The Renormalization Group Far From Equilibrium: Singular Perturbations, Pattern Formation and Hydrodynamics

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Summary

Renormalization and the renormalization group (RG) were originally developed by physicists attempting to understand the divergent terms in perturbation theory and the short distance behaviour of quantum electrodynamics. During the last twenty years, these methods have been used to unify the construction of global approximations to ordinary and partial differential equations. Early examples included similarity solutions and travelling waves, which exhibit the same anomalous scaling properties found in quantum field theories, but here manifested in such problems as flow in porous media, the propagation of turbulence and the spread of advantageous genes. Fifteen years ago, these methods were extended to asymptotic problems with no special power-law scaling structure, enabling a vast generalization that includes and unifies all known singular perturbation theory methods, but with greater accuracy and calculational efficiency. Applications range from cosmology to viscous hydrodynamics.

In this work, RG is applied to differential equations, not field theories. The problems have no stochastic component nor necessarily scale-invariance.

Some uses of RG in applied mathematics

1 Self-similarity, incomplete similarity and asymptotics of nonlinear PDEs

Dimensional analysis; extended dimensional analysis and anomalous exponents in the long-time behaviour of PDEs; modified porous medium equation; propagation of turbulence.

2 Singular perturbations: uniformly valid approximations from RG

Perturbed oscillators, boundary layer problems with log e terms, WKB with turning points, switchback problems; spatially-extended systems and the derivation of amplitude and phase equations near and far from bifurcations.

3 Numerical methods and under-resolved computation

Similarity solutions are fixed points of RG transformations; velocity selection, structural stability and the Kolmogorov-Petrovsky-Piscunov problem; universal scaling phenomena in stochastic PDEs; perfect operators.

Note: large and still growing mathematics literature proving rigorous and formal results about these techniques. Ziane, Temam, DeVille, O'Malley, Kirkinis and many others ...

Motivation: Why RG for PDEs?

We have written the equations of water flow. From experiment, we find a set of concepts and approximations to use to discuss the solution—vortex streets, turbulent wakes, boundary layers. When we have similar equations in a less familiar situation, and one for which we cannot yet experiment, we try to solve the equations in a primitive, halting, and confused way to try to determine what new qualitative features may come out, or what new qualitative forms are a consequence of the equations. Our equations for the sun, for example, as a ball of hydrogen gas, describe a sun without sunspots, without the rice-grain structure of the surface, without prominences, without coronas. Yet, all of these are really in the equations; we just haven't found the way to get them out.

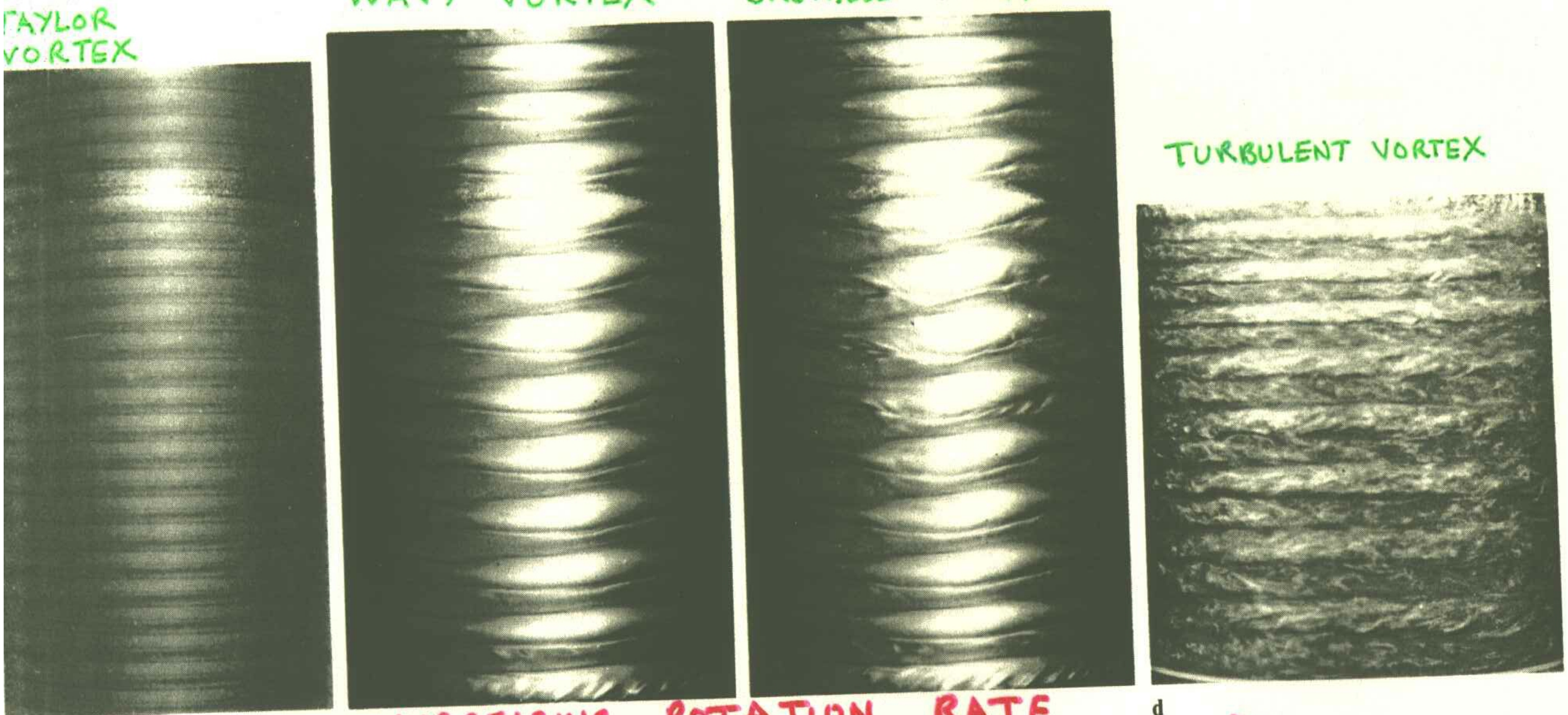
The next great era of awakening of human intellect may well produce a method of understanding the *qualitative* content of equations. Today we cannot. Today we cannot see that the water flow equations contain such things as the barber pole structure of turbulence that one sees between rotating cylinders. Today we cannot see whether Schrödinger's equation contains frogs, musical composers, or morality—or whether it does not. We cannot say whether something beyond it like God is needed, or not. And so we can all hold strong opinions either way.

Feynman Lectures on Physics, vol 2, chapter 41

(a) E. L. KOSCHMIEDER (1979)
(b) D. COLES (1965)
(c) D. COLES (1965)
(d) P. FENSTERMACHER et al (1979)

FLOW BETWEEN CONCENTRIC CYLINDERS WITH INNER CYLINDER ROTATING

TAYLOR VORTEX — WAVY VORTEX — UNSTABLE VORTEX — TURBULENT VORTEX



a b c d

INCREASING ROTATION RATE

Fig. 6.1a–d. Photographs of the flow between concentric cylinders with the inner cylinder rotating. (The radius ratio is 0.88.) **(a)** $R \simeq R_c$; Taylor vortex flow [6.5]. **(b)** $R/R_c = 10.4$; wavy vortex flow [Ref. 6.6, Fig. 19d]. **(c)** $R/R_c = 12.3$; the "first appearance of randomness" in wavy vortex flow [Ref. 6.6, Fig. 19e]. **(d)** $R/R_c = 23.5$; the azimuthal waves have disappeared and the flow is turbulent, although the axial periodicity remains [Ref. 6.7, Fig. 1d]. The visualization of the flow in these experiments was achieved by suspending small flat flakes in the fluid; the flakes align with the flow, and variations in their orientation are observed as variations in the transmitted or reflected intensity

