Solutions of RG flow equations with full momentum dependence

New applications of the Renormalization Groups Method in Nuclear, Particle and Condensed Matter Physics
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Outline

- The eRG
- Approximation schemes
- Critical O(N) models
- Finite temperature

The "exact", "non perturbative", "functional" Renormalization Group

Basic strategy (ex scalar field theory)

$$S = \int \mathrm{d}^d x \left\{ \frac{1}{2} \left(\partial_\mu \varphi(x) \right)^2 + \frac{m^2}{2} \varphi^2(x) + \frac{u}{4!} \varphi^4(x) \right\}$$

Control infrared with "mass-like" regulator

$$\Delta S_\kappa[\varphi] = \frac{1}{2} \int_q R_\kappa(q^2)\, \varphi(q)\varphi(-q)$$

$$R_\kappa(q) = Z_\kappa \kappa^2 r(\tilde{q}), \qquad \tilde{q} = \frac{q}{\kappa} \qquad r(\tilde{q}) = \frac{\alpha \tilde{q}^2}{e^{\tilde{q}^2} - 1}$$

κ runs from microscopic scale Λ to zero (regulator vanishes)

An exact flow equation

Theory at "scale" κ defined by the regulated action

$$S \to S + \Delta S_\kappa$$

or the "effective action" $\Gamma_\kappa[\phi]$

Exact flow equation (Wetterich, 1993)

$$\partial_\kappa \Gamma_\kappa[\phi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \, \partial_\kappa R_\kappa(q) \, G_\kappa(q, -q; \phi) \quad =$$

$$G_\kappa^{-1}[\phi] = \Gamma_\kappa^{(2)}[\phi] + R_\kappa$$

Initial conditions

$$\Gamma_{\kappa=\Lambda}[\phi] \sim S[\phi]$$

An infinite hierarchy of equations for n-point functions

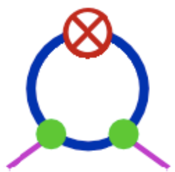
Effective potential

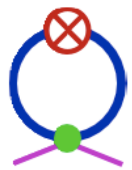
$$\kappa\partial_\kappa V_\kappa(\rho) = \frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\kappa\,\partial_\kappa R_\kappa(q)\,G_\kappa(q,\rho) \quad = $$

$$\rho \equiv \frac{\phi^2}{2}$$

2-point function

$$\partial_\kappa \Gamma^{(2)}_\kappa(p,\rho) \;=\; \int \frac{d^dq}{(2\pi)^d}\,\partial_\kappa R_\kappa(q)\,G^2_\kappa(q,\rho)$$
$$\times\;\left\{\Gamma^{(3)}_\kappa(p,q,-p-q;\phi)G_\kappa(q+p,\rho)\Gamma^{(3)}_\kappa(-p,p+q,-q;\phi) - \frac{1}{2}\Gamma^{(4)}_\kappa(p,-p,q,-q;\phi)\right\}$$





Note pattern of couplings

$$n \rightarrow n+1, n+2 \text{ AND } n=2$$

Approximation schemes

Main interest of the eRG is to offer the possibility to implement approximations at the level of the effective action itself.

The local potential approximation

Assume $\Gamma_\kappa[\phi] = \int d^d x \left\{ \frac{1}{2} (\partial\phi)^2 + V_\kappa(\phi) \right\}$ for all κ

Then

$$\Gamma_\kappa^{(2)}(q;\rho) = q^2 + m_\kappa^2(\rho) \qquad m_\kappa^2(\rho) \equiv \frac{\partial^2 V_\kappa}{\partial\phi^2}$$

and the equation for the effective potential becomes a closed equation

$$\kappa\partial_\kappa V_\kappa(\rho) = \frac{1}{2} I_1$$

$$I_n = J_n(0) \qquad J_n(p) \equiv \int \frac{d^d q}{(2\pi)^d}\, \partial_t R_\kappa(q)\, G_\kappa(p+q) G_\kappa^{n-1}(q)$$

one gets all correlation functions at once (but only at zero external momenta)

$$\Gamma^{(n)}(0, \cdots, 0) = \frac{\partial^n V_\kappa}{\partial\phi^n}$$

Beyond the local potential approximation

J.-P. B, R. Mendez-Galain, N. Wschebor (PLB, 2006)

