# Inverse square potential, scale anomaly, and complex extension

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# Outline

- Introduction and motivation
- Functional renormalization group and RG flow equation
- Complex extension and RG flows on the Riemann sphere
- Connection to large-flavor QCD

# Introduction

- Inverse square potential is classically scale invariant

$$V(r) = -\frac{\kappa}{r^2}$$

- Classical scattering is well defined if

$$b > \sqrt{\frac{\kappa}{E}} \quad \text{for} \quad \kappa > 0$$

otherwise $\rightarrow$ fall to the center

- QM estimate: particle confined to a small ball of radius $r_0$

$$E \approx \frac{1}{r_0^2} - \frac{\kappa}{r_0^2}$$

- It is a border between regular and singular potentials

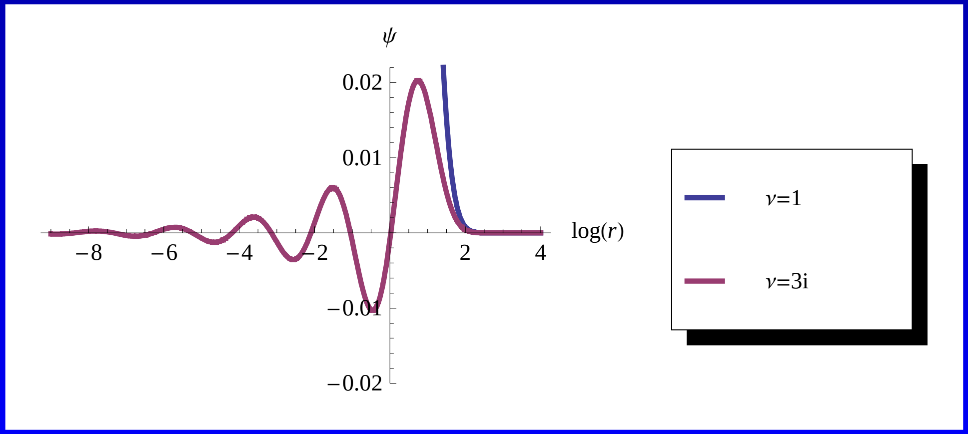
# Introduction

- In QM a critical $\kappa_{cr} > 0$ exists. For $\kappa > \kappa_{cr}$ the Hamiltonian is unbounded from below $\longrightarrow$ problems
- Effective 1D Schrödinger equation

$$\left[-\frac{d^2}{dr^2} - \frac{\kappa}{r^2}\right]\psi(r) = E\psi(r) \qquad E = -\sigma^2 < 0$$

- Solution, which is well-behaved as $r \longrightarrow \infty$

$$\psi(r) = \sqrt{r}K_\nu(\sigma r) \qquad \nu = \sqrt{1/4 - \kappa}$$



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- Infinity of nodes $\longrightarrow$ infinity of bound states for $\kappa > \kappa_{cr}$?
- Proper treatment: regularize around the origin and use RG
- A lot of previous RG studies

Beane et al. 01, Bawin&Coon 03, Barford&Birse 03 …

# Motivation

- Efimov effect for three bosons interacting through a short range attractive potential Efimov 70 . . .

  At unitarity point → effective 1D equation

  $$\left[-\frac{d^2}{dr^2} - \frac{s_0^2 + 1/4}{r^2}\right]\psi(r) = E\psi(r), \quad s_0 \approx 1.0062$$

  Experimentally observed in cold atoms Kraemer et al. 06, . . .

- A neutral polar molecule interacts with an electron via

  $$V(\vec{r}) \sim \frac{\cos\theta}{r^2}$$

  can be reduced to the isotropic form Camblong et al. 01

# Motivation

- Transition from the conformal ($\kappa < \kappa_{cr}$) to the nonconformal ($\kappa > \kappa_{cr}$) regime resembles the BKT phase transition in two dimensions Kaplan et al. 09
- Scalar field near the Reissner-Nordström black hole background Camblong et al. 03
- AdS/CFT correspondence: scalar field in the anti-de Sitter spacetime $AdS_{d+1}$

$$\partial_r^2 \phi - \frac{d-1}{r}\partial_r \phi - \frac{m^2}{r^2}\phi - q^2 \phi = 0, \qquad q^2 = (q^0)^2 + \vec{q}^2$$

