Channel-decomposed one-loop RG for the 2D Hubbard model[1]

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[1] Phys. Rev. B 79, 195125 (2009)
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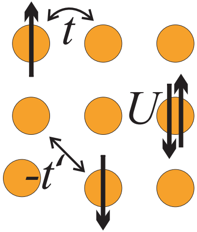
The 2D Hubbard Model

$$\mathcal{H}[a^\dagger, a] = \sum_{\substack{\mathbf{p}\in\Gamma^* \\ \sigma=\pm}} \varepsilon(\mathbf{p}) a^\dagger_{\mathbf{p},\sigma} a_{\mathbf{p},\sigma} + U \sum_{x\in\Gamma} n_+(x) n_-(x)$$



- no apparent dominant energy scale
- no apparent MF order

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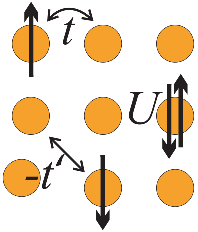

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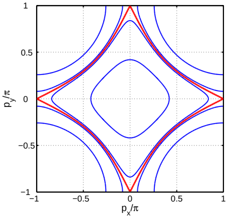

- high-T_c cuprates near half filling (?)

[Anderson 1987, Zhang and Rice 1988]

- van Hove filling: $\nabla\varepsilon = 0$ on Fermi surface
  - logarithmic divergence in density of states
  - may increase T_c
  - interplay FM and SC

The Functional Renormalization Group (or "exact")

Idea [Wilson, 1973]

Integrate all fluctuations systematically step by step.

1. sort fluctuations by energy, inverse length, or temperature scale Λ
2. integrate fluctuations with scale $> \Lambda$
3. calculate the change of the vertex functions as Λ decreases
4. if $\Lambda \to 0$ can be taken, obtain full vertex functions of the model

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Advantages

- manages to treat infrared (also UV) singularities
- does not depend on scaling Ansatzes
- allows to make controllable approximations [Feldman, Knörrer, Trubowitz, Balaban, Gallavotti, Salmhofer]

The Interaction Vertex

$$\mathcal{V}_\Lambda[\Psi] = \frac{1}{2} \int \mathrm{d}p_1 \dots \mathrm{d}p_3 \, V_\Lambda(p_1, p_2, p_3) \sum_{\substack{\sigma,\sigma' \\ \in\{+,-\}}} \overline{\psi}_\sigma(p_1)\overline{\psi}_{\sigma'}(p_2)\psi_{\sigma'}(p_3)\psi_\sigma(p_4)$$

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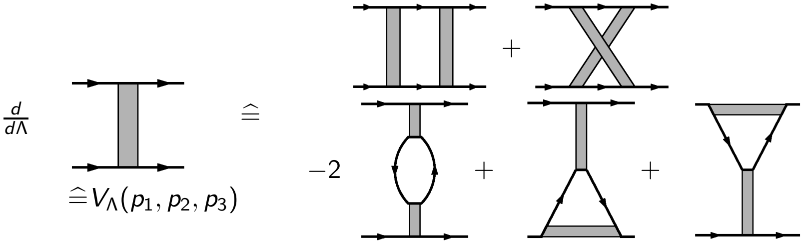
- one–loop approximation



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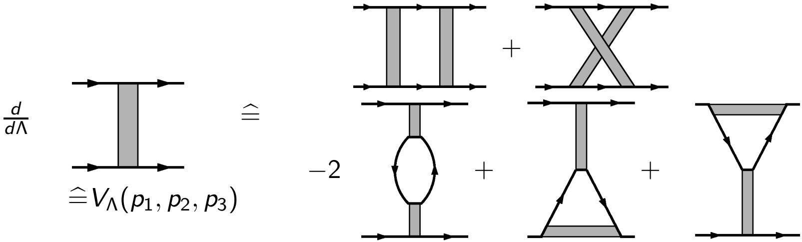
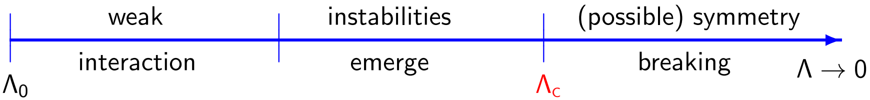