Two observations

The vertex functions depend weakly on the loop momentum

$$\Gamma_\kappa^{(n)}(p_1, p_2, ..., p_{n-1} + q, p_n - q; \phi) \sim \Gamma_\kappa^{(n)}(p_1, p_2, ..., p_{n-1}, p_n; \phi)$$

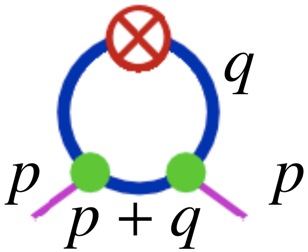
The hierarchy can be closed by exploiting the dependence on the field

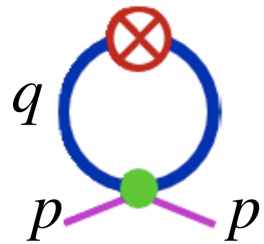
$$\Gamma_\kappa^{(n+1)}(p_1, p_2, ..., p_n, 0; \phi) = \frac{\partial \Gamma_\kappa^{(n)}(p_1, p_2, ...p_n; \phi)}{\partial \phi}$$

BMW Scheme

The 2-point function as an example (LO)

$$\partial_\kappa \Gamma_\kappa^{(2)}(p,\rho) = \int \frac{d^d q}{(2\pi)^d}\, \partial_\kappa R_\kappa(q)\, G_\kappa^2(q,\rho)$$
$$\times \left\{ \Gamma_\kappa^{(3)}(p,q,-p-q;\phi) G_\kappa(q+p,\rho) \Gamma_\kappa^{(3)}(-p,p+q,-q;\phi) - \frac{1}{2}\Gamma_\kappa^{(4)}(p,-p,q,-q;\phi) \right\}$$





$$\partial_\kappa \Gamma_\kappa^{(2)}(p,\rho) = \int \frac{d^d q}{(2\pi)^d}\, \partial_\kappa R_\kappa(q)\, G_\kappa^2(q,\rho)$$
$$\times \left\{ \Gamma_\kappa^{(3)}(p,0,-p;\phi) G_\kappa(p,\rho) \Gamma_\kappa^{(3)}(-p,p,0;\phi) - \frac{1}{2}\Gamma_\kappa^{(4)}(p,-p,0,0;\phi) \right\}$$

$$\Gamma_\kappa^{(3)}(p,-p,0;\phi) = \frac{\partial \Gamma_\kappa^{(2)}(p;\phi)}{\partial \phi} \qquad \Gamma_\kappa^{(4)}(p,-p,0,0;\phi) = \frac{\partial^2 \Gamma_\kappa^{(2)}(p;\phi)}{\partial \phi^2}$$

The equation for the 2-point function becomes a closed equation

$$\kappa\partial_\kappa\Gamma_\kappa^{(2)}(p,\rho) = J_3(p)\left(\frac{\partial\Gamma_\kappa^{(2)}(p,\rho)}{\partial\phi}\right)^2 - \frac{1}{2}I_2\,\frac{\partial^2\Gamma_\kappa^{(2)}(p,\rho)}{\partial\phi^2}$$

Structure of truncation

$$\Gamma^{(n)}(0,\cdots,0) = \frac{\partial^n V_\kappa}{\partial\phi^n} \qquad \text{(LPA)}$$

$$\Gamma^{(n)}(p,-p,0,\cdots,0) = \frac{\partial^{n-2}\Gamma_\kappa^{(2)}(p;\phi)}{\partial\phi^{n-2}} \qquad \text{(BMW-LO)}$$

etc.

The equations to be solved (LO)

$$V_\kappa(\rho) = \frac{1}{2} I_1$$

$$\kappa \partial_\kappa \Gamma_\kappa^{(2)}(p,\rho) = J_3(p) \left(\frac{\partial \Gamma_\kappa^{(2)}(p,\rho)}{\partial \phi} \right)^2 - \frac{1}{2} I_2 \, \frac{\partial^2 \Gamma_\kappa^{(2)}(p,\rho)}{\partial \phi^2}$$

$$I_n = J_n(0) \qquad J_n(p) \equiv \int \frac{d^d q}{(2\pi)^d} \, \partial_t R_\kappa(q) \, G_\kappa(p+q) G_\kappa^{n-1}(q)$$