after change of field $\phi = r^{(d-1)/2}\psi$ one gets

$$-\partial_r^2 \psi + \frac{m^2 + (d^2-1)/4}{r^2}\psi = -q^2 \psi$$

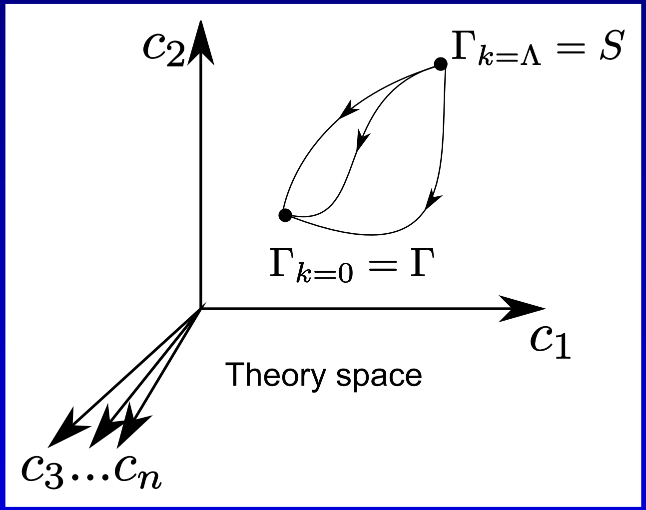
# Functional renormalization group

- Effective average action $\Gamma_k[\phi]$ solves

Wetterich 93

$$\partial_k \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\frac{\partial_k R_k}{\Gamma_k^{(2)}[\phi] + R_k}$$

- RG flow in theory space




- Truncation needed to solve the flow equation

# Model

- Our model in Euclidean QFT formulation

$$\Gamma_k[\psi,\psi^*] = \int_Q \psi^*(Q)[i\omega + \vec{q}^2]\psi(Q) -$$
$$- \kappa \int_{Q_1,\ldots,Q_4} F_d(l)\psi^*(Q_1)\psi(Q_2)\psi^*(Q_3)\psi(Q_4)\delta(-Q_1+Q_2-Q_3+Q_4)$$
$$- \frac{\lambda_\psi}{2} \int_{Q_1,\ldots,Q_4} \psi^*(Q_1)\psi(Q_2)\psi^*(Q_3)\psi(Q_4)\delta(-Q_1+Q_2-Q_3+Q_4),$$

with the Fourier transform of $1/r^2$ potential in $d$ dim

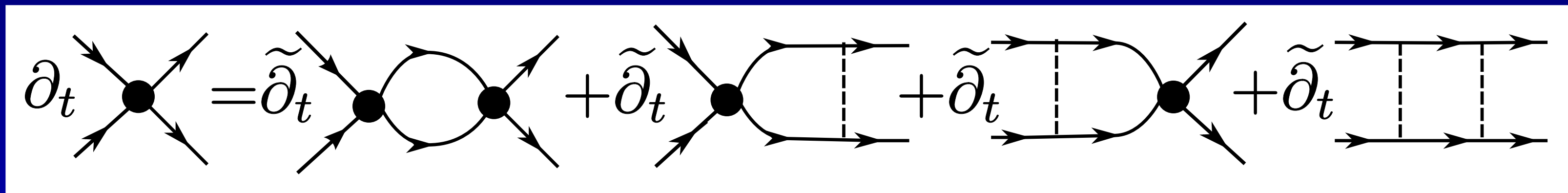
$$F_d(l) = \frac{(4\pi)^{d/2}\Gamma(d/2-1)|\vec{l}|^{2-d}}{4} \qquad d > 2$$

- $\lambda$ is an emergent coupling generated by quantum loops
- We use sharp regulator

$$R_k(L) = (i\omega + l^2)\left(\frac{1}{\theta(l^2-k^2)} - 1\right)$$

# Flow equation

- The propagator is not renormalized in the nonrelativistic vacuum
- The long-range potential coupling $\kappa$ is not renormalized
- RG flow for the coupling $\lambda$




- $\lambda$ is taken to be momentum-independent
- Flow equation for a rescaled dimensionless $\lambda_{\psi R}$