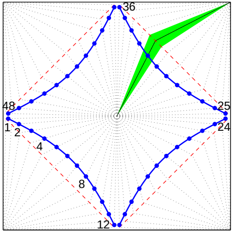
- 3 scale regimes of the RG flow

Honerkamp and Salmhofer, 2001



N–Patch Schemes

- neglect frequency dependence
- divide momentum space into N patches
- solve $\sim N^3$ ordinary differential equations



Zanchi and Schulz, 1998

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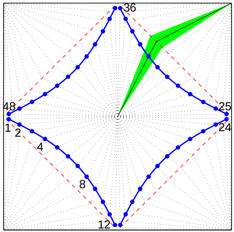
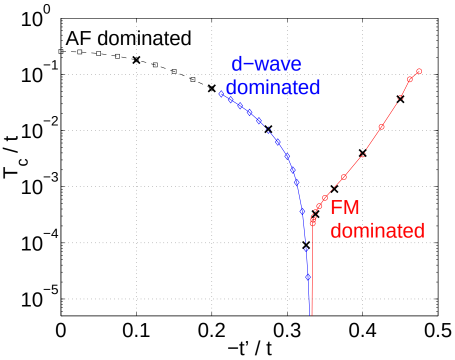


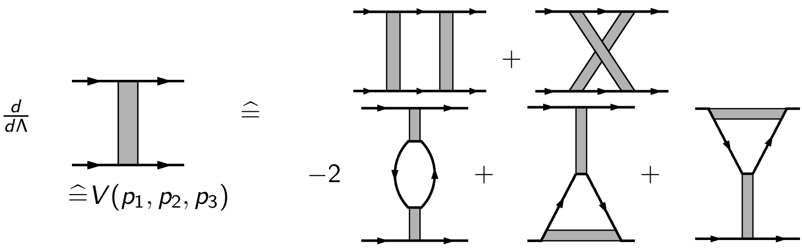
Zanchi and Schulz, 1998

Temperature RG Flow Honerkamp and Salmhofer, 2001



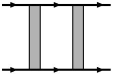
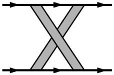
- instabilities at van Hove filling
- ferromagnetism not artifially suppressed

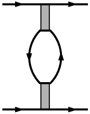
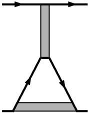
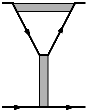
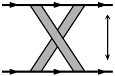
Parametrization of the Vertex Function



Observation: The leading weak coupling instabilities are mainly determined by the *singular* momentum and frequency structure of the flow equation.

Definition of 3 Channels

$\dot{\Phi}^{\Lambda}_{\mathrm{SC}}(p_1, p_3, p_1 + p_2) \quad \widehat{=} -$  , $\dot{\Phi}^{\Lambda}_{\mathrm{M}}(p_1, p_2, p_3 - p_1) \quad \widehat{=}$ 

$\dot{\Phi}^{\Lambda}_{\mathrm{K}}(p_1, p_2, p_2 - p_3) \quad \widehat{=} \quad 4$  -2  -2  $+$ 

mit $\Phi^{\Lambda_0}_{\mathrm{SC}} = \Phi^{\Lambda_0}_{\mathrm{M}} = \Phi^{\Lambda_0}_{\mathrm{K}} = 0$

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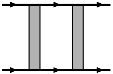
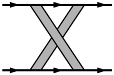
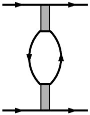
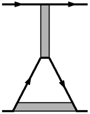
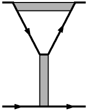
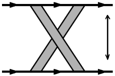
$\dot{\Phi}^{\Lambda}_{\mathrm{SC}}(p_1, p_3, p_1 + p_2) \;\widehat{=}\; -$  , $\dot{\Phi}^{\Lambda}_{\mathrm{M}}(p_1, p_2, p_3 - p_1) \;\widehat{=}$ 

$\dot{\Phi}^{\Lambda}_{\mathrm{K}}(p_1, p_2, p_2 - p_3) \;\widehat{=}\; 4$  -2  -2  $+$ 

mit $\Phi^{\Lambda_0}_{\mathrm{SC}} = \Phi^{\Lambda_0}_{\mathrm{M}} = \Phi^{\Lambda_0}_{\mathrm{K}} = 0$