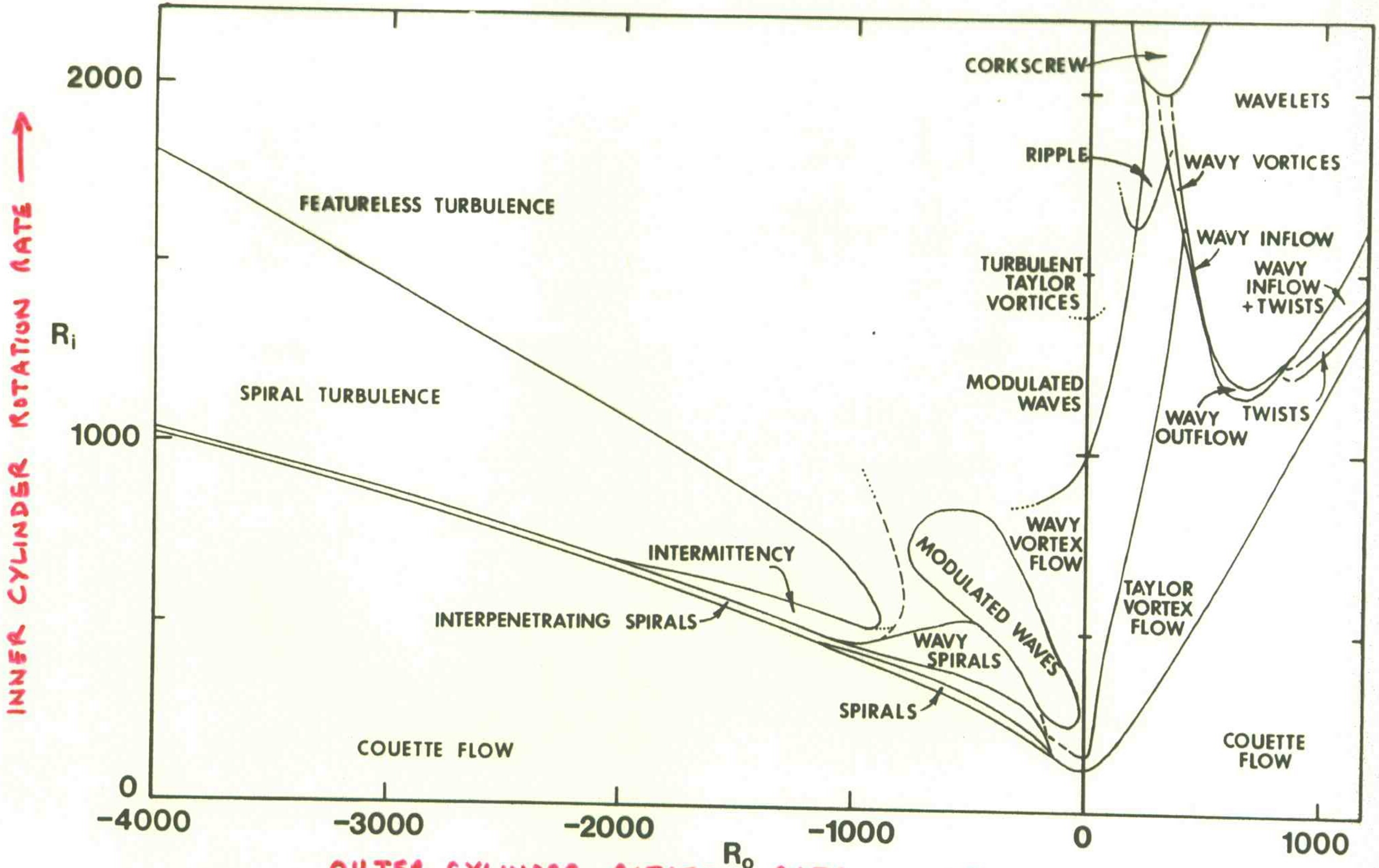


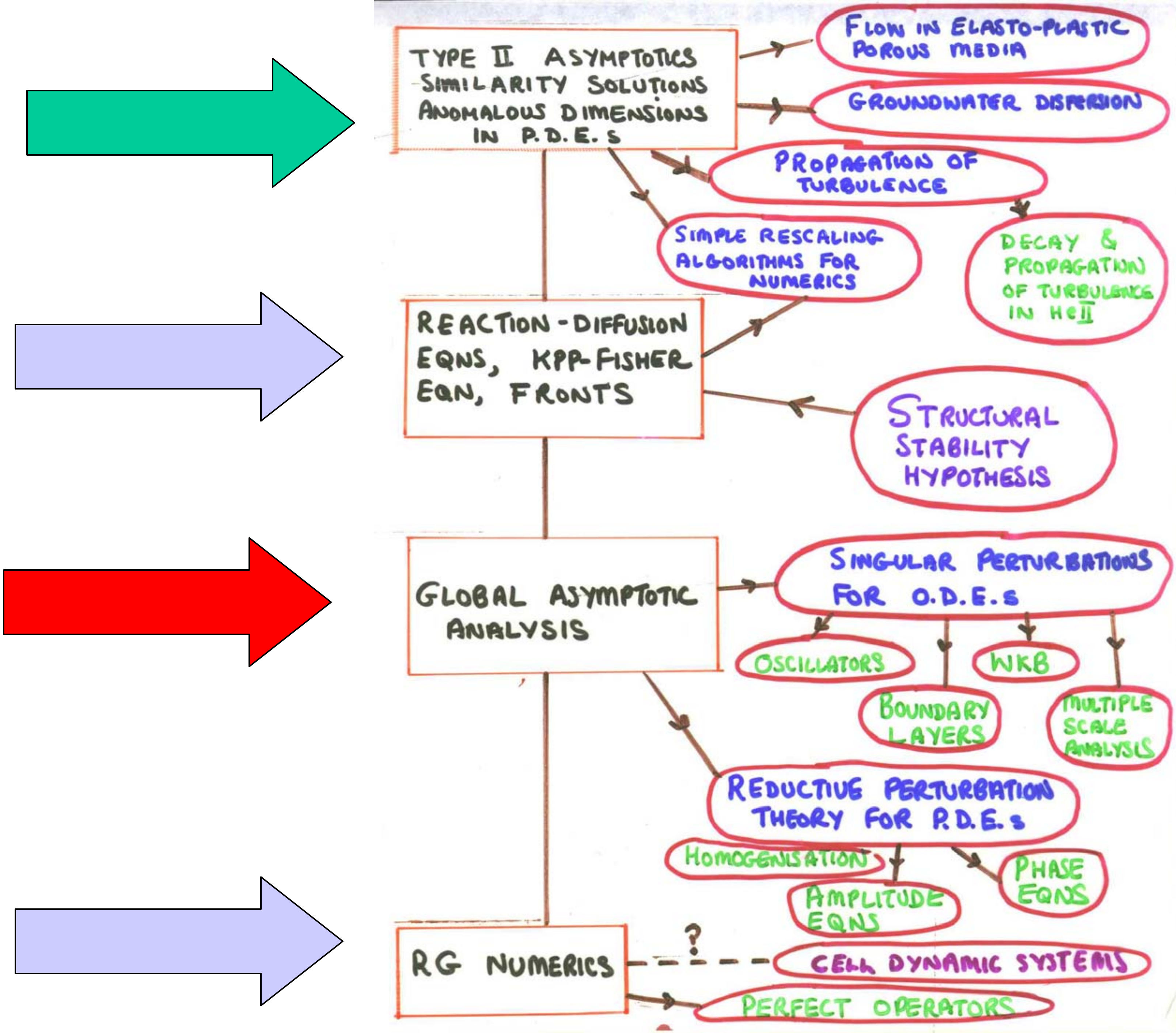
Fig. 10.1. Flow regimes observed in flow between independently rotating cylinders with radius ratio 0.883 and aspect ratio 30. (Adapted from [10.40])

C.D. ANDERECK et al (1985).

N.B. PRECISE STATE IS NOT A UNIQUE FUNCTION OF R_i, R_o, R_i/R_o and ASPECT RATIO

Development of RG methods at Illinois 1989-present

Historical overview


TYPE II ASYMPTOTICS SIMILARITY SOLUTIONS ANOMALOUS DIMENSIONS IN P.D.E.S
FLOW IN ELASTO-PLASTIC POROUS MEDIA
GROUNDWATER DISPERSION
PROPAGATION OF TURBULENCE
SIMPLE RESCALING ALGORITHMS FOR NUMERICS
DECAY & PROPAGATION OF TURBULENCE IN HeII
REACTION-DIFFUSION EQNS, KPP-FISHER EQN, FRONTS
STRUCTURAL STABILITY HYPOTHESIS
GLOBAL ASYMPTOTIC ANALYSIS
SINGULAR PERTURBATIONS FOR O.D.E.S
OSCILLATORS
WKB
BOUNDARY LAYERS
MULTIPLE SCALE ANALYSIS
REDUCTIVE PERTURBATION THEORY FOR P.D.E.S
HOMOGENISATION
AMPLITUDE EQNS
PHASE EQNS
RG NUMERICS
?
CELL DYNAMIC SYSTEMS
PERFECT OPERATORS

Anomalous dimensions in partial differential equations

Similarity, Self-Similarity, and Intermediate Asymptotics

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LECTURES ON PHASE TRANSITIONS AND THE RENORMALIZATION GROUP

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see ch. 10 especially

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SIMILARITY SOLUTIONS

IN NON-EQUILIBRIUM PROBLEMS, WE ARE OFTEN INTERESTED IN SIMILARITY SOLUTIONS

$$\boxed{u(x,t) = t^{\alpha} f(x t^{\beta})}$$

OR TRAVELLING WAVES

$$\boxed{u(x,t) = f(x - vt)}$$

REASON: THESE SOLUTIONS OFTEN DESCRIBE LONG TIME BEHAVIOUR

GOAL: COMPUTE EXPONENTS α, β
VELOCITY v
SCALING FUNCTION f

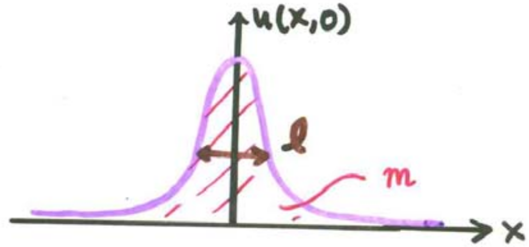
SUFFICES TO CONSIDER SIMILARITY SOLUTIONS ONLY: SUBSTITUTION $x = \log X$ $t = \log T$ CONVERTS

$$\underbrace{f(x - vt)}_{\text{TRAVELLING WAVE}} \longrightarrow \underbrace{F\left(\frac{X}{T^{v}}\right)}_{\text{SIMILARITY SOLUTION}}$$