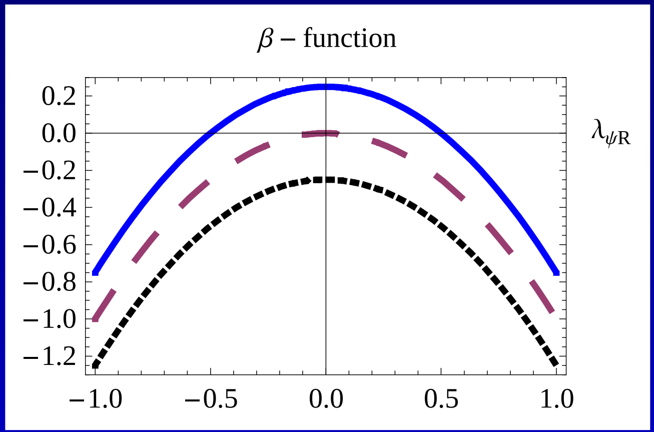
$$\partial_t \lambda_{\psi R} = -\lambda_{\psi R}^2 + \left( -\frac{2\kappa}{d-2} + d - 2 \right) \lambda_{\psi R} - \frac{\kappa^2}{(d-2)^2}$$

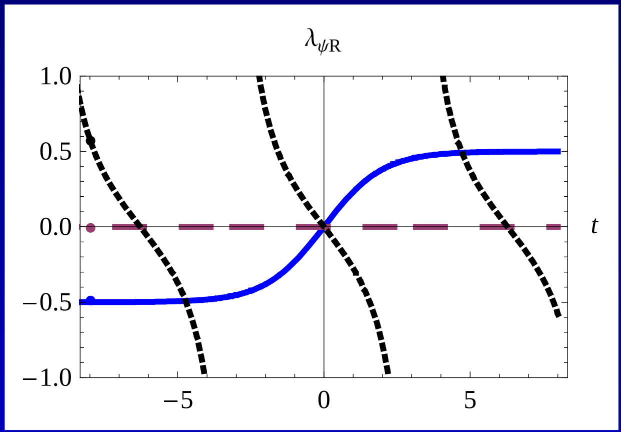
# Solution of the flow equation

$$\frac{d}{dt}\lambda_{\psi R}(t) = \alpha\lambda_{\psi R}(t)^2 + \beta\lambda_{\psi R}(t) + \gamma$$

- The solution is determined by the sign of the discriminant

$$D = \beta^2 - 4\alpha\gamma$$




- $\kappa < \kappa_{cr} \rightarrow D > 0 \rightarrow$ two fixed points (CFT)
- $\kappa = \kappa_{cr} \rightarrow D = 0 \rightarrow$ single fixed point $\rightarrow \kappa_{cr} = \frac{(d-2)^2}{4}$
- $\kappa > \kappa_{cr} \rightarrow D < 0 \rightarrow$ no fixed points (limit cycle)

# Complex extension– analytical treatment

- For a deeper mathematical understanding we perform a complex extension

$$\lambda \to \lambda_1 + i\lambda_2$$

$$\partial_t \lambda = \alpha\lambda^2 + \beta\lambda + \gamma \to \begin{cases} \partial_t \lambda_1 = \alpha\lambda_1^2 - \alpha\lambda_2^2 + \beta\lambda_1 + \gamma \\ \partial_t \lambda_2 = 2\alpha\lambda_1\lambda_2 + \beta\lambda_2 \end{cases}$$

