Unitary Fermi Gas: Quarky Methods

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Outline

- Fermion Lagrangian
- Monte Carlo calculation of Tc
- Superfluid EFT
- Random matrix theory

Fermion L

Dilute Fermi gas, 2 spins

Hamiltonian

$$\mathcal{H} = -\frac{1}{2m}\sum_{\sigma}\bar{\psi}_\sigma \nabla^2 \psi_\sigma - \frac{g^2}{2}\left(\bar{\psi}_1\bar{\psi}_2\psi_2\psi_1\right)$$

Lagrangian, real time

$$\mathcal{L}_t = \bar{\psi}\, i\partial_t \psi + \frac{1}{2m}\bar{\psi}\nabla^2\psi + \frac{g^2}{2}\left(\bar{\psi}\psi\right)^2$$

Galilean invariance $t \rightarrow t, \quad \vec{x} \rightarrow \vec{x} + \vec{v}t \quad \Rightarrow$

$$\psi(t,\vec{x}) \rightarrow \psi'(t,\vec{x}) = e^{im\vec{v}\cdot\vec{x}}\psi(t,\vec{x}-\vec{v}t)$$

SU(2) spin symmetry $\psi \rightarrow e^{i\vec{\beta}\cdot\vec{\sigma}}\psi, \quad \bar{\psi} \rightarrow \bar{\psi}e^{-i\vec{\beta}\cdot\vec{\sigma}}$

U(1) fermion number $\psi \rightarrow e^{i\alpha}\psi, \quad \bar{\psi} \rightarrow \bar{\psi}e^{-i\alpha}$

Transformations

Hubbard-Stratonovich, as in Chen & Kaplan, PRL (2003).

$$\mathcal{L}_t \;=\; \bar{\psi}\, i\partial_t \psi + \frac{1}{2m}\bar{\psi}\nabla^2\psi + g\bar{\psi}\phi\psi - \frac{1}{2}\phi^2$$

Imaginary time for finite temperature formulation

$$\mathcal{Z} \;=\; \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}\phi\, \exp\left(-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\, \mathcal{L}_\tau\right)$$

with

$$\begin{aligned}\mathcal{L}_\tau \;&=\; \bar{\psi}\,\partial_\tau\psi - \frac{1}{2m}\bar{\psi}\nabla^2\psi - g\bar{\psi}\phi\psi + \frac{1}{2}\phi^2 \\ &\equiv\; \bar{\psi}K\psi + \frac{1}{2}\phi^2\end{aligned}$$

Gor'kov basis

$$\mathcal{L}_\tau = \frac{1}{2}(\psi^T, -\bar{\psi}(i\sigma_2)) \begin{pmatrix} 0 & K^\dagger(i\sigma_2) \\ K(i\sigma_2) & 0 \end{pmatrix} \begin{pmatrix} \psi \\ i\sigma_2\bar{\psi}^T \end{pmatrix} + \frac{1}{2}\phi^2$$

$$\equiv \frac{1}{2}\Psi^T \mathcal{K} \Psi + \frac{1}{2}\phi^2$$

SU(2) spin symmetry $\quad \Psi \to V\Psi = \Psi + i\vec{\beta}\cdot \underbrace{\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}}_{\gamma_k\gamma_5} \Psi + \ldots$

U(1) fermion number $\quad \Psi \to U\Psi = \Psi + i\alpha \underbrace{\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}}_{\gamma_0} \Psi + \ldots$

Chen & Kaplan, PRL (2003).

Pairing source

$$+\frac{1}{2}\left(J\psi^T\sigma_2\psi \;+\; J^*\bar{\psi}\,\sigma_2\bar{\psi}^T\right)$$

$$\mathcal{J} \;=\; \begin{pmatrix} J\sigma_2 & 0 \\ 0 & J^*\sigma_2 \end{pmatrix}$$

$$\mathcal{A} \;=\; \mathcal{K} \;+\; \mathcal{J}$$

$$\{\gamma_0, \mathcal{K}\} \;=\; 0\,, \qquad \{\gamma_0, \mathcal{J}\} \;\neq\; 0\,, \qquad \{\gamma_k\gamma_5, \mathcal{A}\} \;=\; 0$$

conserves U(1) — breaks U(1) — conserves SU(2)

