_1)\psi_1(q_2)\psi_2^*(q_3)\psi_2(q_4)\,\delta(q_1-q_2+q_3-q_4)$$

The classical inverse boson propagator is given by

$$P_\phi(q) = iq_0 - \frac{1}{2}\vec{q}^2 + \nu - 2\mu$$

On microscopic level the interaction between the fermions is described by tree level process

$$\lambda_{\psi,\text{eff}} = -\frac{h^2}{-\omega + \frac{1}{2}\vec{q}^2 + \nu - 2\mu}$$



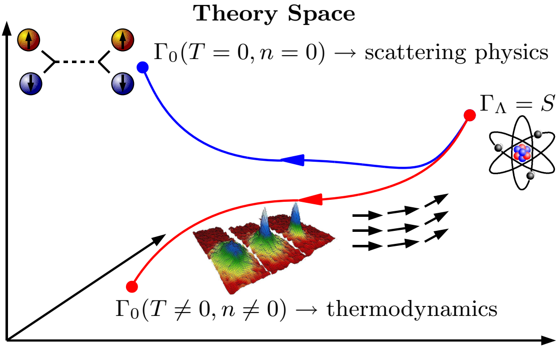
Flow equation and theory space

- Average action $\Gamma_k[\chi]$ interpolates between

$$\begin{array}{lccc} \text{microscopic action } (k \to \Lambda): & \Gamma_k[\chi] & \to & S[\chi] \\ \text{full effective action } (k \to 0): & \Gamma_k[\chi] & \to & \Gamma[\chi] \end{array}$$

- The scale dependence of $\Gamma_k[\chi]$ is given by exact flow equation (Wetterich 1993)

$$\partial_k \Gamma_k[\chi] = \frac{1}{2}\mathrm{STr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1} \partial_k R_k\right]$$



Truncation

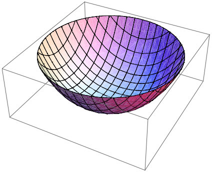
$$
\begin{aligned}
\Gamma_k[\chi] = \int_0^{1/T} d\tau \int d^3x \Big\{ \quad &+ \quad \bar{\psi}^\dagger Z_\psi(\partial_\tau - \Delta)\bar{\psi} + \bar{m}_\psi{}^2 \bar{\psi}^\dagger \bar{\psi} \\
&+ \quad \bar{\phi}^*(\bar{Z}_\phi \partial_\tau - \frac{A_\phi}{2}\Delta)\bar{\phi} \\
&+ \quad \bar{U}(\bar{\phi}^*\bar{\phi}, \mu) + \frac{\bar{\lambda}_\psi}{2}(\bar{\psi}^\dagger\bar{\psi})^2 \\
&- \quad \bar{h}(\bar{\phi}^*\bar{\psi}_1\bar{\psi}_2 + \bar{\phi}\bar{\psi}_2^*\bar{\psi}_1^*) + \bar{\lambda}_{\phi\psi}\bar{\phi}^*\bar{\phi}\bar{\psi}^\dagger\bar{\psi} \Big\}
\end{aligned}
$$

Effective potential: Expansion around the k-dependent location of the minimum $\rho_0(k)$

$$
\begin{aligned}
U_k(\rho, \mu) \quad = \quad & m^2(\rho - \rho_0) + \frac{1}{2}\lambda(\rho - \rho_0)^2 \\
& + U(\rho_0, \mu_0) - n(\mu - \mu_0) + \alpha(\mu - \mu_0)(\rho - \rho_0)
\end{aligned}
$$

We classify the thermodynamic phases of the system

$$
\begin{aligned}
\text{Symmetric regime}: &\quad \rho_0 = 0, \quad m^2 > 0 \\
\text{Symmetry broken regime}: &\quad \rho_0 > 0, \quad m^2 = 0 \\
\text{Phase transition}: &\quad \rho_0 = 0, \quad m^2 = 0
\end{aligned}
$$