In fact, one needs to separate p=0 sector

$$\Gamma_\kappa^{(2)}(p,\rho) = p^2 + \Delta_\kappa(p,\rho) + m_\kappa^2(\rho) \qquad m_\kappa^2(\rho) = \frac{\partial^2 V_\kappa}{\partial \phi^2} = \Gamma_\kappa^{(2)}(p=0,\rho)$$

and solve equation for $\Delta_\kappa(p,\rho)$

NB. $Z_\kappa = \left. \frac{\partial \Gamma_\kappa^{(2)}(p;\rho)}{\partial p^2} \right|_{p=0;\rho=\rho_0}$

Applications

- Bose-Einstein condensation
- Critical O(N) models
- Finite temperature

shift of BEC transition temperature due to weak repulsive interactions

$$\frac{\Delta T_c}{T_c^0} = \frac{T_c - T_c^0}{T_c^0} = c\, a\, n^{1/3}$$

The coefficient c is obtained from the change in the fluctuations in a classical field theory at criticality.

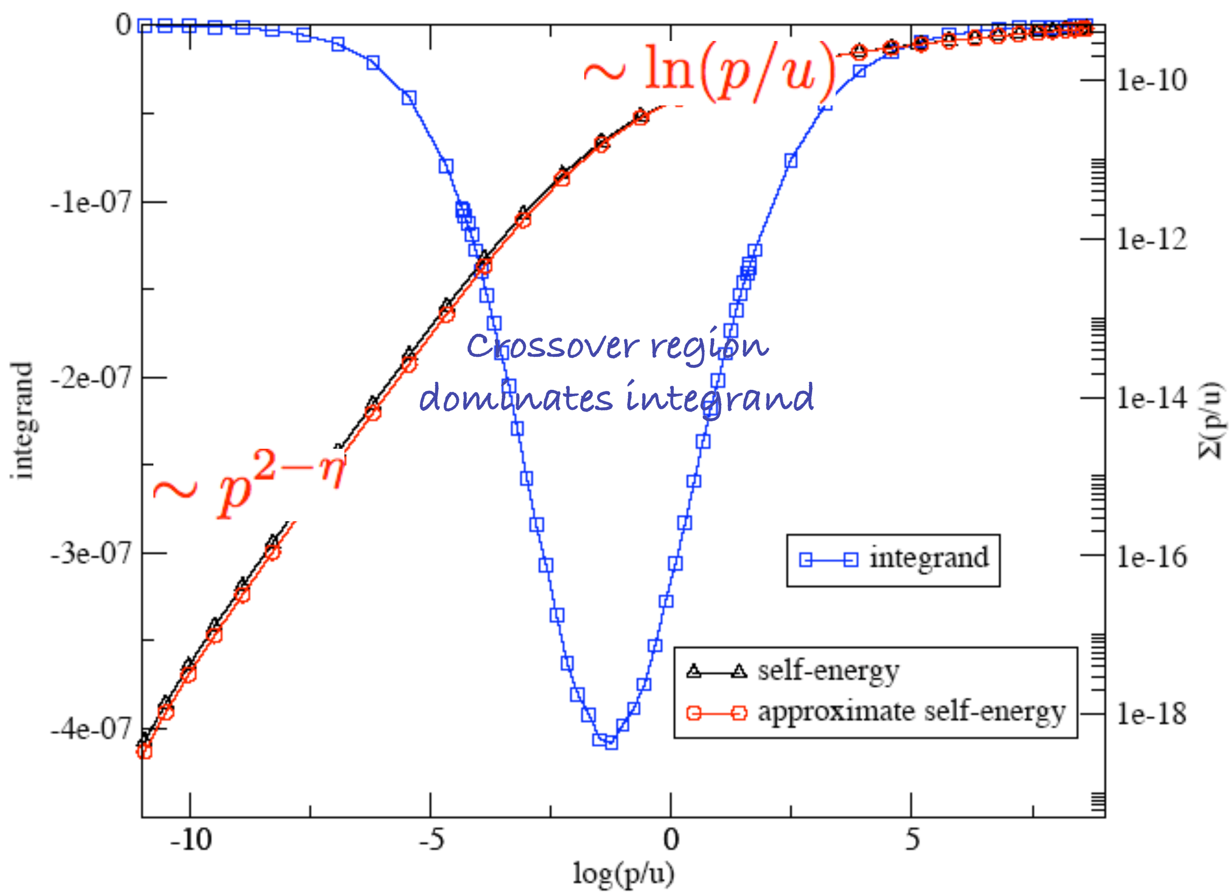
$$\int \mathrm{d}^d x \left\{ \frac{1}{2} \left[\nabla \varphi(x)\right]^2 + \frac{1}{2} r \varphi^2(x) + \frac{u}{4!} \left[\varphi^2(x)\right]^2 \right\}$$

$$c \propto \frac{\Delta \langle \varphi_i^2 \rangle}{N u}$$

G Baym, J-P Blaizot, M Holzmann, F Laloe, and D Vautherin, PRL (1999)