DIFFUSION EQUATION

INITIAL VALUE PROBLEM:

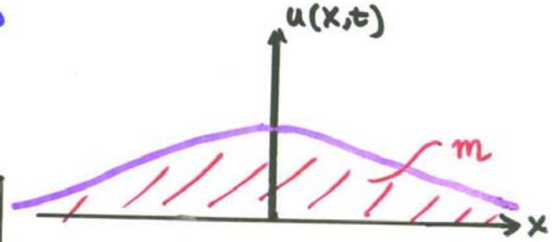
$$\boxed{\partial_t u = \frac{1}{2}\partial_x^2 u}$$



GAUSSIAN OF WIDTH l AND MASS m

$\downarrow t$

$$\boxed{u(x,t) = \frac{m e^{-x^2/2(t+l^2)}}{\sqrt{2\pi(t+l^2)}}}$$



LONG TIME BEHAVIOUR:

$$u(x,t) \xrightarrow[l \text{ fixed}]{t \to \infty} \frac{m e^{-x^2/2t}}{\sqrt{2\pi t}}$$

OR EQUIVALENTLY

$$u(x,t) \xrightarrow[t \text{ fixed}]{l \to 0} \frac{m e^{-x^2/2t}}{\sqrt{2\pi t}}$$

i.e. ASYMPTOTIC BEHAVIOUR OF INITIAL VALUE PROBLEM GIVEN BY SIMILARITY SOLUTION — THE SOLUTION CORRESPONDING TO DELTA FUNCTION INITIAL CONDITION.

DIMENSIONAL ANALYSIS

DIFFUSION EQUATION EXAMPLE OF COMMON PHENOMENON IN PHYSICS.

EXPRESS PHYSICAL PROBLEM IN DIMENSIONLESS VARIABLES $\Pi, \Pi_0, \Pi_1, \Pi_2, \ldots \Pi_n$

THEN SOLUTION IS OF FORM

$$\boxed{\Pi = f(\Pi_0, \Pi_1, \Pi_2, \cdots, \Pi_n)}$$

IF ONE VARIABLE (e.g.) Π_0 IS SMALL, THEN USUALLY SET $\Pi_0 = 0$.

i.e.

$$\Pi_0 = \frac{\text{characteristic dimension of apparatus}}{\text{radius of moon}} \approx 0$$

THEN WE HAVE

$$\boxed{\Pi = f(0, \Pi_1, \Pi_2, \ldots, \Pi_n)}$$

"COMMON SENSE" = CASE 1

IN DIFFUSION EQUATION EXAMPLE

$$\Pi = \frac{u}{m}\sqrt{t} \; ; \quad \Pi_0 = \frac{\ell}{\sqrt{t}} \; ; \quad \Pi_1 = \frac{x}{\sqrt{t}}$$

$$\boxed{u = \frac{m}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right) \quad \text{as } \Pi_0 \to 0}$$

DIMENSIONAL ANALYSIS (2)

WE MADE A STRONG ASSUMPTION THAT THE LIMIT $\Pi_0 \to 0$ EXISTS. BARENBLATT HAS GIVEN SEVERAL EXAMPLES WHERE THIS ASSUMPTION BREAKS DOWN.

CLASSIFY ASYMPTOTICS:

CASE 1: $\boxed{\Pi \sim f(0, \Pi_1, \ldots \Pi_n) \quad \text{as } \Pi_0 \to 0}$

COMMONPLACE (BY CONSTRUCTION)

CASE 2: $\boxed{\Pi \sim \Pi_0^{-\alpha} g\left(\frac{\Pi_1}{\Pi_0^{\alpha_1}}, \cdots \frac{\Pi_n}{\Pi_0^{\alpha_n}}\right) \quad \text{as } \Pi_0 \to 0}$

PRESENTS PROBLEMS WHEN IT OCCURS. FUNCTION g AND THE EXPONENTS $\alpha, \alpha_1, \ldots \alpha_n$ MUST BE DETERMINED.

CASE 3: NONE OF THE ABOVE

CASE 2 EXAMPLES IN FLUID MECHANICS, CRITICAL PHENOMENA, ELECTROMAGNETISM, ...

THESE PROBLEMS CAN BE ANALYSED USING THE RENORMALISATION GROUP.

BARENBLATT EQUATION

SEEMINGLY INNOCUOUS MODIFICATION TO DIFFUSION EQN.

$$\boxed{\partial_t u = D\,\partial_x^2 u \qquad D = \begin{cases} \frac{1}{2} & \partial_x^2 u > 0 \\ \frac{1}{2}(1+\epsilon) & \partial_x^2 u < 0 \end{cases}} \quad (B)$$

DESCRIBES PRESSURE IN A FLUID PASSING THROUGH A POROUS MEDIUM WHICH CAN EXPAND AND CONTRACT IRREVERSIBLY (P2C2E).

PARAMETER ϵ DEPENDS UPON ELASTIC CONSTANTS OF FLUID, POROUS MEDIUM.

(B) IS NOT DERIVABLE FROM CONTINUITY EQN

$$\partial_t u + \nabla \cdot \underline{J} = 0$$

SO MASS OF DISTRIBUTION NOT CONSERVED:

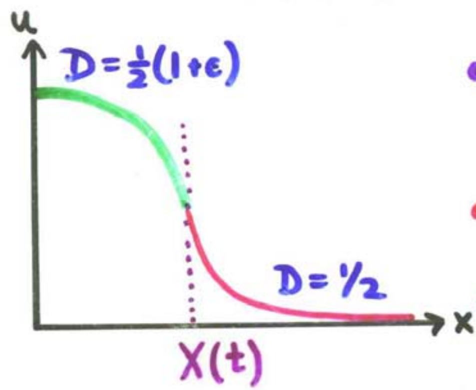
$$\boxed{m(t) \neq m(0)}$$

BARENBLATT EQN (2)

Q/ WHAT IS LONG TIME BEHAVIOUR OF (B) ?

A/ ¿ $u(x,t) \xrightarrow[t \to \infty]{} \frac{1}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}, \epsilon\right)$?

NO !



- SUBSTITUTE PROPOSED FORM INTO (B).
- GIVES TWO O.D.E.'s
- CANNOT MATCH 1^{st} + 2^{nd} DERIVATIVES AT $X(t)$

BUT THERE EXISTS A UNIQUE SOLUTION OF THE INITIAL VALUE PROBLEM WITH CONTINUOUS SECOND DERIVATIVES (KAMENOMOSTSKAYA, 1957)

WE WILL SEE THAT LONG TIME BEHAVIOUR IS

$$\boxed{u(x,t) \xrightarrow[t \to \infty]{} \frac{1}{t^{\frac{1}{2}+\alpha}} f\left(\frac{x}{\sqrt{t}}, \epsilon\right)}$$

anomalous dimension, $\alpha = \alpha(\epsilon)$

HEURISTIC DERIVATION

WRITE SOLUTION AS

$$\boxed{u(x,t) = \frac{m(t)}{\sqrt{2\pi(t+\ell^2)}}\, e^{-x^2/2(t+\ell^2)}}$$

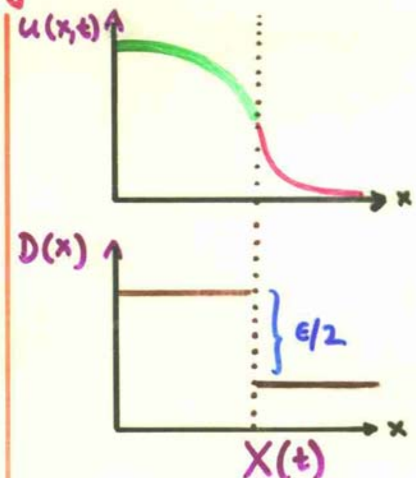
IF ϵ IS SMALL, REMOVAL OF MASS OCCURS "SLOWLY" AND DISTRIBUTION ADIABATICALLY ADJUSTS TO THE GAUSSIAN FORM ABOVE (STRICTLY VALID FOR $\epsilon = 0$).