Truncation

$$\Gamma_k[\chi] = \int_0^{1/T} d\tau \int d^3x \Big\{ \quad + \quad \bar{\psi}^\dagger Z_\psi (\partial_\tau - \Delta)\bar{\psi} + \bar{m}_\psi{}^2 \bar{\psi}^\dagger \bar{\psi}$$
$$+ \quad \bar{\phi}^* (\bar{Z}_\phi \partial_\tau - \frac{A_\phi}{2}\Delta)\bar{\phi}$$
$$+ \quad \bar{U}(\bar{\phi}^*\bar{\phi}, \mu) + \frac{\bar{\lambda}_\psi}{2}(\bar{\psi}^\dagger \bar{\psi})^2$$
$$- \quad \bar{h}(\bar{\phi}^* \bar{\psi}_1 \bar{\psi}_2 + \bar{\phi}\bar{\psi}_2^* \bar{\psi}_1^*) + \bar{\lambda}_{\phi\psi}\bar{\phi}^*\bar{\phi}\bar{\psi}^\dagger\bar{\psi} \Big\}$$

Effective potential: Expansion around the k-dependent location of the minimum $\rho_0(k)$

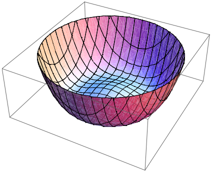
$$U_k(\rho,\mu) \quad = \quad m^2(\rho - \rho_0) + \frac{1}{2}\lambda(\rho - \rho_0)^2$$
$$+U(\rho_0,\mu_0) - n(\mu - \mu_0) + \alpha(\mu - \mu_0)(\rho - \rho_0)$$

We classify the thermodynamic phases of the system

Symmetric regime : $\rho_0 = 0, \quad m^2 > 0$

Symmetry broken regime : $\rho_0 > 0, \quad m^2 = 0$

Phase transition : $\rho_0 = 0, \quad m^2 = 0$



Truncation

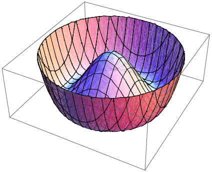
$$\Gamma_k[\chi] = \int_0^{1/T} d\tau \int d^3x \Big\{ \quad + \quad \bar{\psi}^\dagger Z_\psi(\partial_\tau - \Delta)\bar{\psi} + \bar{m}_\psi{}^2 \bar{\psi}^\dagger \bar{\psi}$$
$$+ \quad \bar{\phi}^*(\bar{Z}_\phi \partial_\tau - \frac{A_\phi}{2}\Delta)\bar{\phi}$$
$$+ \quad \bar{U}(\bar{\phi}^*\bar{\phi}, \mu) + \frac{\bar{\lambda}_\psi}{2}(\bar{\psi}^\dagger\bar{\psi})^2$$
$$- \quad \bar{h}(\bar{\phi}^*\bar{\psi}_1\bar{\psi}_2 + \bar{\phi}\bar{\psi}_2^*\bar{\psi}_1^*) + \bar{\lambda}_{\phi\psi}\bar{\phi}^*\bar{\phi}\bar{\psi}^\dagger\bar{\psi} \Big\}$$

Effective potential: Expansion around the k-dependent location of the minimum $\rho_0(k)$

$$U_k(\rho,\mu) \quad = \quad m^2(\rho - \rho_0) + \frac{1}{2}\lambda(\rho - \rho_0)^2$$
$$+U(\rho_0,\mu_0) - n(\mu - \mu_0) + \alpha(\mu - \mu_0)(\rho - \rho_0)$$

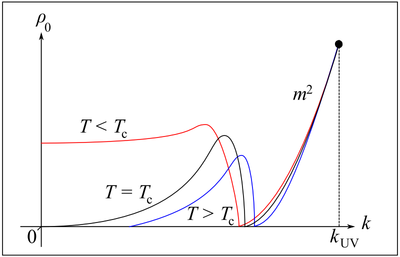
We classify the thermodynamic phases of the system