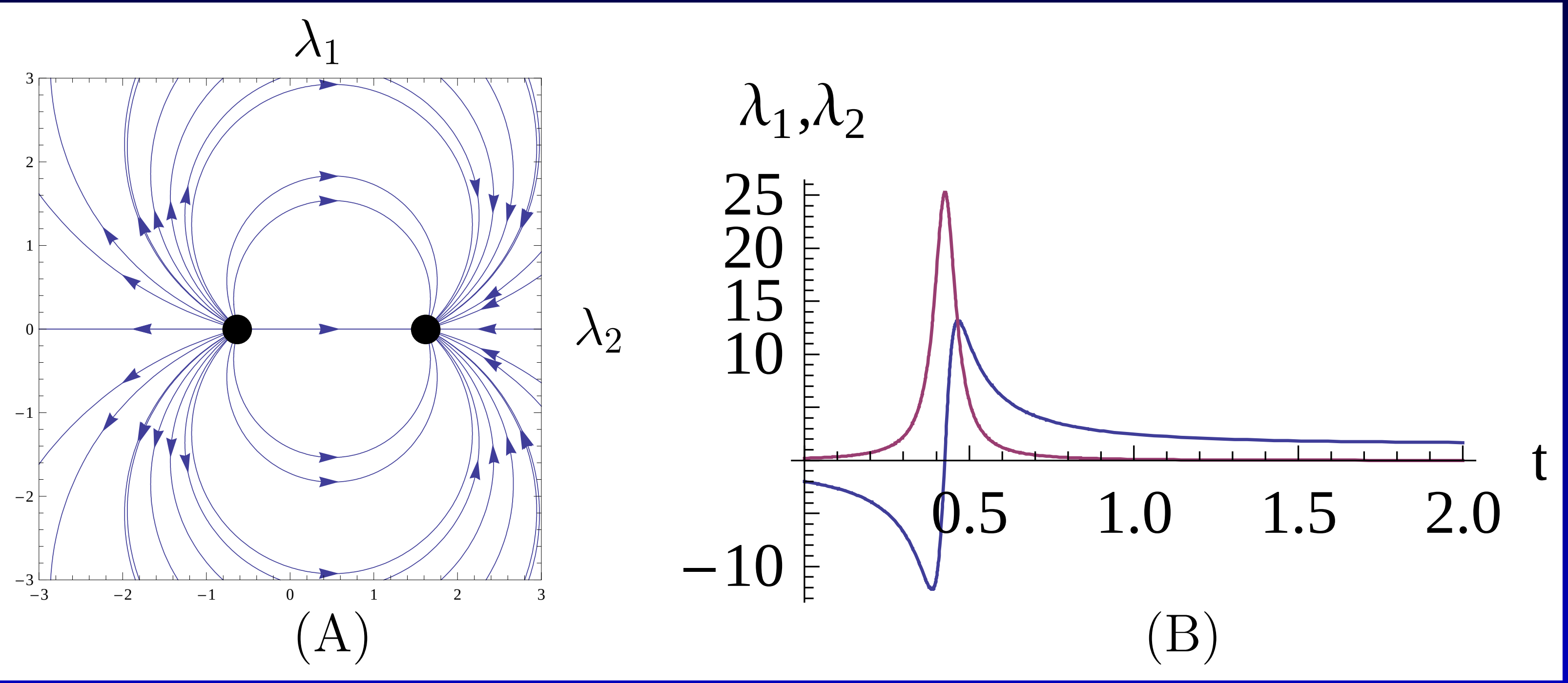
- The analytical solution is

$$\lambda(t) = \frac{1}{2\alpha}\left(-\beta - \sqrt{D}\frac{e^{\frac{\sqrt{D}t}{2}} - Ce^{-\frac{\sqrt{D}t}{2}}}{e^{\frac{\sqrt{D}t}{2}} + Ce^{-\frac{\sqrt{D}t}{2}}}\right)$$

- $C$ determines initial condition for $\lambda$ in complex plane
- RG trajectories have a constant positive curvature

# Complex extension– numerical treatment

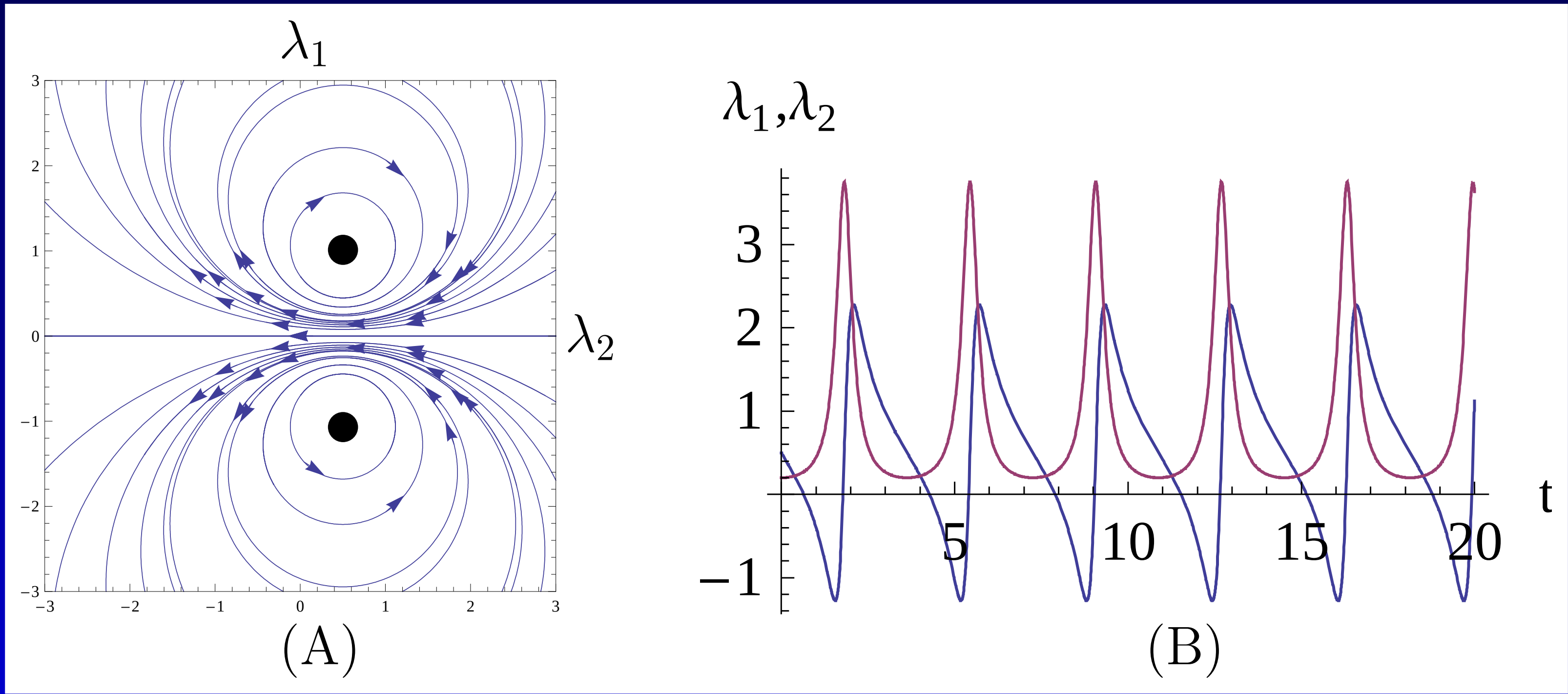
- $D > 0$: two real fixed points with real eigenvalues