EQUATION OF MOTION FOR $m(t)$:

$$\partial_t m(t) = \partial_t \int_{-\infty}^{\infty} u(x,t)\, dx = \int D(x)\, \partial_x^2 u(x,t).dx$$

$$= -\int \partial_x D \cdot \partial_x u \cdot dx$$

$$\boxed{\partial_t m(t) = \epsilon\, \partial_x u\big(X(t),t\big)}$$



$X(t) = \sqrt{t+\ell^2} + O(\epsilon)$

SUBSTITUTE SOLUTION

$$\partial_t m = -\frac{\epsilon\, m(t)}{\sqrt{2\pi}}\, \frac{e^{-1/2}}{(t+\ell^2)}$$

$$\boxed{\begin{aligned} m(t) &= m(0)\, \frac{\ell^{2\alpha}}{(t+\ell^2)^{\alpha}} \\ \alpha &= \epsilon/\sqrt{2\pi e} \end{aligned}}$$

HEURISTIC DERIVATION (2)

SOLUTION IN FORM

$$u(x,t) = \frac{m(t)}{\sqrt{2\pi(t+\ell^2)}} e^{-x^2/2(t+\ell^2)}$$

TIME VARIATION OF MASS

$$m(t) \cong m(0) \frac{\ell^{2\alpha}}{(t+\ell^2)^\alpha}, \qquad \alpha = \frac{\epsilon}{\sqrt{2\pi e}}$$

SOLUTION

$$\boxed{u(x,t) = \frac{m(0)\,\ell^{2\alpha} \cdot e^{-x^2/2(t+\ell^2)}}{\sqrt{2\pi}\,(t+\ell^2)^{\frac{1}{2}+\alpha}}}$$

- MORE CAREFUL RENORMALISATION GROUP ANALYSIS SHOWS THAT

$$\alpha = \frac{\epsilon}{\sqrt{2\pi e}} - 0\cdot 101\cdots \epsilon^2 + O(\epsilon^3)$$

AND FORM OF $u(x,t)$ CORRECT TO $O(\epsilon)$.

- EXPANSION FOR $\alpha(\epsilon)$ IS ANALYTIC (ARONSON + VAZQUEZ)
- LIMIT $\ell \to 0$ SINGULAR
- NO NOISE IN BARENBLATT EQN OR PARTITION FUNCTION

INTERPRETATION

$\epsilon = 0$ MEASUREMENT AT LONG TIMES OF $m(t)$ IMPLIES KNOWLEDGE OF INITIAL VALUE $m(0)$.

$\ell \to 0$ LIMIT O.K. SYSTEM "FORGETS" INITIAL CONDITION AFTER SUFFICIENTLY LONG TIME. $u(x,t) \xrightarrow[t/\ell^2 \to \infty]{} \dfrac{m e^{-x^2/2t}}{(2\pi t)^{1/2}}$

$\epsilon \neq 0$ AT LATE TIMES CANNOT INFER $m(0)$ FROM $m(t)$ ALONE. INDEED, ONE CANNOT EVEN TELL HOW MUCH TIME HAS ELAPSED!

$\ell \to 0$ LIMIT SINGULAR. SYSTEM "REMEMBERS" EXISTENCE OF INITIAL CONDITION WITH NON-ZERO WIDTH. BUT ANOMALOUS DIMENSION IS INDEPENDENT OF ℓ. $u(x,t) \xrightarrow[t/\ell^2 \to \infty]{} \dfrac{m(0)\, \ell^{2\alpha}\, e^{-x^2/2t}}{(2\pi)^{1/2}\, t^{1/2+\alpha}}$

ANOMALOUS DIMENSIONS AT CRITICAL POINTS

TWO POINT CORRELATION FUNCTION

$$\boxed{G(\underline{x}-\underline{y}) = \langle \phi(\underline{x})\, \phi(\underline{y}) \rangle}$$

ORDER PARAMETER

AT $T = T_c$

$$\boxed{\hat{G}(k, T_c) \sim k^{-2+\eta}} \quad \circledast$$

ANOMALOUS DIMENSION

BUT DIMENSIONAL ANALYSIS GIVES

$$\boxed{[\phi] = L^{1-d/2} \Rightarrow [\hat{G}(k,T_c)] = L^2}$$

DIMENSION OF ....

LENGTH

DIMENSIONALITY OF SPACE

DOES $\circledast$ VIOLATE DIMENSIONAL ANALYSIS?

NO! MUST INCLUDE LATTICE SPACING ℓ

$$\boxed{\hat{G}(k,T_c) \sim \ell^{\eta}\, k^{-2+\eta}}$$

EVEN WHEN CORRELATION LENGTH $\rightarrow \infty$, SYSTEM "REMEMBERS" EXISTENCE OF LATTICE.

PERTURBATIVE RENORMALISATION

1. WRITE BARENBLATT EQUATION AS

$$\left[\partial_t - \tfrac{1}{2}\partial_x^2\right] u(x,t) = \frac{\epsilon}{2}\,\Theta\big(X(t,\epsilon) - |x|\big)\,\partial_x^2 u$$

WHERE $\partial_t u(X(t,\epsilon),t) = 0$

2. SOLUTION IS

$$u(x,t) = \int_{-\infty}^{\infty} dy\, G(x-y,t)\, u(y,0)$$

$$G(x,y) \equiv \frac{1}{\sqrt{2\pi y}}\, e^{-x^2/2y}$$

$$+\ \frac{\epsilon}{2}\int_0^t ds \int_{-\infty}^{\infty} dy\, G(x-y,t-s)\,\Theta\big(-\partial_s u(y,s)\big)\,\partial_y^2 u(y,s)$$

3. EVALUATION OF INTEGRALS AND ISOLATION OF DIVERGENCES GIVES

$$\boxed{u(x,t) = \frac{m_0}{\sqrt{2\pi t}}\, e^{-x^2/2t}\left[1 - \frac{\epsilon}{\sqrt{2\pi e}}\log\frac{t}{\ell^2} + O(\epsilon^2)\right] + O(\ell,\epsilon)}$$

MASS ASSOCIATED WITH INITIAL CONDITION OF WIDTH ℓ

REGULAR AS $\ell \to 0$

4. PERTURBATIVE RENORMALISATION

IN LIMIT $\ell \to 0$ m_0 MAY GO TO ZERO OR INFINITY.
BUT DISTRIBUTION MASS AT TIME t STILL EXISTS AND IS OBSERVABLE.

$$m = Z^{-1}\left(\frac{\ell}{\mu},\epsilon\right) m_0$$

N.B. $m = m(\mu,\epsilon)$

Z IS DIMENSIONLESS $\Rightarrow$ CANNOT DEPEND ON ℓ ALONE:
NEED ANOTHER LENGTH SCALE μ.

PERTURBATIVE RENORMALISATION (2)

5. POWER SERIES EXPANSION OF Z

$$Z = 1 + \sum_{n=1}^{\infty} a_n(\ell/\mu)\,\epsilon^n$$

CHOSE a_n ORDER BY ORDER IN ϵ SO THAT $u(x,t)$ IS FINITE

$$a_1(\ell/\mu) = \frac{1}{\sqrt{2\pi e}} \log\left(\frac{C_1\mu^2}{\ell^2}\right), \quad C_1 \text{ arbitrary}$$

$$u(x,t) = \frac{m}{\sqrt{2\pi t}} e^{-x^2/2t}\left[1 + \frac{\epsilon}{\sqrt{2\pi e}}\log\frac{C_1\mu^2}{\ell^2} + O(\epsilon^2)\right] \times$$

$$\times\left[1 - \frac{\epsilon}{\sqrt{2\pi e}}\log\frac{t}{\ell^2} + O(\epsilon^2)\right] + O(\ell,\epsilon)$$

6. CANCELLATION OF DIVERGENCE AS $\ell \to 0$

MULTIPLYING OUT $[\cdots] \times [\cdots]$

$$= 1 - \frac{\epsilon}{\sqrt{2\pi e}}\left[\log\frac{t}{\cancel{\ell^2}} + \log\frac{\cancel{\ell^2}}{C_1\mu^2} + O(\epsilon)\right]$$

7. RENORMALIZATION GROUP EQUATION

THE SCALE μ IS ARBITRARY SO $u(x,t)$ INDEPENDENT OF μ.

$$\frac{du}{d\mu} = 0 \quad\Rightarrow\quad \frac{dm}{m} = -\frac{2\epsilon}{\sqrt{2\pi e}}\frac{d\mu}{d\mu} \quad\Rightarrow\quad m \sim m_0\,\mu^{-2\epsilon/\sqrt{2\pi e}}$$

i.e. $\quad m(\mu) = m(\sigma)\left(\frac{\sigma}{\mu}\right)^{2\epsilon/\sqrt{2\pi e}}$

RELATES m AT TWO DIFFERENT SCALES μ AND σ

PERTURBATIVE RENORMALISATION (3)

8. ELIMINATE LOG TERM BY SUITABLE CHOICE OF μ

$$u(x,t) = m(\mu)\frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\left(1 - \frac{\epsilon}{\sqrt{2\pi e}}\log\frac{t}{C_1\mu^2} + O(\epsilon^2)\right) + O(\cancel{\ell}\,\epsilon)$$