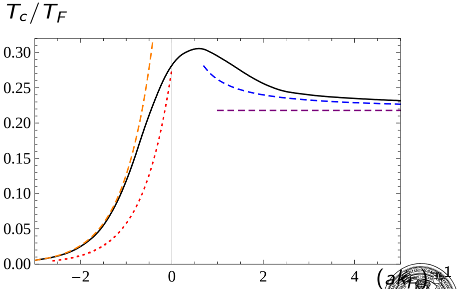
$$\begin{aligned} \text{Symmetric regime}: &\quad \rho_0 = 0, \quad m^2 > 0 \\ \text{Symmetry broken regime}: &\quad \rho_0 > 0, \quad m^2 = 0 \\ \text{Phase transition}: &\quad \rho_0 = 0, \quad m^2 = 0 \end{aligned}$$



Basic truncation & critical temperature

- Solution of the two-body problem: Scattering physics of the fermionic system in vacuum ($T = 0$ and $n = 0$) yields microscopic parameters (see also talk by Boris Krippa)
- Start the flow in the UV at defined T and look in the IR if it ends up in the symmetric phase or the spontaneously broken phase.
- The temperature for which $m^2 \to 0$ and $\rho_0 \to 0$ as $k \to 0$ is T_c $(\lambda_{\psi,eff} \propto \frac{-h^2}{m^2} \to \infty)$



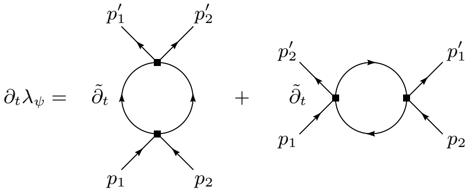


Momentum Dependent Four-Fermion Interaction

- In purely fermionic language the fermion interaction is momentum dependent

$$\int_{p_1,p_2,p_1',p_2'} \lambda_\psi(p_1', p_1, p_2', p_2)\psi_1^*(p_1')\psi_1(p_1)\psi_2^*(p_2')\psi_2(p_2)$$

- The flow of λ_ψ has two contributions:



- BCS theory: Only particle-particle fluctuations (first loop) $\rightarrow$ phase transition to superfluidity:

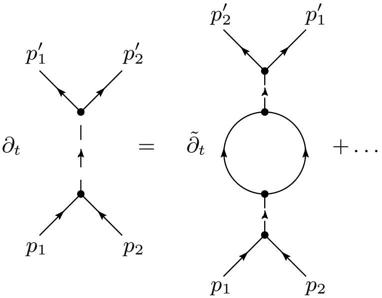
$$T_{c,\mathrm{BCS}} \approx 0.61 e^{\pi/2k_F a} T_F$$

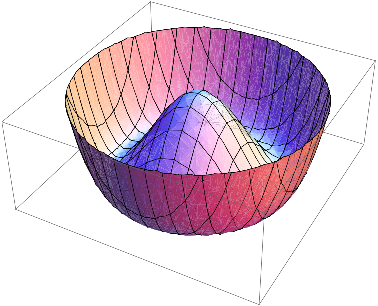
- Screening of the interaction between two fermions by the particle-hole fluctuations (second loop) is a quantitative effect and lowers the critical temperature as compared to BCS theory

$$T_c = \frac{1}{(4e)^{1/3}} T_{c,\mathrm{BCS}} \approx \frac{1}{2.2} T_{c,\mathrm{BCS}} \quad (\text{Gorkov}, 1963)$$

Bosonization

In a bosonized language, the fermionic interaction is described by boson exchange





- The phase transition to the superfluid phase is indicated by the vanishing of the bosonic mass term $m^2 = 0$ (SSB)
- For vanishing external momenta: $\lambda_{\psi,\mathrm{eff}} = \frac{-h^2}{m^2}$
- In this setting, where the bosonization took place only on the microscopic scale, we do not account for particle-hole fluctuations

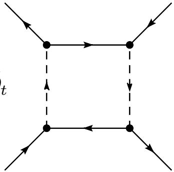
Bosonization is destroyed by the RG flow