The calculation of c involves the 2-point function at all momenta

$$\frac{\Delta\langle\varphi_i^2\rangle}{N} = \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{p^2+\Sigma(p)} - \frac{1}{p^2}\right)$$



Application to critical O(N) model

(from F. Benitez, J.-P. Blaizot, H. Chate, B. Delamotte, R. Mendez-Galain, N. Wschebor, arXiv:0901.0128 [Phys. Rev. E80 (2009)])

TABLE I. Coefficient c and critical exponents of the $O(N)$ models for $d=3$.

	BMW				Resummed perturbative expansions					Monte-Carlo and high-temperature series				
N	η	ν	ω	c	η	ν	ω	c	Ref.[a]	η	ν	ω	c	Ref.[a]
0	0.034	0.589	0.83		0.0284(25)	0.5882(11)	0.812(16)		[13]	0.030(3)	0.5872(5)	0.88		[14] [15]
1	0.039	0.632	0.78	1.15	0.0335(25)	0.6304(13)	0.799(11)	1.07(10)	[13] [16]	0.0368(2)	0.6302(1)	0.821(5)	1.09(9)	[17] [18]
2	0.041	0.674	0.75	1.37	0.0354(25)	0.6703(15)	0.789(11)	1.27(10)	[13] [16]	0.0381(2)	0.6717(1)	0.785(20)	1.32(2)	[19] [20]
3	0.040	0.715	0.73	1.50	0.0355(25)	0.7073(35)	0.782(13)	1.43(11)	[13] [16]	0.0375(5)	0.7112(5)	0.773		[21,22]
4	0.038	0.754	0.72	1.63	0.035(4)	0.741(6)	0.774(20)	1.54(11)	[13] [16]	0.0365(10)	0.749(2)	0.765	1.6(1)	[22] [18]
10	0.022	0.889	0.80		0.024	0.859			[23]					

results for d=2, N=1

$$\eta = 0.254, \;\; \nu = 1$$

(exact results $\eta = 1/4, \;\; \nu = 1$)

Anomalous dimension

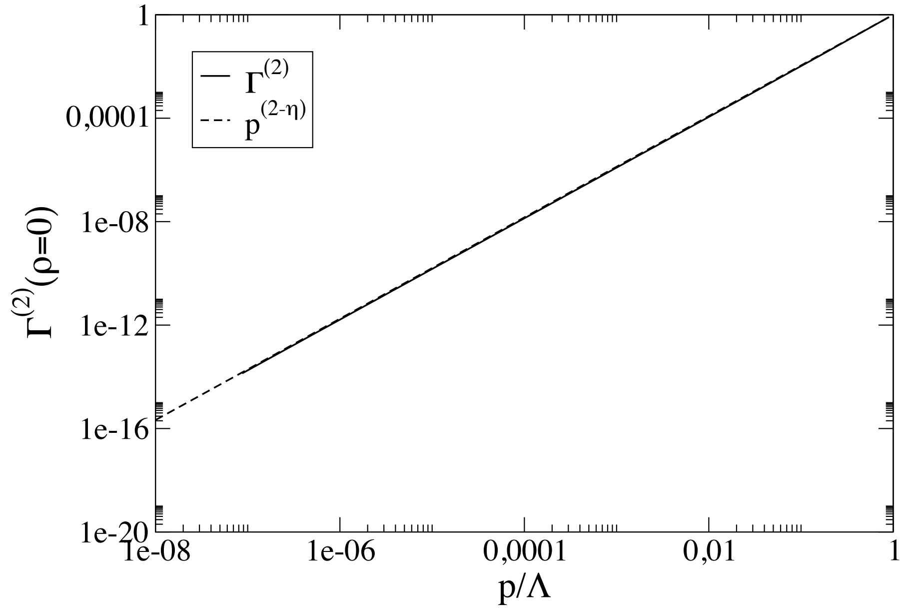
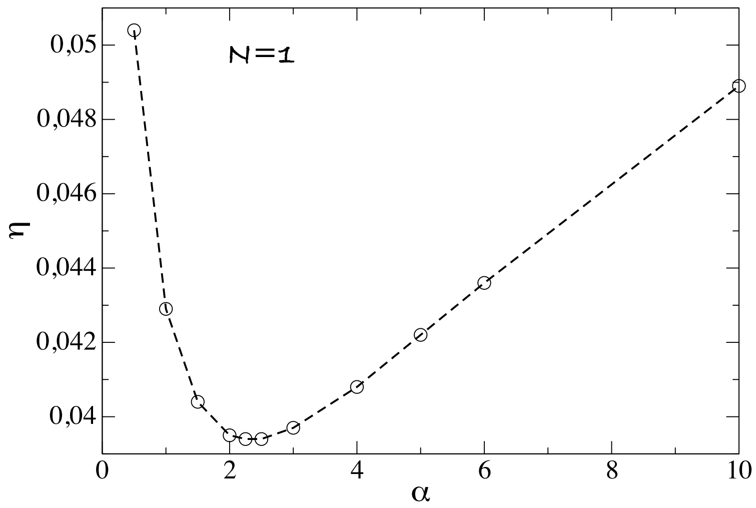