CHOSE $\mu^2 = t/C_1 \implies \log t/C_1\mu^2 = 0$

$$u(x,t) = m_0\frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\left(\frac{C_1}{t}\right)^{\epsilon/\sqrt{2\pi e}}\left(1 + O(\epsilon^2)\right) + O(\epsilon)$$

$$\boxed{u(x,t) \sim t^{-(\alpha + 1/2)} \qquad \alpha = \frac{\epsilon}{\sqrt{2\pi e}} + O(\epsilon^2)}$$

9. APPLY INITIAL CONDITIONS

KEEPING FACTORS OF ℓ

$$u(x,t) = m(\mu)\frac{e^{-x^2/2(t+\ell^2)}}{\sqrt{2\pi(t+\ell^2)}}\left(1 - \frac{\epsilon}{\sqrt{2\pi e}}\log\frac{t+\ell^2}{C_1\mu^2} + O(\epsilon^2)\right) + O(\epsilon)$$

AT $t=0$ $\quad \mu = \sigma \equiv \ell/\sqrt{C_1}$ AND $m(\sigma) = m_0$

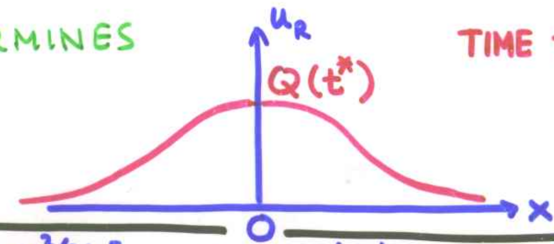
$$\boxed{u(x,t) = m_0\left(\frac{\ell^2}{t}\right)^{\epsilon/\sqrt{2\pi e}}\frac{e^{-x^2/2(t+\ell^2)}}{\sqrt{2\pi(t+\ell^2)}}}$$

Anomalous Dimensions (5)

(d) CHOSE ONE FROM FAMILY OF SOLUTIONS

THIS DETERMINES TIME t^*

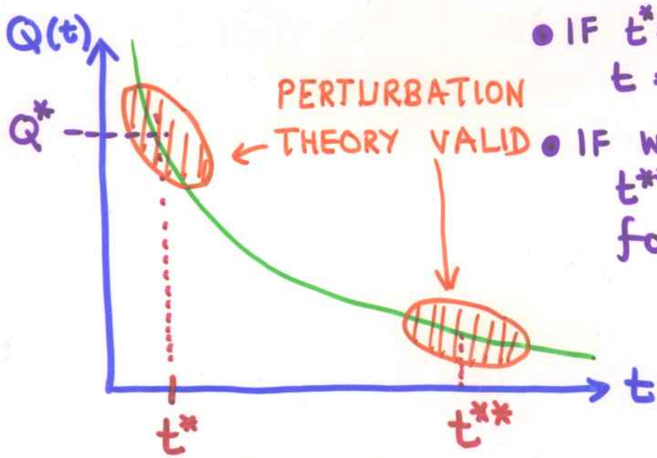
$C_1, \ldots$



$$\boxed{u_R(x,t) = Q(t^*)\sqrt{\frac{t^*}{t}}\, e^{-x^2/2t}\left[1 - \frac{\epsilon}{\sqrt{2\pi e}} \ln\left(\frac{t}{t^*}\right) + O(\epsilon^2)\right]}$$

THIS PERTURBATIVE SOLUTION VALID FOR $t \approx t^*$.

(e) BUT HAVE NOT YET SPECIFIED $t^*, Q(t^*)$



- IF $t^* = 5$ secs, PT poor for $t = 10^6$ secs.
- IF WE KNEW $Q(t^{**})$, WITH $t^{**} \doteq 9 \times 10^5$ secs., PT good for $t = 10^6$ secs.

$\Rightarrow$ CHOSE $t^* = 9 \times 10^5$ secs. BUT INSIST THAT u_R STAYS ON THE PARTICULAR SOLUTION WITH $Q = Q^*$ AT $t = 5$ secs.

ANOMALOUS DIMENSIONS (6)

(f) GELLMANN–LOW TRICK:

$u_R(x,t)$ IS INDEPENDENT OF t^*.

$$\Rightarrow \boxed{\frac{\partial u_R}{\partial t^*} + \frac{\partial u_R}{\partial Q}\frac{dQ}{dt^*} = 0}$$

$$\therefore \beta(Q) \equiv t^* \frac{dQ}{dt^*} = \frac{-t^* \frac{\partial u_R}{\partial t^*}}{\frac{\partial u_R}{\partial Q}}$$

$$\boxed{\beta(Q) = -Q\left[\frac{1}{2} + \frac{\epsilon}{\sqrt{2\pi e}} + O(\epsilon^2)\right]}$$

(g) INTEGRATE β-FUNCTION

$$Q(t^*) = (At^*)^{-\left[\frac{1}{2} + \epsilon/\sqrt{2\pi e} + O(\epsilon^2)\right]}$$

SET $t^* = t$:

$$\boxed{\begin{aligned} U_R(x,t) &= \frac{A}{t^{\frac{1}{2}+\alpha}} e^{-x^2/2t}(1+O(\epsilon^2)) \\ \alpha &= \epsilon/\sqrt{2\pi e} + O(\epsilon^2) \end{aligned}}$$

RG and singular perturbations

PDEs with no scale invariance

MOTIVATION

1. Multiple scales analysis requires choice of scales as $\epsilon \to 0$
   - often not obvious a priori
   - often justified post hoc, by 'artistry'.

2. Boundary layer + WKB methods
   - often require difficult matching of inner and outer expansions

3. Reductive perturbation theory
   - systematic way to derive amplitude and phase equations, preserving symmetries.

Goal: make all this 'mechanical'.

RESULTS

1. RG IS A METHOD TO EXTRACT STRUCTURALLY STABLE FEATURES OF MODELS OR EQUATIONS
   - EX: AMPLITUDE EQNS NEAR BIFURCATION POINTS
2. RENORMALISATION REMOVES DIVERGENCES IN PERTURBATION THEORY
   - EX: SECULAR TERMS
3. RG USES NAIVE PERTURBATION THEORY ONLY.
   - DO NOT NEED TO GUESS $\epsilon^{1/2}$, $\log \epsilon$ TERMS ETC.
4. RG GENERATES PROBLEM-SPECIFIC ASYMPTOTIC SEQUENCE
   - PRACTICALLY SUPERIOR TO CONVENTIONAL EXPANSION
   - TYPICALLY AS INTEGRAL REPRESENTATIONS, WHICH CAN BE EXPANDED TO REPRODUCE, IF DESIRED, CONVENTIONAL EXPANSIONS WITH ALL FRACTIONAL POWERS, LOGS ETC.
5. RG USES ONLY INNER EXPANSION
   - NO ASYMPTOTIC MATCHING NEEDED

BOUNDARY LAYER PROBLEMS

- SIMPLE LINEAR PROBLEM
- SIMPLE LINEAR PROBLEM DONE
  BY WILSON ITERATED MAP RG
  (DYNAMICAL SYSTEM FORMULATION)
- EXAMPLE WITH LOG ε GENERATED
  BY INNER EXPANSION
  (TRICKY TO DO BY MATCHED ASYMPTOTICS)
- NONLINEAR PROBLEM
- MULTIPLE BOUNDARY LAYERS

SIMPLE LINEAR BOUNDARY LAYER

$$\boxed{\epsilon y'' + y' + y = 0 \qquad \epsilon \ll 1}$$

DOMINANT BALANCE — BOUNDARY LAYER OF THICKNESS $\delta = O(\epsilon)$ AT $t = 0$

RESCALE TO INNER COORDINATE — $t = \epsilon \tau$

$$\ddot{y} + \dot{y} + \epsilon y = 0$$

NAIVE EXPANSION

$$y(\tau) = A_0 + B_0 e^{-\tau} + \epsilon\left[-A_0(\tau-\tau_0) + B_0(\tau-\tau_0)e^{-\tau}\right] + O(\epsilon)$$

TERMS REGULAR AS $\tau - \tau_0 \to \infty$ TERMS $O(\epsilon^2)$

RENORMALISED EXPANSION

$$y(\tau) = A(\mu) - \epsilon A(\mu)(\tau-\mu) + \left[B(\mu) + \epsilon B(\mu)(\tau-\mu)\right]e^{-\tau} + O(\epsilon)$$

RG EQN

$$\frac{dA}{d\mu} + \epsilon A + \left[\frac{dB}{d\mu} - \epsilon B\right]e^{-\tau} + O(\epsilon^2) = 0$$

$$\forall \tau \Rightarrow \quad \frac{dA}{d\mu} + \epsilon A = O(\epsilon^2)\ ; \quad \frac{dB}{d\mu} = \epsilon B + O(\epsilon^2)$$

$O(\epsilon^2)$ RG EQN

$$\boxed{\begin{aligned} \frac{dA}{d\mu} &= -(\epsilon A + \epsilon^2 A) + O(\epsilon^3) \\ \frac{dB}{d\mu} &= \epsilon B + \epsilon^2 B + O(\epsilon^3) \end{aligned}}$$

SIMPLE BL (2)