Reduced Gor'kov basis

$$\mathcal{L} = (\psi_1, \bar{\psi}_2) \begin{pmatrix} -iJ & \tilde{K}^\dagger \\ \tilde{K} & -iJ^* \end{pmatrix} \begin{pmatrix} \psi_2 \\ -\bar{\psi}_1 \end{pmatrix} + \frac{1}{2}\phi^2$$

$$\equiv \bar{\eta}\,\widetilde{\mathcal{A}}\,\eta + \frac{1}{2}\phi^2$$

$$\widetilde{\mathcal{A}} = \widetilde{\mathcal{K}} + \widetilde{\mathcal{J}}$$

$$\{\sigma_3, \widetilde{\mathcal{K}}\} = 0\,, \qquad \{\sigma_3, \widetilde{\mathcal{J}}\} \neq 0$$

conserves U(1) — breaks U(1)

$$\mathcal{Z} = \int \mathcal{D}\phi \,\det\widetilde{\mathcal{A}}\, e^{-S_\phi}$$

Pseudofermions

$$\mathcal{Z} = \int \mathcal{D}\phi \, \mathcal{D}\zeta \, \exp\left[-\left(\int \zeta^\dagger \widetilde{\mathcal{A}}^{-1} \zeta + \frac{1}{2}\phi^2\right)\right]$$

- HMC: Molecular dynamics + accept/reject step
- Requires inversion of sparse matrix
- But matrix becomes singular in physical limit

Banks-Casher for superfluidity

Order parameter (negl. signs, limits...)

$$\Sigma \equiv \left.\frac{\partial \log Z}{\partial J}\right|_{J=0} = \frac{1}{2}\langle \psi^T \sigma_2 \psi + \bar{\psi}\sigma_2 \bar{\psi}^T \rangle$$

$$= \frac{1}{2}\langle \Psi^T \sigma_{31} \Psi \rangle$$

$$= \langle \bar{\eta}\eta \rangle$$

$$\widetilde{\mathcal{K}}\eta_n = \lambda_n \eta_n$$ e.v. real, paired if nonzero $$\det \widetilde{\mathcal{A}} = \prod_n (\lambda_n - iJ)$$

$$\Sigma = \lim_{J\to 0} \lim_{V\to\infty} \left\langle \frac{1}{V} \sum_n \frac{1}{\lambda_n - iJ} \right\rangle$$

Banks-Casher for superfluidity

$$\Sigma = \lim_{J\to 0}\lim_{V\to\infty}\left\langle \frac{1}{V}\sum_n \frac{1}{\lambda_n - iJ}\right\rangle$$

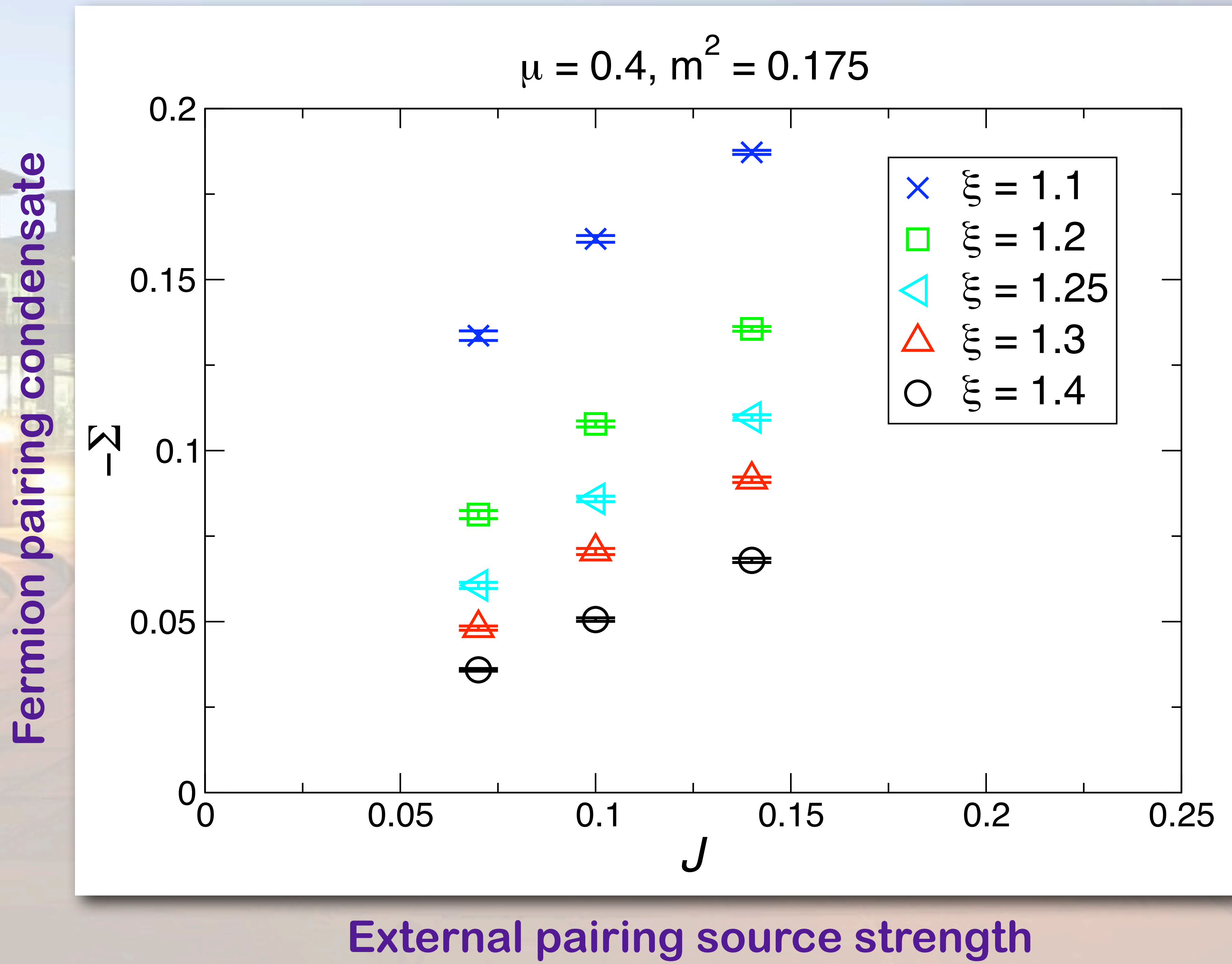
$$= \lim_{J\to 0}\int_0^\infty \frac{2i\,J\,\rho(\lambda)}{\lambda^2 + J^2}\,d\lambda$$