Illustration of cut-off dependence and PMS



Scaling functions

$$G^{(2)}_{\pm}(q) = \chi g_{\pm}(x) \qquad x = q\xi \qquad \chi^{-1} = \Gamma^{(2)}(q=0)$$

small x

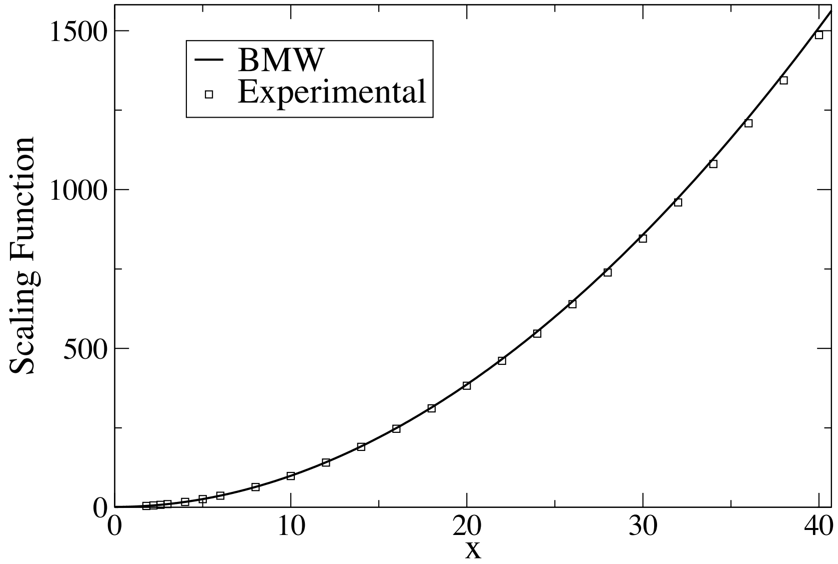
$$g^{-1}_{\pm}(x) = g^{-1}_{OZ}(x) + \sum_{n=2} c^{\pm}_{n} x^{2n} \qquad g_{OZ}(x) = \frac{1}{1+x^2}$$

(Ornstein-Zernicke)

large x (Fisher-Langer)

$$g_{\pm}(x) = \frac{C_1}{x^{2-\eta}}\left(1 + \frac{C^{\pm}_{2}}{x^{(1-\alpha)/\nu}} + \frac{C^{\pm}_{3}}{x^{1/\nu}}\right)$$

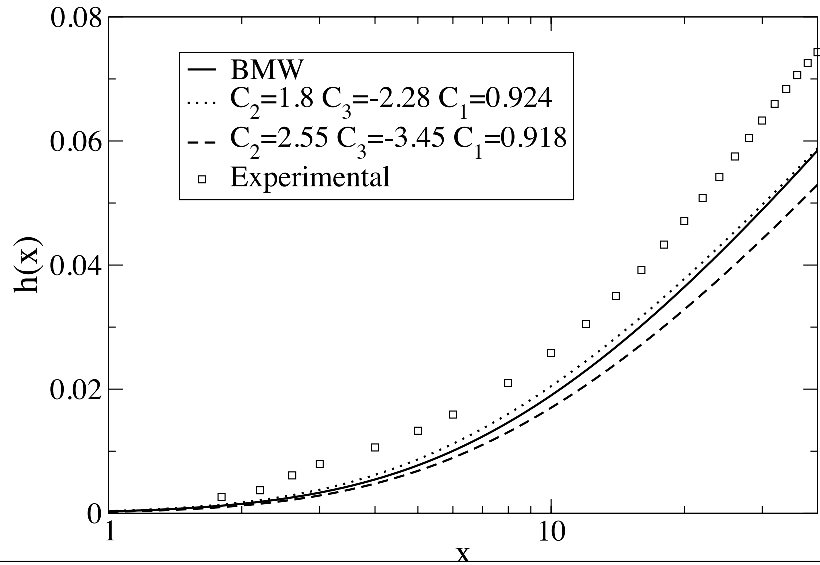
interpolation formula (Bray)