SOLVE RG EQNS
TO $O(\epsilon^2)$; $\mu = \tau$
$\tau = t/\epsilon$

$$\boxed{y(t) = C_1 e^{-(1+\epsilon)t} + C_2 e^{-t/\epsilon + (1+\epsilon)t} + O(\epsilon^2)}$$

COMMENT

- REPRODUCES STANDARD RESULT
- NO MATCHING REQUIRED

BL PROBLEM WITH LOGS

$$\boxed{\epsilon y'' + x y' - x y = 0 \qquad y(0) = 0 \qquad y(1) = e}$$

DOMINANT BALANCE   BL OF THICKNESS $\delta = O(\epsilon^{1/2})$ AT $x = 0$

RESCALE $x = \epsilon^{1/2} X \qquad Y(X) = y(x)$

RENORMALISED PERTURBATION EXPANSION

$$Y(X) = \left\{ A(\mu) + \epsilon^{1/2} A(X-\mu) + \frac{\epsilon}{2} A(X-\mu)^2 \right\}$$

$$+ \left\{ B(\mu) + \epsilon^{1/2} B(X-\mu) + \frac{\epsilon}{2} B(X-\mu)^2 - \epsilon\left(\frac{2}{\sqrt{\pi}} A + B\right) \log\left(\frac{X}{\mu}\right) \right\} \int_0^X ds\, e^{-s^2/2}$$

UNIFORMLY VALID FINAL RESULT

$$\boxed{y(x) = e^{x} x^{-\epsilon} \left\{ 1 - \sqrt{\frac{2}{\pi}} \int_{x/\sqrt{\epsilon}}^{\infty} ds\, e^{-s^2/2} \right\}}$$

COMMENT

- EXPANDING RG RESULT IN ϵ YIELDS $\epsilon \log \epsilon$ TERMS FROM $x^{-\epsilon}$. THESE TERMS NEEDED AD HOC TO SUCCESSFULLY DO ASYMPTOTIC MATCHING
- RG APPROACH MECHANICAL — NO INSIGHT NEEDED!

SWITCHBACK PROBLEMS

- EMERGENCE OF TERMS SUCH AS

  $\epsilon \log \epsilon$ ETC

- SOMETIMES NEED TO CALCULATE INFINITE NUMBER OF TERMS TO PERFORM EVEN FIRST ORDER MATCHING

DRAG ON A SPHERE AT LOW REYNOLDS NUMBER

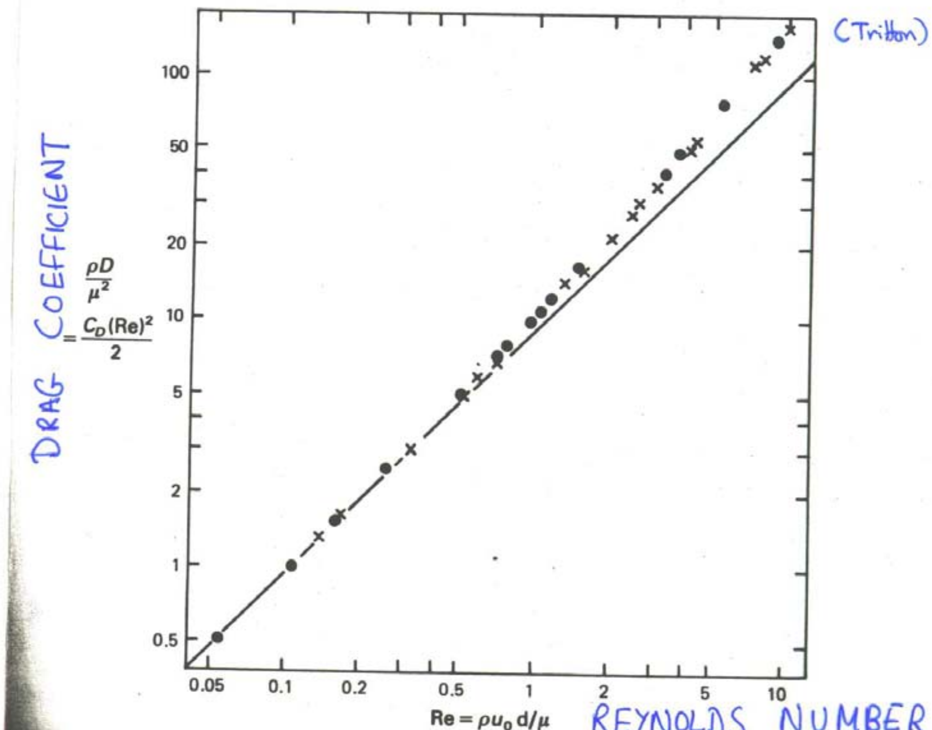


FIG. 9.3 Drag on a sphere at low Reynolds numbers. Experimental points from Refs. [248](×) and [336](●), both using the falling sphere method. The line represents eqn (9.17). (Liebster 1927, Schmiedel 1928)

$$C_D = \frac{6\pi}{R}\left[1 + \frac{3}{8}R + \frac{9}{40}R^2 \log R + \frac{9}{40}\left(\gamma + \frac{5}{3}\log_e 2 - \frac{323}{360}\right)R^2 + \frac{27}{80}R^3 \log R + O(R^3)\right]$$

(Chester & Breach 1969) o.k. for $0 \leq R \leq 0.5$

DRAG ON A CYLINDER AT LOW REYNOLDS NUMBER

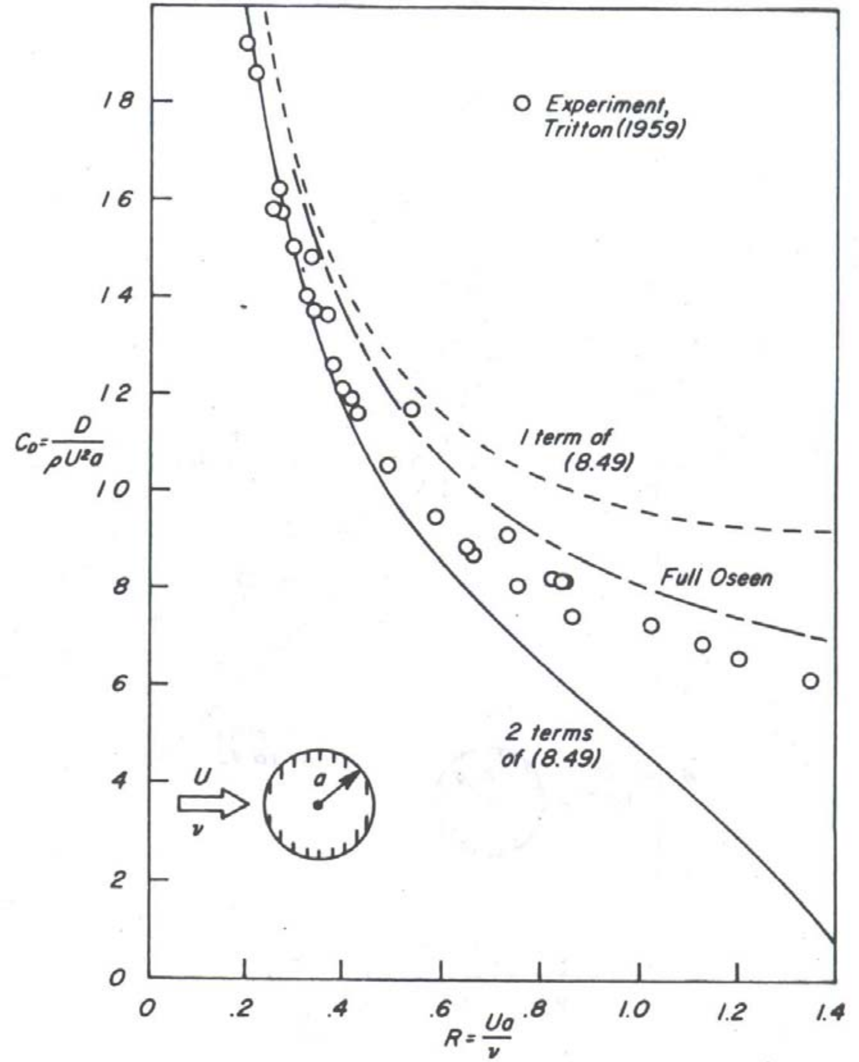


Fig. 8.5. Drag of circular cylinder at low Reynolds number.

$$C_D = \frac{4\pi}{R}\left[\Delta_1(R) - 0.87\,\Delta_1^3(R) + O(\Delta_1^4(R))\right]$$

(Kaplun, Lagerstrom; Proudman, Pearson 195?)

$$\Delta_1(R) \equiv \left(\log\frac{4}{R} - \gamma + 1/2\right)^{-1}$$

SWITCHBACK PROBLEM

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + \varepsilon u \frac{du}{dr} = 0$$

$$u(1) = 0 \qquad u(\infty) = 1$$

Conventional analysis involves terms such as $\varepsilon \log \varepsilon$

Boundary layer of thickness $\delta = O(\varepsilon)$ near $r = \infty$

Inner eqn $\quad x = \varepsilon r$

$$\frac{d^2u}{dx^2} + \frac{2}{x}\frac{du}{dx} + u\frac{du}{dx} = 0 \qquad u(x=\varepsilon) = 0$$

$$u(x=\infty) = 1$$

RG

$$u(r,\varepsilon) = 1 - \frac{e_2(\varepsilon r)}{e_2(\varepsilon)} + O\left(1/e_2(\varepsilon)^2\right)$$

$$e_2(t) \equiv \int_t^\infty dx\, x^{-2} e^{-x} \sim \frac{1}{t} + \ln t + (\gamma - 1) - \frac{t}{2} + O(t^2)$$

- Expansion sequence defined by requirement to satisfy b.c.'s order by order, not matching conditions as in usual approach.