The particle-hole fluctuations are not yet included, since we neglected so far, that the term

$$\int_{\tau,\vec{x}} \lambda_\psi \psi_1^\dagger \psi_1 \psi_2^\dagger \psi_2$$

is re-generated by the flow.

$$\partial_t \lambda_\psi \quad = \quad \tilde{\partial}_t$$



λ_ψ contributes to the effective interaction between fermions

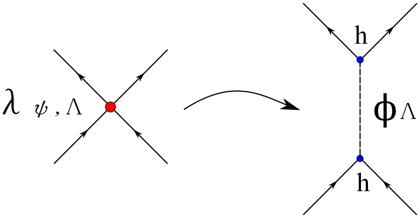
$$\lambda_{\psi,\text{eff}} = \frac{-h^2}{m^2} + \lambda_\psi$$

The physical picture, that the divergence of $\lambda_{\psi,eff}$ is connected to the onset of a nonvanishing expectation value of the bosonic field ρ_0 does not hold anymore.

Rebosonization I

Idea:

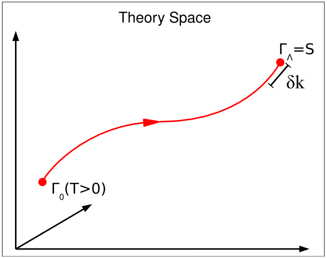
1. Bosonize at microscopic scale with a field ϕ_Λ, $\Rightarrow \lambda_{\psi,\Lambda} = 0$



Rebosonization I

Idea:

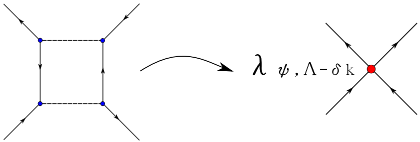
1. Bosonize at microscopic scale with a field ϕ_Λ, $\Rightarrow \lambda_{\psi,\Lambda} = 0$
2. Perform one renormalization step δk



Rebosonization I

Idea:

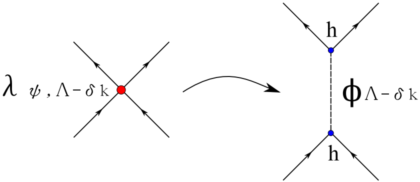
1. Bosonize at microscopic scale with a field ϕ_Λ, $\Rightarrow \lambda_{\psi,\Lambda} = 0$
2. Perform one renormalization step δk
3. The boxdiagram regenerates a nonvanishing $\lambda_{\psi,\Lambda-\delta k}$



Rebosonization I

Idea:

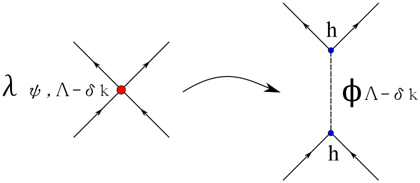
1. Bosonize at microscopic scale with a field ϕ_Λ, $\Rightarrow \lambda_{\psi,\Lambda} = 0$
2. Perform one renormalization step δk
3. The boxdiagram regenerates a nonvanishing $\lambda_{\psi,\Lambda-\delta k}$
4. Bosonize again (with a field $\phi_{\Lambda-\delta k}$), $\Rightarrow \lambda_{\psi,\Lambda-\delta k} = 0$



Rebosonization I

Idea:

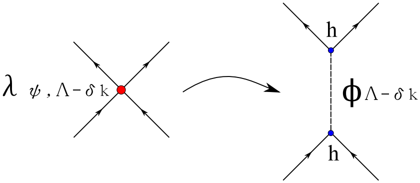
1. Bosonize at microscopic scale with a field ϕ_Λ, $\Rightarrow \lambda_{\psi,\Lambda} = 0$
2. Perform one renormalization step δk
3. The boxdiagram regenerates a nonvanishing $\lambda_{\psi,\Lambda-\delta k}$
4. Bosonize again (with a field $\phi_{\Lambda-\delta k}$), $\Rightarrow \lambda_{\psi,\Lambda-\delta k} = 0$
5. Repeat the steps 2 - 4 until we reach $k = 0$