Expt: P. Damay et al, PRB58 (1998) [n scatt. on CO2 near critical pt]

A closer look :

$$h(x) = \log\left[\frac{g(x)}{g_{OZ}(x)}\right]$$

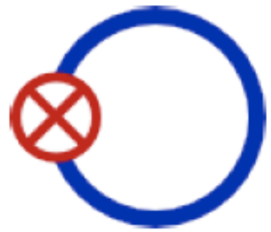


Non perturbative renormalization group at finite temperature

(J.-P B, A. Ipp, N. Wschebor, in preparation)

effective action

$$\partial_t \Gamma_\kappa[\phi] = \frac{1}{2} T \sum_n \int \frac{d^3\mathbf{q}}{(2\pi)^3} \partial_t R_\kappa(Q) G_\kappa(Q;\rho) \quad =$$



propagator $G_\kappa^{-1}(Q;\rho) \equiv \Gamma_\kappa^{(2)}(\mathbf{Q},-\mathbf{Q};\rho) + R_\kappa(Q)$

regulator

four-momentum vectors:

$\mathbf{Q} = (\omega_n, \mathbf{q})$

$Q = |\mathbf{Q}|$

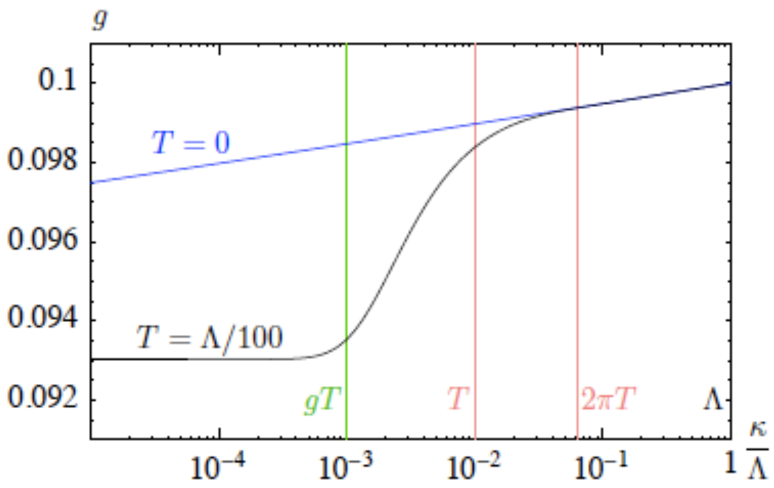
Matsubara frequency $\omega_n = 2\pi n T$

momentum derivative: $\partial_t \equiv \kappa\, \partial_\kappa$

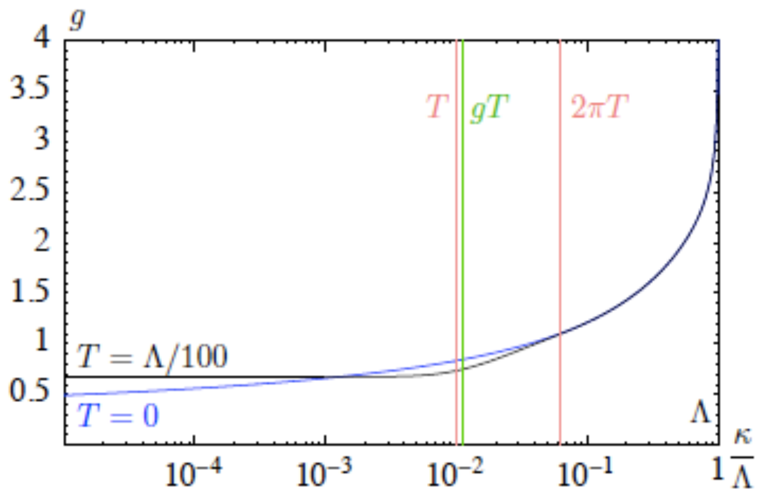
scalar field: $\rho \equiv \frac{1}{2}\phi^2$