$$= 2i\pi\rho(0)$$

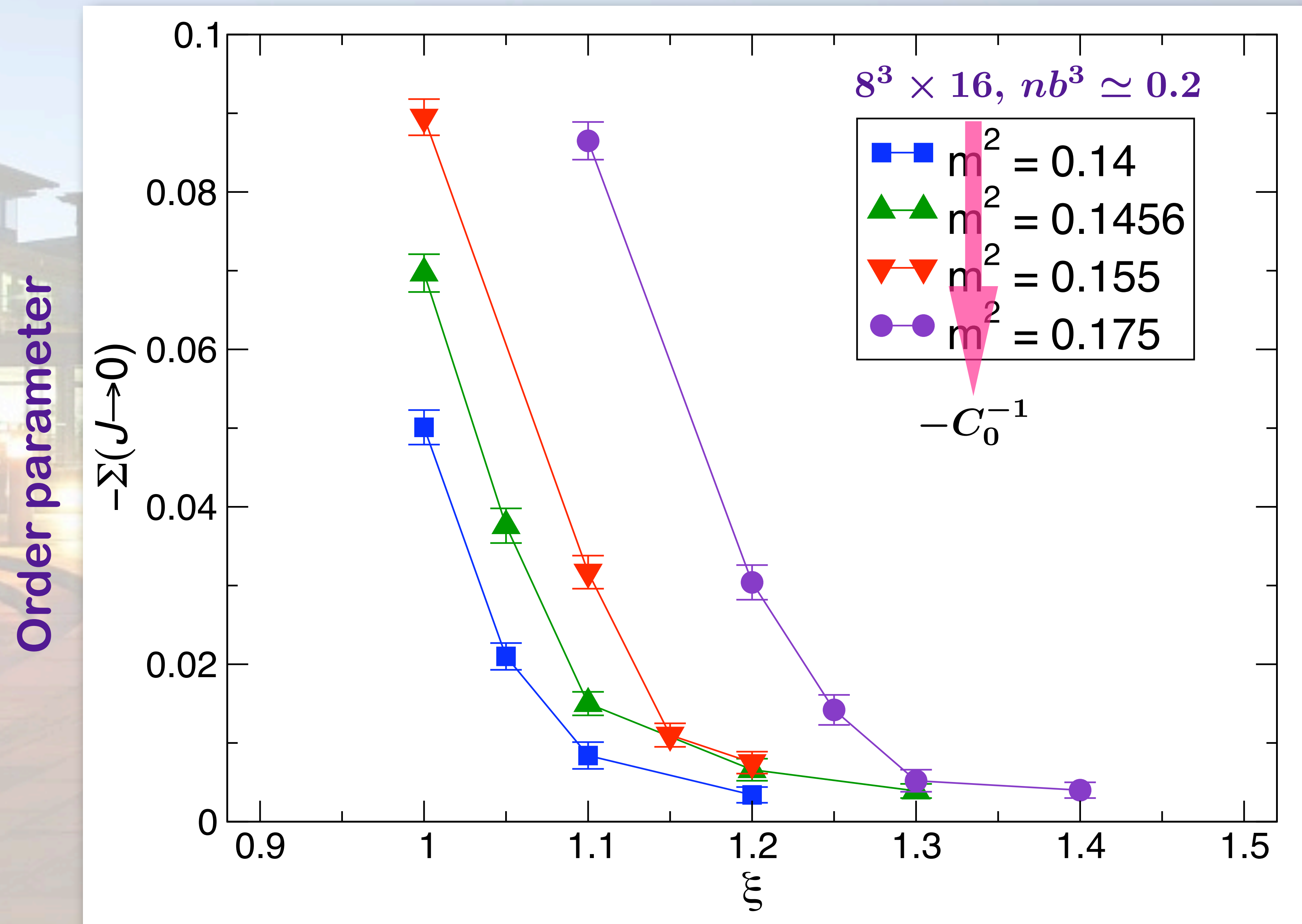
- Non-vanishing condensate implies accumulation of zero-modes in the thermodynamic limit
- Interesting, but makes inversions for HMC increasingly difficult