The vertex function

$$V(p_1, p_2, p_3) = U - \Phi^{\Lambda}_{\mathrm{SC}}(p_1, p_3, p_1 + p_2) + \Phi^{\Lambda}_{\mathrm{M}}(p_1, p_2, p_3 - p_1)$$
$$+ \frac{1}{2}\Phi^{\Lambda}_{\mathrm{M}}(p_1, p_2, p_2 - p_3) - \frac{1}{2}\Phi^{\Lambda}_{\mathrm{K}}(p_1, p_2, p_2 - p_3)$$

The Effective Two-Fermion Interaction

$$\mathcal{V}_\Lambda[\Psi] = \mathcal{V}_{\text{H.M.}}[\Psi] + \mathcal{V}^\Lambda_{\text{SC}}[\Psi] + \mathcal{V}^\Lambda_{\text{M}}[\Psi] + \mathcal{V}^\Lambda_{\text{K}}[\Psi]$$

$$\mathcal{V}^\Lambda_{\text{SC}}[\Psi] = -\frac{1}{4}\int \mathrm{d}q\mathrm{d}q'\mathrm{d}l\ \Phi^\Lambda_{\text{SC}}(q,q',l)\sum_{J=0}^{3}\left(\overline{\Psi}(q)\sigma^{(J)}\overline{\Psi}(l-q)\right)\left(\Psi(q')\sigma^{(J)}\Psi(l-q')\right)$$

"superconductivity"

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$$\mathcal{V}_{\text{M}}^\Lambda[\Psi] = -\frac{1}{4}\int \mathrm{d}q\mathrm{d}q'\mathrm{d}l\ \Phi_{\text{M}}^\Lambda(q,q',l)\sum_{j=1}^{3}\Big(\overline{\Psi}(q)\sigma^{(j)}\Psi(q+l)\Big)\Big(\overline{\Psi}(q')\sigma^{(j)}\Psi(q'-l)\Big)$$

"magnetism"

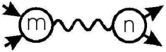
$$\mathcal{V}_{\text{K}}^\Lambda[\Psi] = -\frac{1}{4}\int \mathrm{d}q\mathrm{d}q'\mathrm{d}l\ \Phi_{\text{K}}^\Lambda(q,q',l)\Big(\overline{\Psi}(q)\Psi(q+l)\Big)\Big(\overline{\Psi}(q')\Psi(q'-l)\Big)$$

"forward scattering"

Decomposition of the Superconducting Channel

$$\Phi^{\Lambda}_{\mathrm{sc}}(q,q',l) = \sum_{mn} D_{mn}(l) f_m(\tfrac{\mathbf{l}}{2}-\mathbf{q}) f_n(\tfrac{\mathbf{l}}{2}-\mathbf{q}') + R_{\mathrm{sc}}(q,q',l)$$

$$\widehat{=} \;\sum_{mn}$$



$$+ R_{\mathrm{sc}}(q,q',l)$$

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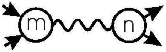
$$\widehat{=} \quad \sum_{mn}$$  $$+ R_{\mathrm{sc}}(q,q',l)$$

that is,

$$\mathcal{V}^{\Lambda}_{\mathrm{sc}}[\Psi] = -\frac{1}{4}\sum_{mn}\int \mathrm{d}l \; D_{mn}(l) \sum_{J=0}^{3} \int \mathrm{d}q \Big(\overline{\Psi}(q)\sigma^{(J)}\overline{\Psi}(l-q)\Big) f_m(\tfrac{l}{2}-q)$$
$$\int \mathrm{d}q' \Big(\Psi(q')\sigma^{(J)}\Psi(l-q')\Big) f_n(\tfrac{l}{2}-q') + \mathcal{R}_{\mathrm{sc}}[\Psi]$$

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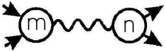
$$\widehat{=} \;\; \sum_{mn}$$  $$+ R_{\mathrm{sc}}(q, q', l)$$

For a *curved* and *regular* Fermi surface

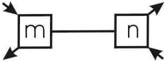
- particle–hole graphs are marginal
- the most attractive eigenvalue of $D(0)$ determines symmetry of the gap
- particle–hole graphs induce attractive eigenvalues of $D(0)$

Magnetic and Forward Scattering Channel

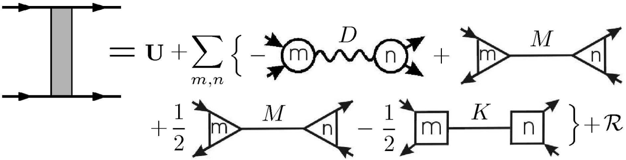
$$\Phi^{\Lambda}_{\mathrm{M}}(q,q',l) = \sum_{mn} M_{mn}(l) f_m(\mathbf{q}+\tfrac{\mathbf{l}}{2}) f_n(\mathbf{q}'-\tfrac{\mathbf{l}}{2}) + R_{\mathrm{M}}(q,q',l)$$