Rebosonization I

Idea:

1. Bosonize at microscopic scale with a field ϕ_Λ, $\Rightarrow \lambda_{\psi,\Lambda} = 0$
2. Perform one renormalization step δk
3. The boxdiagram regenerates a nonvanishing $\lambda_{\psi,\Lambda-\delta k}$
4. Bosonize again (with a field $\phi_{\Lambda-\delta k}$), $\Rightarrow \lambda_{\psi,\Lambda-\delta k} = 0$
5. Repeat the steps 2 - 4 until we reach $k = 0$



- (Re-)appearance of a λ_ψ by the flow of the box diagrams can be absorbed by the introduction of scale dependent fields ϕ_k
- Context of QCD → talk by J. M. Pawlowski: dynamical hadronisation
- Scale dependent fields → modified flow equation (Gies & Wetterich 2001, Pawlowski 2005, Floerchinger & Wetterich 2009)

$$\partial_k \Gamma_k[\chi_k] = \frac{1}{2}\mathrm{STr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1} \partial_k R_k\right] + \int \frac{\delta \Gamma_k}{\delta \chi_k} \partial_k \chi_k$$

- k-dependence can be chosen arbitrarily

Rebosonization II

- We choose the following scale dependence for the bosonic fields

$$\partial_k \bar{\phi}_k(q) \quad = \quad (\psi_1 \psi_2)(q) \partial_k \upsilon$$

- $\partial_k \upsilon$ is to be determined for our purposes
- Flow equations are modified

$$\begin{aligned} \partial_k \bar{h} &= \partial_k \bar{h}\big|_{\bar{\phi}_k} - \bar{m}^2 \partial_k \upsilon \\ \partial_k \lambda_\psi &= \partial_k \lambda_\psi\big|_{\bar{\phi}_k} - 2\bar{h} \partial_k \upsilon \end{aligned}$$

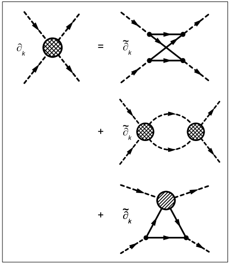
- Choose $\partial_k \upsilon$ such that the flow of λ_ψ vanishes $\Rightarrow \lambda_\psi = 0$ on all scales
- The modified flow of the Yukawa coupling reads

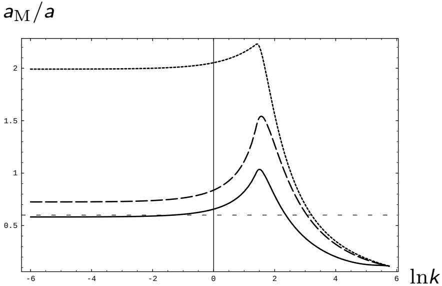
$$\partial_k \bar{h} = \partial_k \bar{h}\Big|_{\phi_k} - \frac{\bar{m}^2}{2\bar{h}} \partial_k \lambda_\psi \Big|_{\phi_k}$$

- Four-fermion interaction is now purely given by the boson exchange and ph-fluctuations are incorporated via the second term in the latter equation

Atom-dimer vertex

- Atom-dimer vertex $\lambda_{\phi\psi}$ contributes to vacuum dimer-dimer scattering length a_{M}/a





- Taking into account $\lambda_{\phi\psi}$ yields $a_{\mathrm{M}}/a = 0.59$
- The exact result is $a_{\mathrm{M}}/a = 0.60$ (cf. Petrov *et al.* 2004)
- In the SSB regime $\lambda_{\phi\psi}$ also has an effect on the Fermi-surface