- Divergences are regularized → metastable resonances
- Flow of the imaginary part determines the decay width

# Complex extension– numerical treatment

- $D < 0$: two complex fixed points with imaginary eigenvalues




- For $D = 0$ two fixed points merge
- Intuitive understanding of the transition

# Complex extension– physical motivation

- Partial wave expansion for a scattering amplitude

$$f(p,\theta) = \sum_{l=0}^{l=\infty} (2l+1) f_l(p) P_l(\cos(\theta))$$

- For short-range forces s-wave dominates at low energies

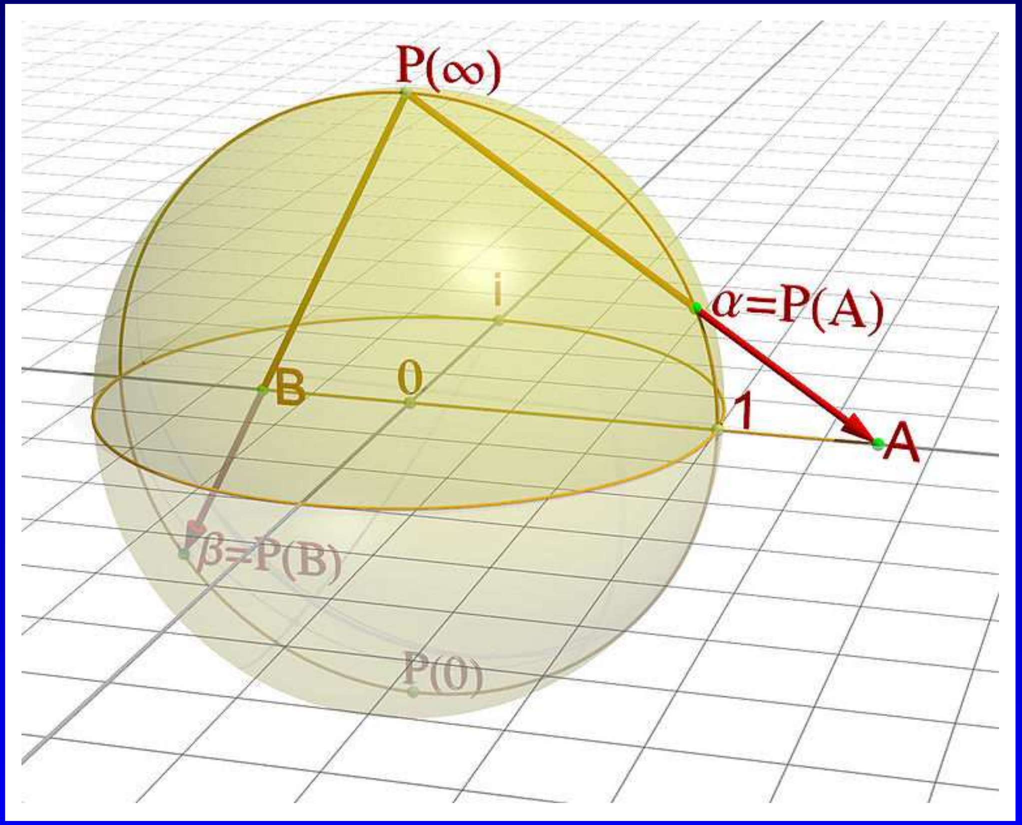
$$f_0 = \frac{1}{g_0(p^2) - ip} \approx \frac{1}{-a^{-1} + \frac{1}{2} r_{\mathrm{eff}} p^2 - ip}$$