$\widehat{=} \; \sum_{mn}$  $+ R_{\mathrm{M}}(q,q',l)$

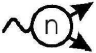
$$\Phi^{\Lambda}_{\mathrm{K}}(q,q',l) = \sum_{mn} K_{mn}(l) f_m(\mathbf{q}+\tfrac{\mathbf{l}}{2}) f_n(\mathbf{q}'-\tfrac{\mathbf{l}}{2}) + R_{\mathrm{K}}(q,q',l)$$

$\widehat{=} \; \sum_{mn}$  $+ R_{\mathrm{K}}(q,q',l)$

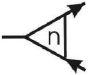
Channel Decomposition: Summary



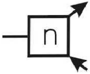
bilinears:



$$= \int \mathrm{d}q\ \overline{\Psi}(q)\sigma^{(J)}\overline{\Psi}(l-q)f_n(\tfrac{l}{2}-q)$$



$$= \int \mathrm{d}q\ \overline{\Psi}(q)\sigma^{(j)}\Psi(q+l)f_n(q+\tfrac{l}{2})$$



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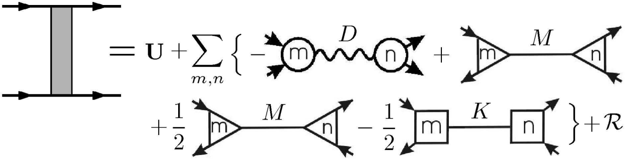
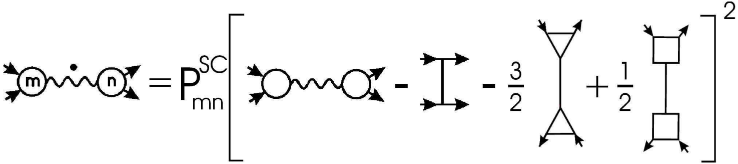


Remainder term $\mathcal{R}$

- under control for curved and regular Fermi surfaces
- in general: choose f's such that they are "small"

. . . is dropped at first.

The Boson Propagator Flow



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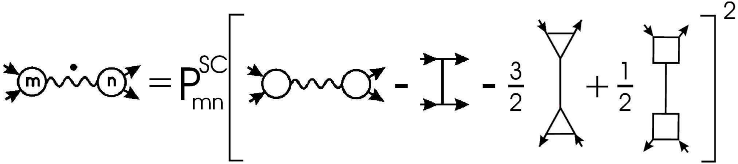
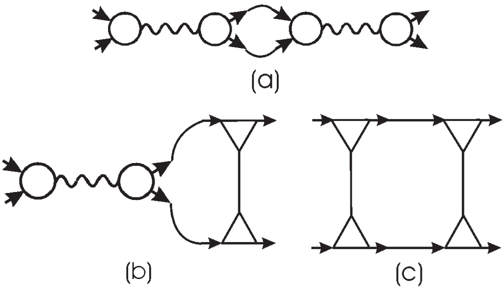


three examples, how the square is taken:



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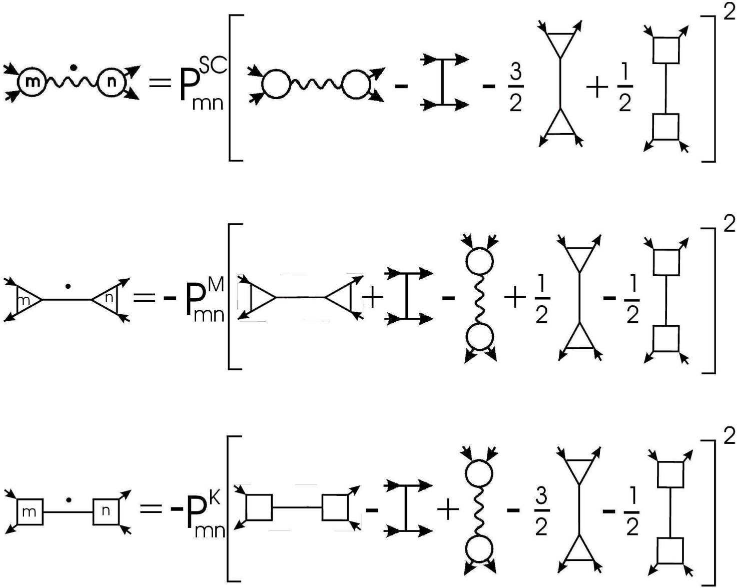