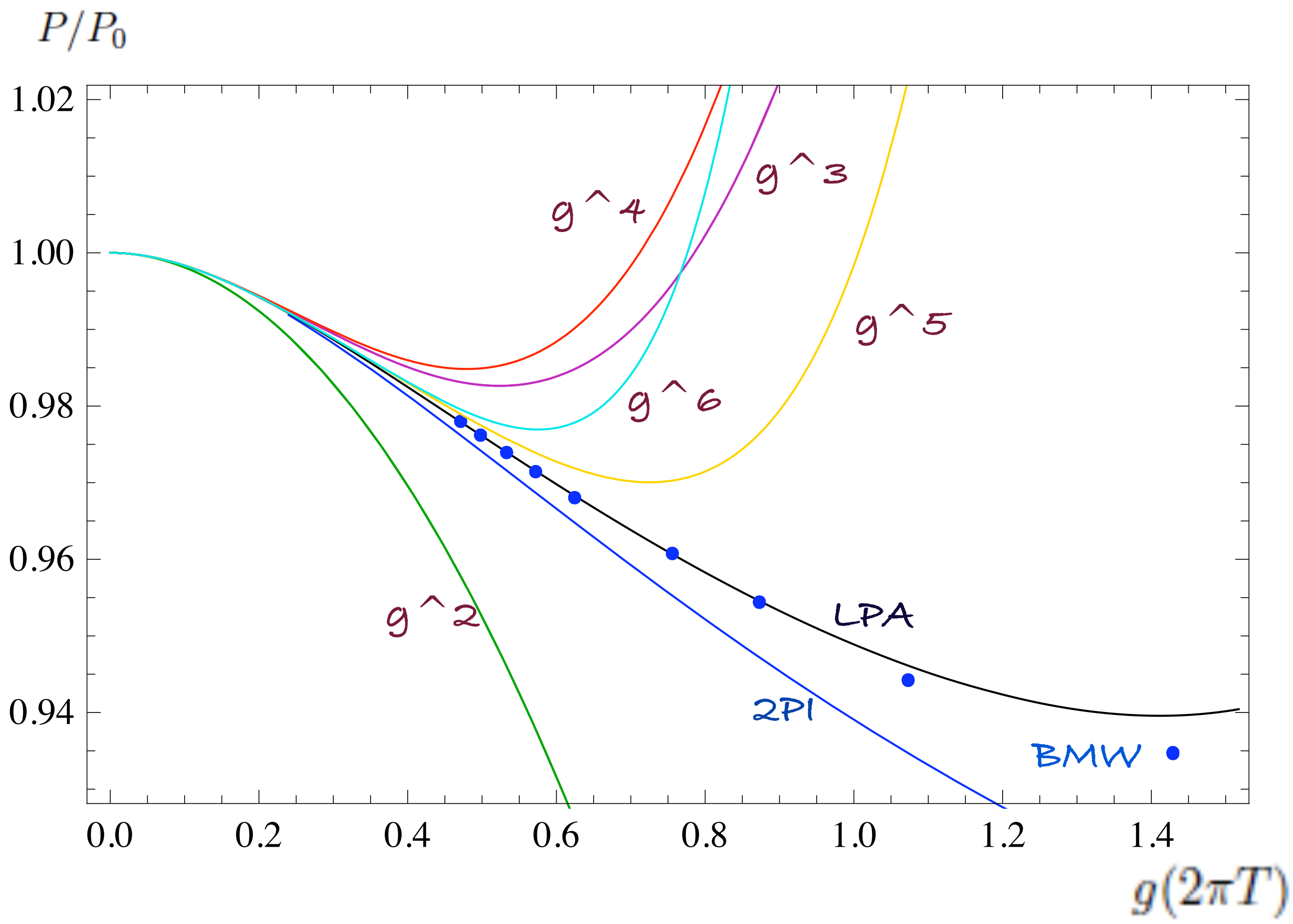
Flow of coupling constant (LPA)

weak coupling



strong coupling





(g ^ 6 from A. Gynther, et al ,hep-ph/0703307)

Conclusions

eRG is a nice tool

why does it work so well ?