- Complex scattering length $a = \alpha + i\beta$ opens an inelastic scattering channel
- Bound states with finite decay width $\longrightarrow$ resonances

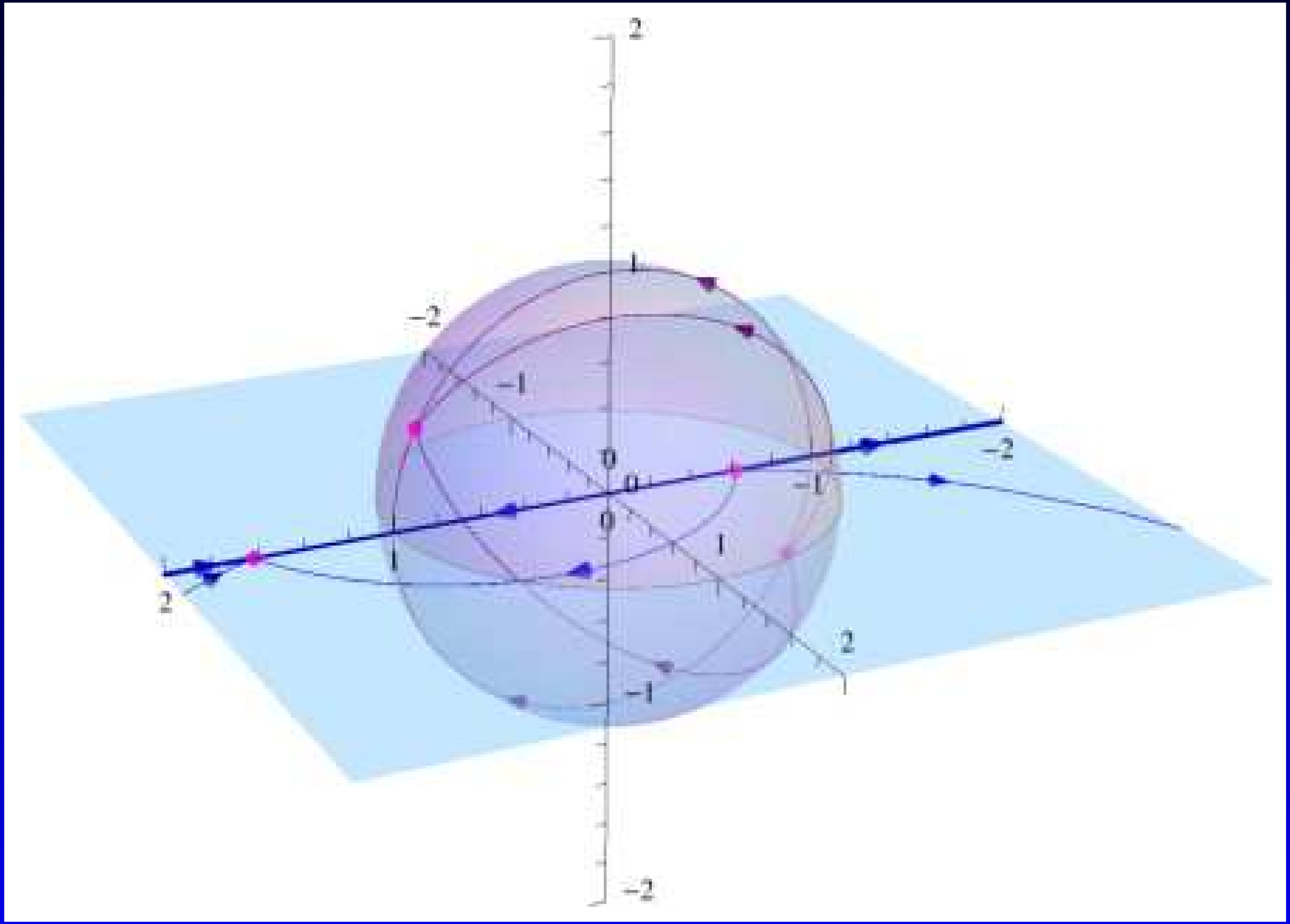
# Riemann sphere

- The complex plane $\mathbb{C}$ can be extended by an additional point $\infty$
- The extended complex plane is mapped onto the Riemann sphere via the stereographic projection