Specific Set-up of the RG Flow

- RG scale $\Lambda = \Omega$ is decreased in the flow

$$\frac{1}{ip_0 - \varepsilon(\mathbf{p}) + \mu} \quad \frac{p_0^2}{p_0^2 + \Omega^2}$$

- treat scales $\Omega > \Omega_0$ by perturbation theory in $\frac{U}{\Omega_0}$

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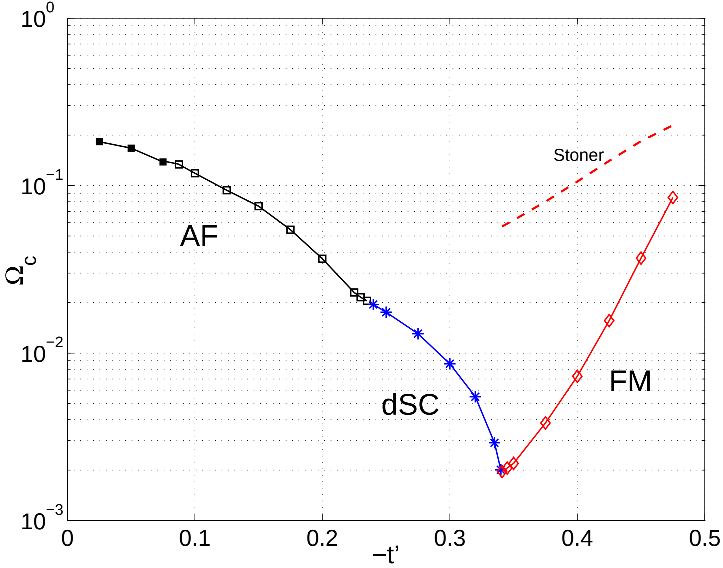
- form factors: only

$$f_1(\mathbf{p}) = 1 \qquad (s\text{-wave})$$
$$f_2(\mathbf{p}) = \cos p_x - \cos p_y \qquad (d_{x^2-y^2}\text{-wave})$$

- boson propagators:
  - neglect frequency dependence
  - approximate momentum dependence numerically [step functions with high accuracy for transfer momenta $\approx (0,0), (\pi,\pi)$]

Instabilities at Van Hove Filling, $U = 3t$, and $T = 0$



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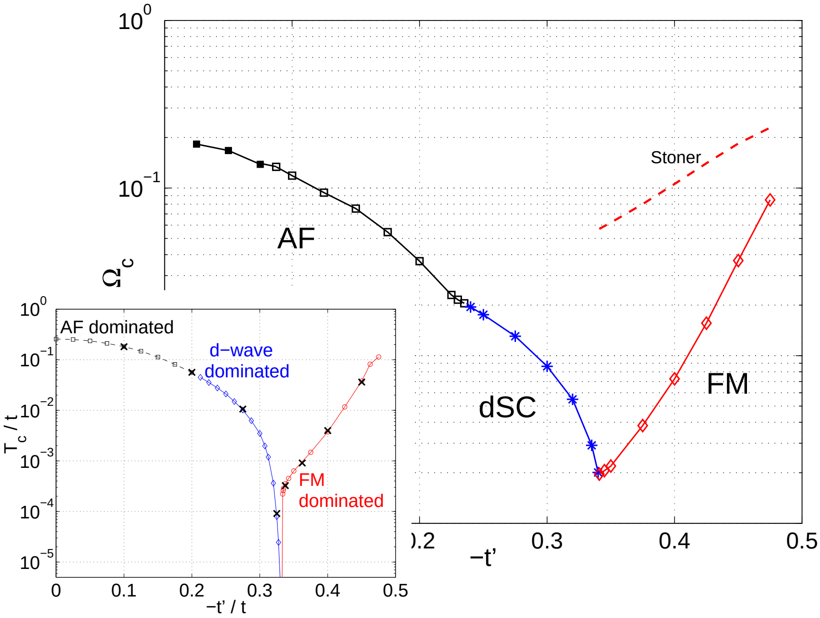
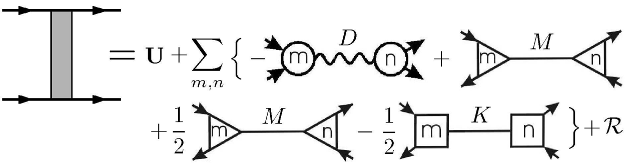
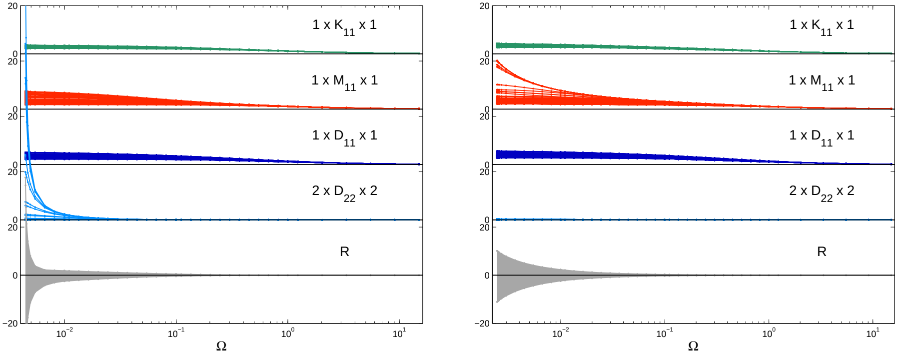