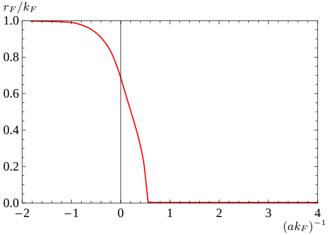
Running fermion sector

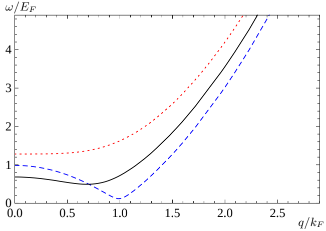
Running $Z_\psi, m_\psi^2, \lambda_{\phi\psi} \to$ dispersion relation:
Renormalized propagator of the fermionic field after analytical continuation to real frequencies

$$G_\psi^{-1} = \begin{pmatrix} -h\phi_0\epsilon & -\omega - (\vec{q}^2 + m_\psi^2 + \lambda_{\phi\psi}\rho_0) \\ -\omega + (\vec{q}^2 + m_\psi^2 + \lambda_{\phi\psi}\rho_0) & h\phi_0\epsilon \end{pmatrix}$$

The dispersion relation follows from $\det G_\psi^{-1} = 0$

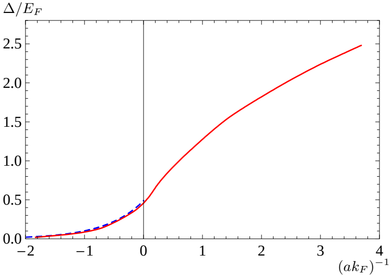
$\omega = \pm\sqrt{\Delta^2 + (\vec{q}^2 - r_F^2)^2}$ where $\Delta = h\sqrt{\rho_0}$ is the gap and $r_F = \sqrt{-m_\psi^2 - \lambda_{\phi\psi}\rho_0}$ the effective radius of the Fermi sphere





Single-particle gap at $T = 0$

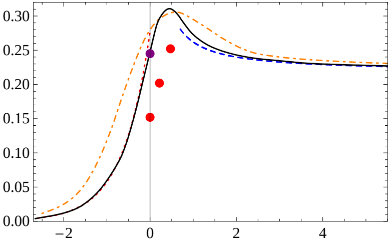
Gap in units of the Fermi energy Δ/E_F



- For comparison the result found by Gorkov and Melik-Bakhudarov
- At the unitarity point $\Delta_{\mathrm{GMB}}/E_F = 0.49$.

	μ/E_F	Δ/E_F
Carlson *et al.* 2004	0.43	0.54
Haussmann *et al.* 2007	0.36	0.46
Bartosch *et al.* 2009	0.32	0.61
present work	0.51	0.46

Critical Temperature



On BEC-side our result approaches the critical temperature of a free Bose gas in the form

$$\frac{T_c - T_{c,\mathrm{BEC}}}{T_{c,\mathrm{BEC}}} = c\frac{a_M}{a}\frac{ak_F}{(6\pi^2)^{1/3}},$$

a_M is the scattering length between the molecules. We use our result $a_M/a = 0.59$. and find $c = 1.39$. Arnold *et al.* (2001) and Baym *et al.* (2006) give $c \approx 1.31$.

	T_c/T_F	μ_c/T_F
Burovski *et al.* 2006	0.15	0.49
Akkineni *et al.* 2006	0.245	-
Bulgac *et al.* 2008	< 0.15	0.43
present work	0.248	0.55

Conclusions & Outlook

Conclusions:

- We established a unified description of the whole crossover region
- We can assess scattering physics as well as thermodynamics
- A simple truncation gives a good qualitative picture
- Extended truncation recovers quantitatively the well-known weak-coupling limits (TD and scattering physics)
- At the unitarity point we are in reasonable quantitative agreement with QMC data and other methods for $T = 0$, deviations for $T \neq 0$

Outlook:

- Understand the difference of the results at unitarity for $T \neq 0$ from different methods
- More involved frequency and momentum dependence of the propagators and vertices?
- Put the system in a finite volume and study the volume dependence to make contact with QMC simulations (with Jens Braun and Sebastian Diehl)
- Study effects of a trap & imbalanced Fermi gas

Thanks to...

- Jens Braun
- Sebastian Diehl
- Stefan Flörchinger
- Holger Gies
- Jan Martin Pawlowski
- Christof Wetterich