Wikipedia

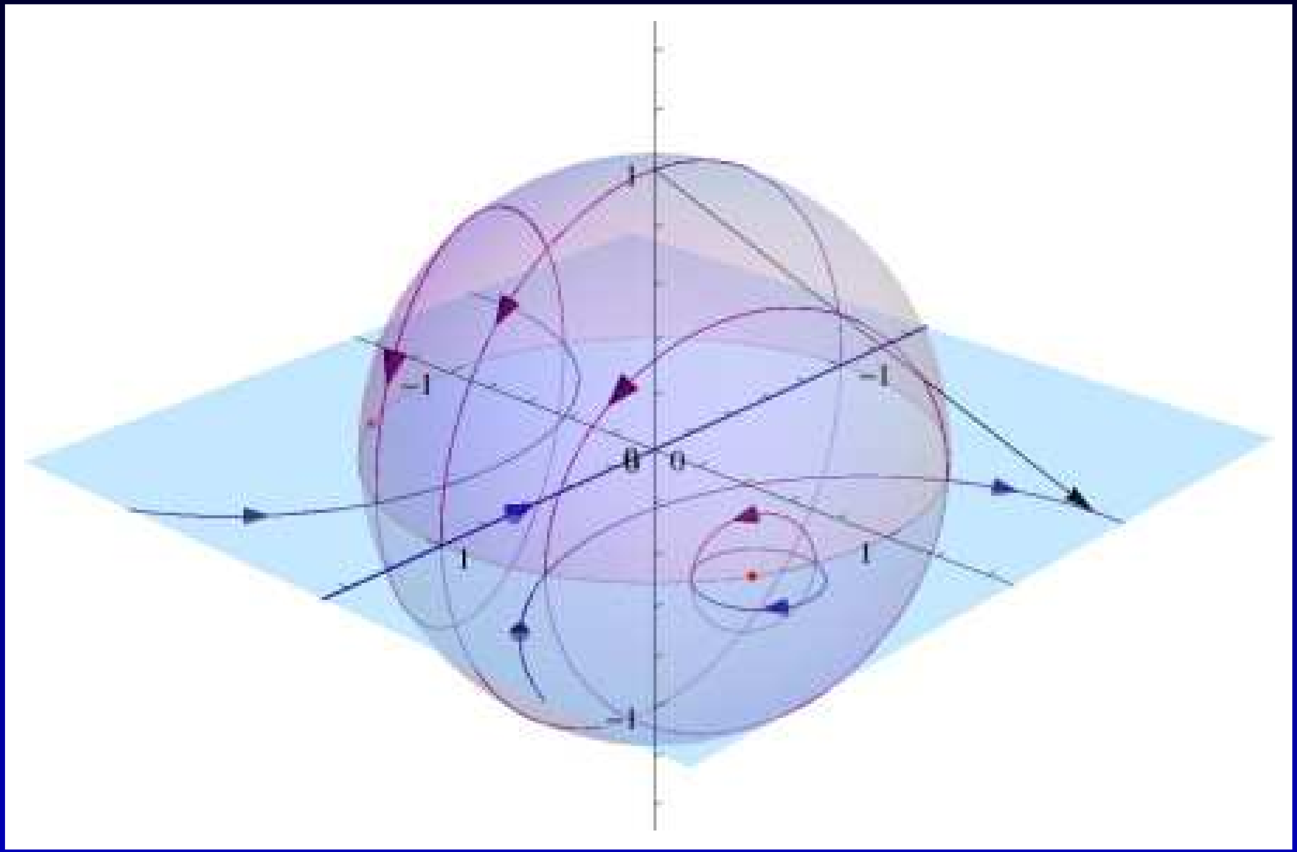


# Flows on the Riemann sphere: $D > 0$




- Two fixed point on the great real circle

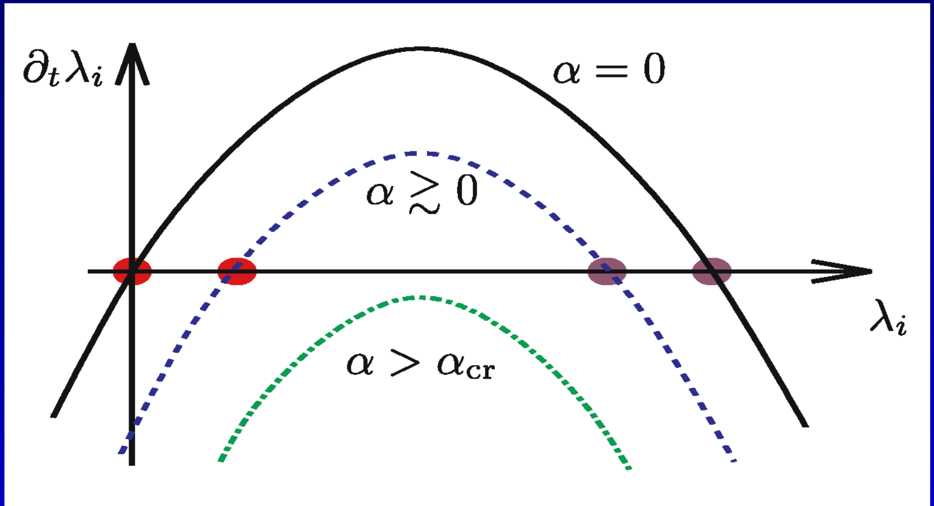
# Flows on the Riemann sphere: $D < 0$



- Real flow forms an infinite limit cycle in the complex plane with periodic discontinuities
- On the Riemann sphere the flow periodically traverses the great real circle

# Connection to large-flavor QCD

- FRG studies of conformal windows and chiral phase transition in many flavor massless QCD  Gies, Jaeckel 06
- RG $\beta$-functions of the fermionic self-interactions $\lambda_i$ at a fixed gauge coupling $\alpha$




- $\beta$-function are similar to our problem
- However, gauge coupling $\alpha$ is running in QCD
- No limit cycles, but chiral symmetry breaking for $\alpha > \alpha_{cr}$

# Conclusions

- Nonrelativistic inverse square potential has different physical applications
- It is a paradigm for nonrelativistic conformal symmetry and scale anomaly
- It must be regularized at origin $\longrightarrow$ need for RG
- Complex extension provides a deeper mathematical understanding and is physically motivated
- Geometric description $\longrightarrow$ flows on the Riemann sphere

More can be found in Annals of Physics **325**, 491 (2010)

# Extra slides

# Functional renormalization group

- We use quantum field theory methods to investigate nonrelativistic $1/r^2$ problem
- In quantum field theory in Euclidean formulation