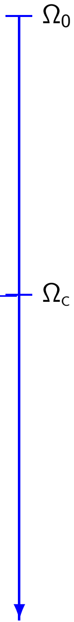
Remainder Estimation

J. Ortloff, C.H., C. Honerkamp, M. Salmhofer





And Now?



The proposed parametrization of the vertex function

- allows to identify leading instabilities,
- shows that the vertex function is singular only at points, and
- allows Hubbard–Stratonovich transformations.

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Ω_0

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Ω_c

Continuation of the RG flow into an ordered phase

- fermions Salmhofer et al.
- bosons Metzner et al.
- fermions and bosons Wetterich et al., Metzner et al., Kopietz et al.

Effective Model at Scale Ω_c

eg. keep only $d_{x^2-y^2}$-wave superconductivity

$$Z = \int \mathcal{D}[\Psi] \exp\Big\{ -\frac{1}{2}(\Psi, G_{\Omega_c}^{-1}\Psi) - (\overline{X}, D_{11} X) \Big\}$$

with

$$\overline{X}(l) = \frac{1}{2} \int \mathrm{d}q \; \overline{\Psi}(q)\epsilon\overline{\Psi}(l-q) \; f_1(\tfrac{l}{2} - q)$$

Effective Model at Scale Ω_c

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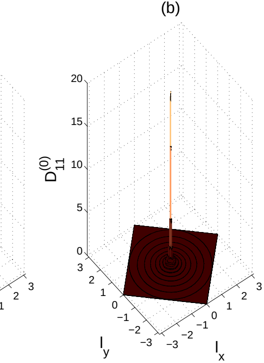
$$Z = \int \mathcal{D}[\Psi] \exp\Big\{ -\frac{1}{2}(\Psi, G_{\Omega_c}^{-1}\Psi) - (\overline{X}, D_{11}X)\Big\}$$

with



$$\overline{X}(l) = \frac{1}{2}\int \mathrm{d}q\ \overline{\Psi}(q)\epsilon\overline{\Psi}(l-q)\ f_1(\tfrac{l}{2}-q)$$

$$G_{\Omega_c}(p) = \frac{1}{ip_0 - \epsilon(\mathbf{p}) + \mu}\ \frac{\Omega_c^2}{p_0^2 + \Omega_c^2}$$

↑
suppresses all fermions that have already been integrated out

Conclusion and Outlook

The proposed decomposition

- preserves the essential structure of the one-loop RG,
- reduces computing cost, $O(N^3) \to O(N)$ ODEs,
- reveals point-like singularities of the interaction vertex, and
- gives effective vertices near Ω_c that can be directly transformed into an effective theory of order parameter fields.

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Work in progress:

- adaptive flow of Ω-dependent form factors
- frequency dependence of the boson propagators
- $\Omega < \Omega_c$: competition of superconductivity and ferromagnetism?
pseudo-gap phase?