- Not an asymptotic series in ε^n. But asymptotic series are unique only once asymptotic sequence is fixed. Ours seems superior to conventional one.

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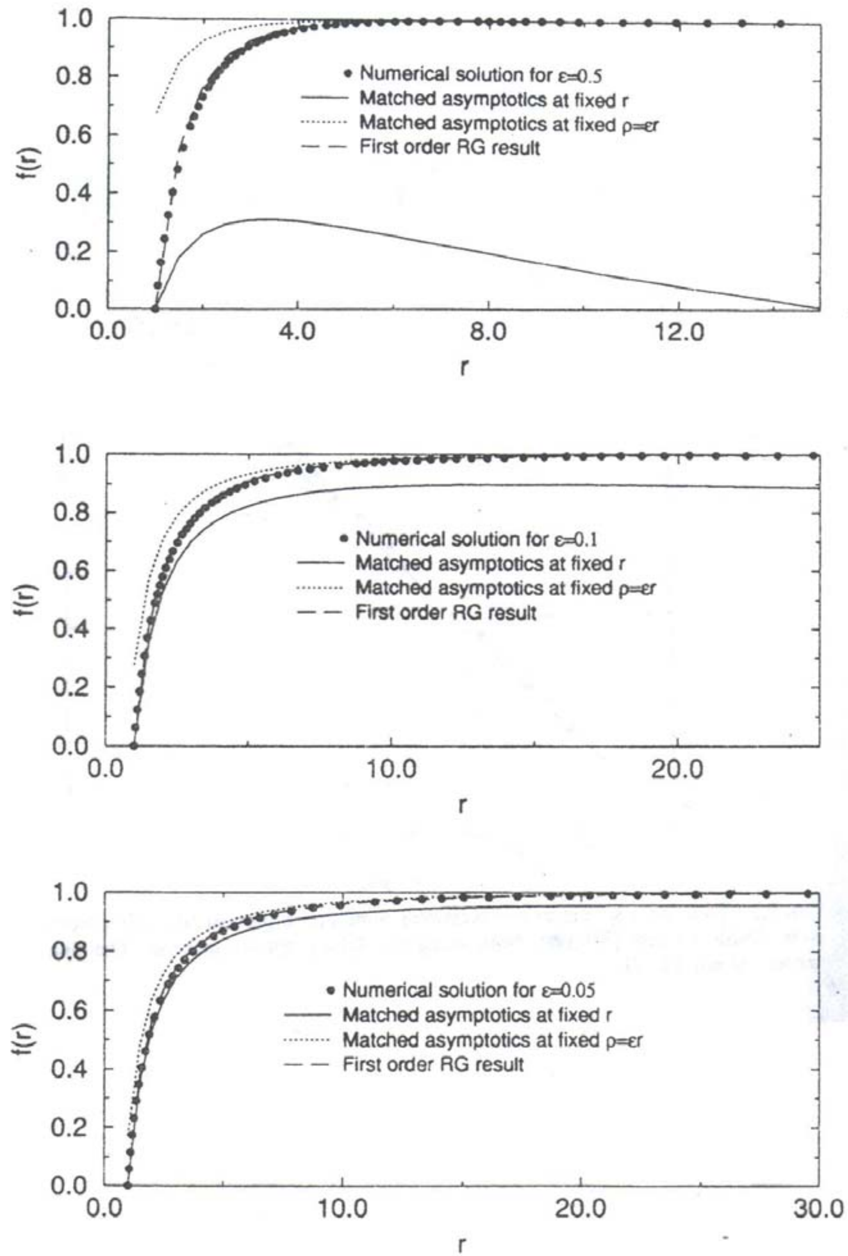

FIG. 2. Comparison between the numerical solution of Eq. (4.2)

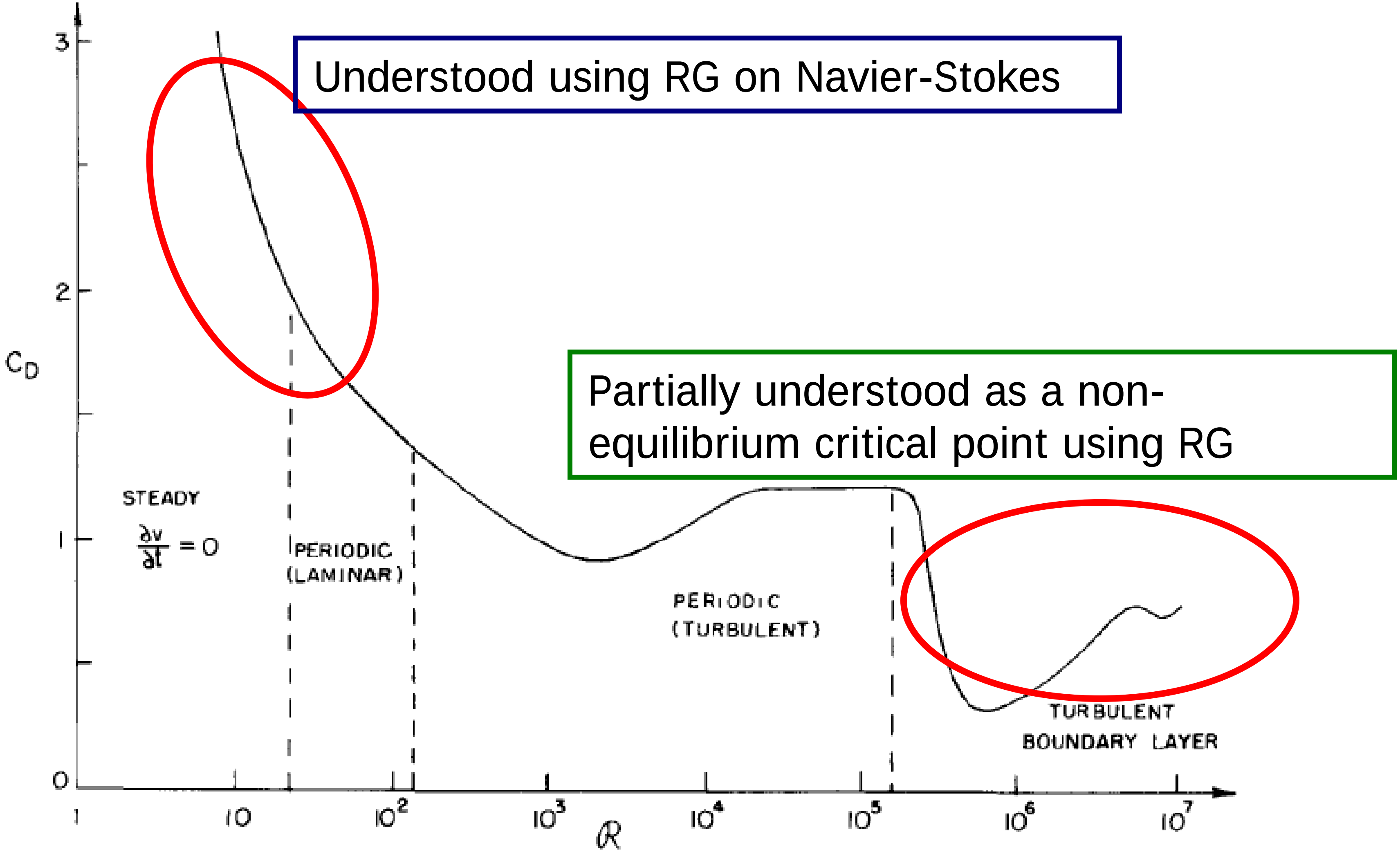

Fig. 41–4. The drag coefficient C_D of a circular cylinder as a function of the Reynolds number.

Feynman Lectures on Physics, vol 2, chapter 41

Why use RG to solve singular PDEs?

- Do not need to guess how characteristic lengths or times scale with the small parameter
  - method is reasonably mechanical: even a physicist can do it!