$$e^{W[J]} = \int \mathscr{D}\varphi e^{-S[\varphi]+\int J\varphi}$$

- From $W[J]$ we can extract connected correlation functions

$$\Downarrow$$

scattering amplitudes, bound state energies

- However, the functional integral for $W[J]$ is difficult to evaluate

# Functional renormalization group

- Renormalization group idea- introduce regulator

$$e^{W_k[J]} = \int \mathscr{D}\varphi \exp\left(-S[\varphi] - \frac{1}{2}\int \varphi R_k \varphi + \int J\varphi\right)$$

  and study sliding scale $k$ dependence
- Effective average action $\Gamma_k[\phi]$ is a Legandre transform of $W_k[J]$
- Regulator $R_k$ introduces scheme dependence in the problem
- For $k = 0$ we recover the effective action $\Gamma[\phi] \Rightarrow$ 1PI vertices $\Rightarrow$ correlation functions

# Solution of the flow equation

- $D > 0$ and $\lambda_{\psi R}^{IR} < \lambda_{\psi R} < \lambda_{\psi R}^{UV}$: smooth interpolation between two fixed points

$$\lambda_{\psi R}(t) = \frac{-\beta - \sqrt{D}\tanh\left[\frac{\sqrt{D}}{2}(t+\eta)\right]}{2\alpha}$$

- $D = 0$: logarithmic running $\longrightarrow$ Landau pole

$$\lambda_{\psi R}(t) = \lambda_{\psi R}^{*} - \frac{1}{\alpha t + \eta}, \qquad \lambda_{\psi R}^{*} = -\frac{\beta}{2\alpha}$$

- $D < 0$: periodic infinities $\longrightarrow$ geometric bound spectrum

$$\lambda_{\psi R}(t) = \frac{-\beta + \sqrt{-D}\tan\left[\frac{\sqrt{-D}}{2}(t+\eta)\right]}{2\alpha}$$