- Result is practically superior to standard matched asymptotics, boundary layer, multiple scales analysis methods
  - RG approximant reveals the source of the weird non-analyticities that plague traditional methods

- RG automatically preserves symmetries of underlying equations
  - Important in deriving amplitude equations near bifurcations in spatially-extended (i.e. pattern forming) dynamical systems in hydrodynamics and materials science

Efimov states in low Reynolds number fluid dynamics

K. Moffatt, Viscous and resistive eddies near a sharp corner, J. Fluid Mech. **18**, 1 (1964)

S. Taneda, Visualization of separating Stokes flows, J. Phys. Soc. Jpn. **46**, 1935 (1979)

Similarity solutions with complex exponents

- PDEs sometimes have similarity solutions with complex exponents
  - corresponds to discrete scale invariance

- Examples include:
  - scalar field collapse in general relativity (Choptuik 1993)
  - Low Reynolds number fluid dynamics: Stokes flow in wedge geometry

The interesting feature of the solution that is implied by the complex exponent is the sequence of eddies that must be induced near the origin. To see this it is simply necessary to write the asymptotic stream function in the form

$$\begin{aligned}\psi &\sim r^{\lambda_1}(A\cos\lambda_1\theta + C\cos(\lambda_1-2)\theta)\\ &= A'\left(\frac{r}{r_0}\right)^{\lambda_1}[\cos\lambda_1\theta\cos(\lambda_1-2)\alpha - \cos(\lambda_1-2)\theta\cos\lambda_1\alpha],\end{aligned} \tag{3.9}$$

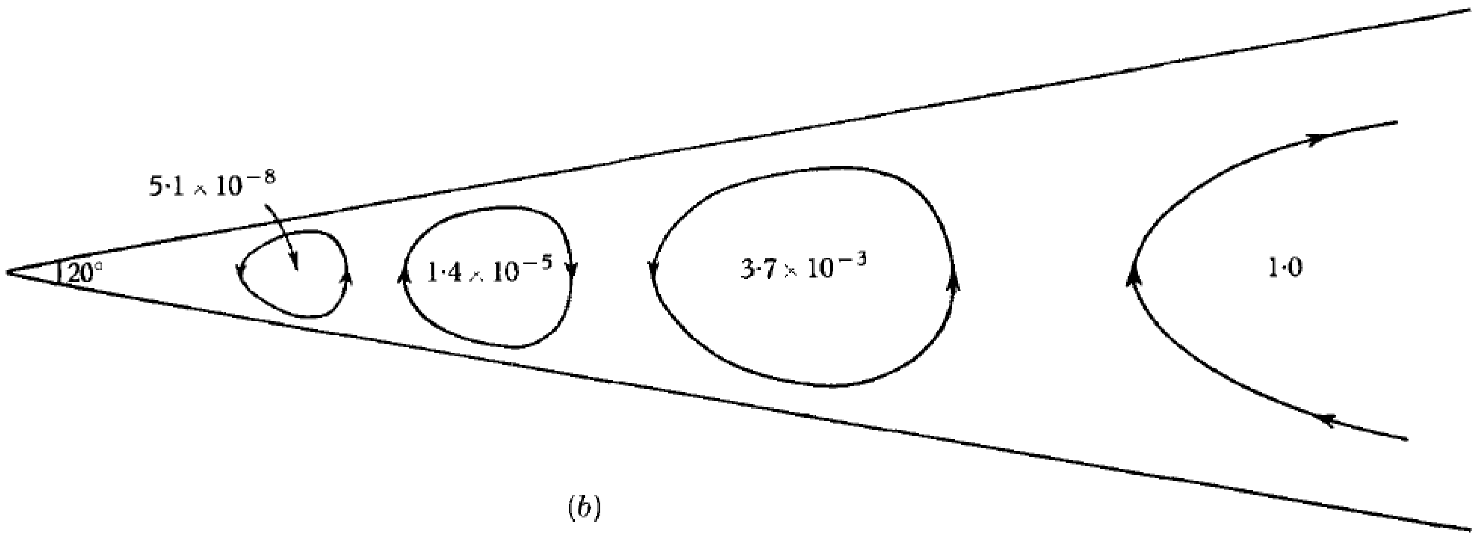

FIGURE 6. Sketch of streamlines in corner eddies (*a*) for $2\alpha = 60°$, (*b*) for $2\alpha = 20°$; the relative dimensions of these eddies are approximately correct, and the relative intensities are as indicated.

K. Moffatt, Viscous and resistive eddies near a sharp corner, J. Fluid Mech. **18**, 1 (1964)

S. Taneda, Visualization of separating Stokes flows, J. Phys. Soc. Jpn. **46**, 1935 (1979)

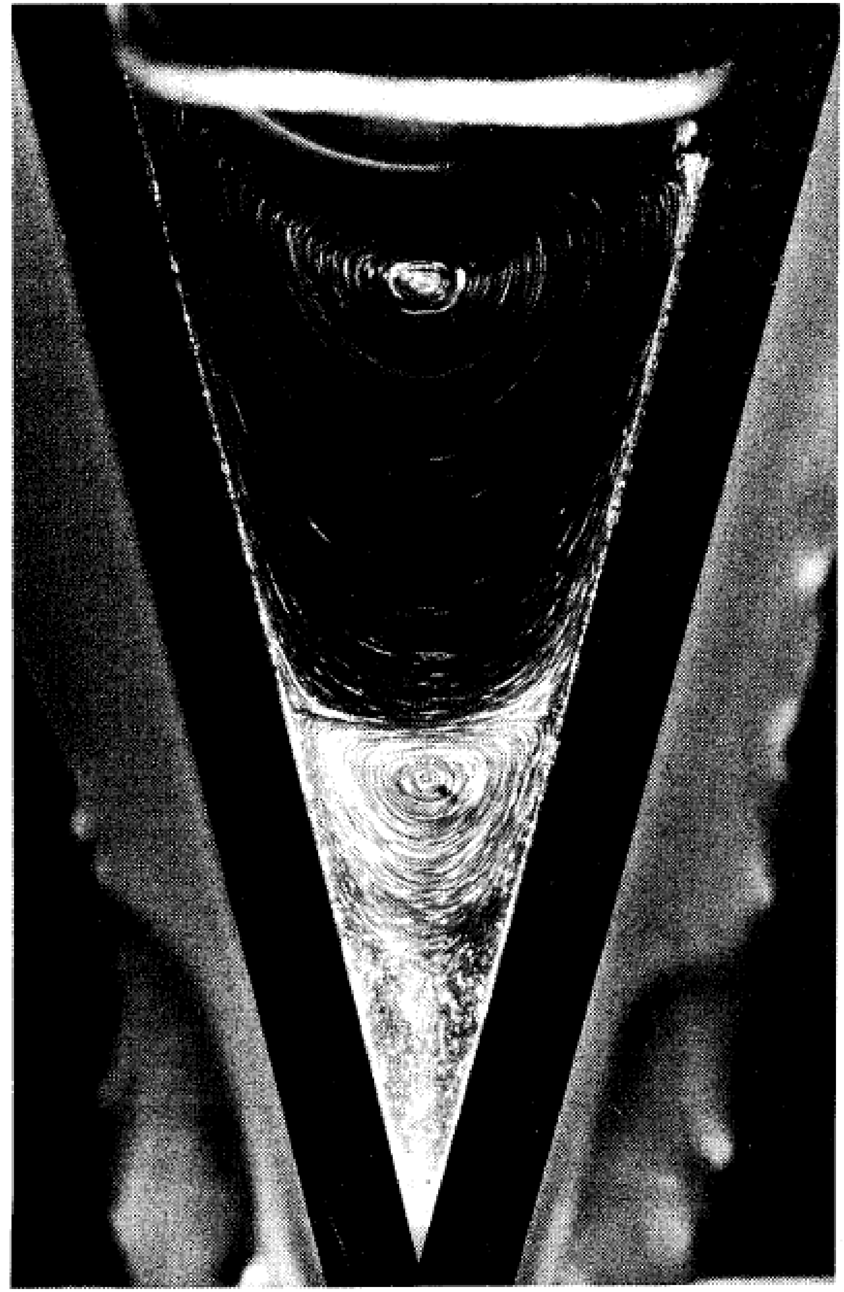

Fig. 19. Streamline pattern in a wedge-shaped region (Reynolds number 1.7×10^{-1}).

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All my RG papers can be obtained in reprint form from http://guava.physics.uiuc.edu/~nigel/articles/RG

Summary

Renormalization and the renormalization group (RG) were originally developed by physicists attempting to understand the divergent terms in perturbation theory and the short distance behaviour of quantum electrodynamics. During the last twenty years, these methods have been used to unify the construction of global approximations to ordinary and partial differential equations. Early examples included similarity solutions and travelling waves, which exhibit the same anomalous scaling properties found in quantum field theories, but here manifested in such problems as flow in porous media, the propagation of turbulence and the spread of advantageous genes. Fifteen years ago, these methods were extended to asymptotic problems with no special power-law scaling structure, enabling a vast generalization that includes and unifies all known singular perturbation theory methods, but with greater accuracy and calculational efficiency. Applications range from cosmology to viscous hydrodynamics.

In this work, RG is applied to differential equations, not field theories. The problems have no stochastic component nor necessarily scale-invariance.