$$T_c$$

Extrapolation to zero external source



Phase transition

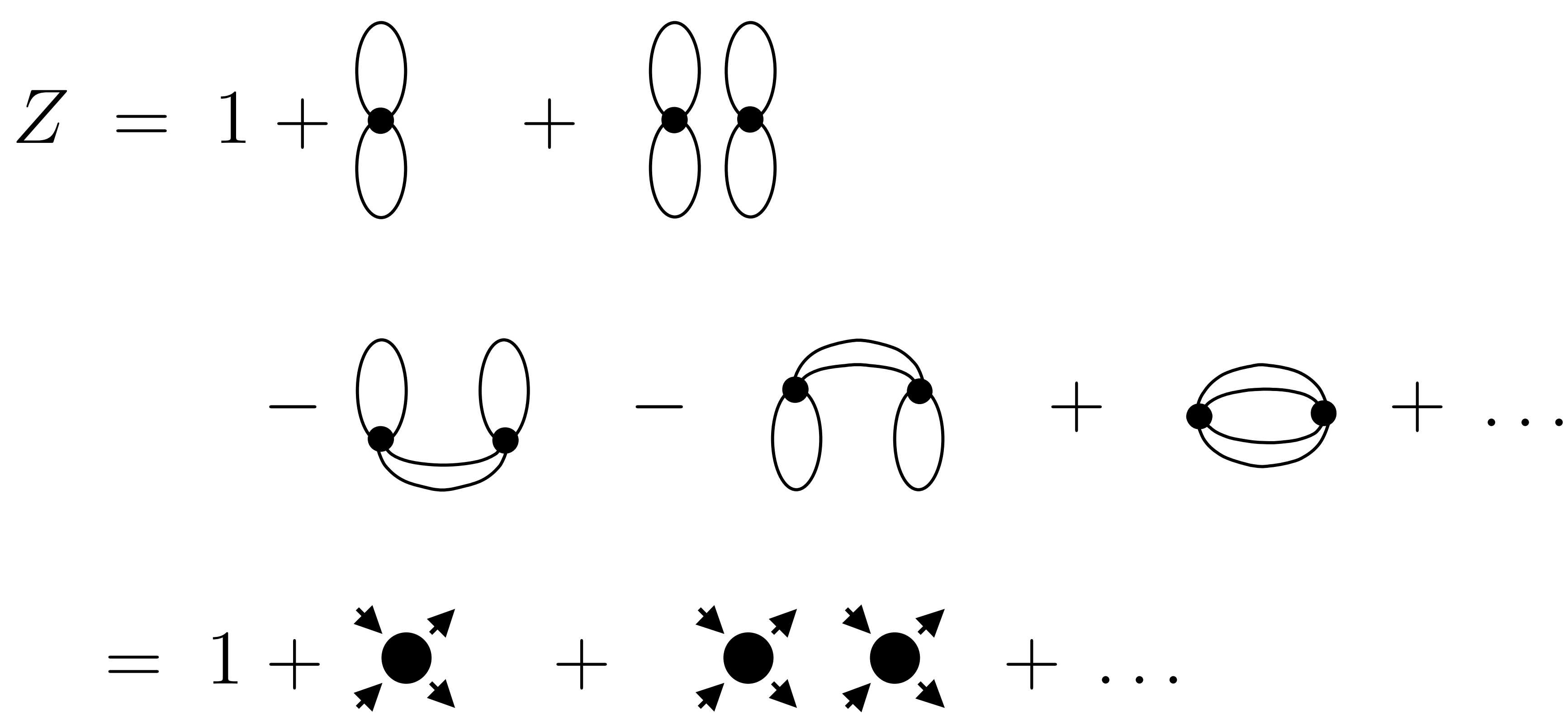


Anisotropy (≈Temperature)

MW, cond-mat/0502372

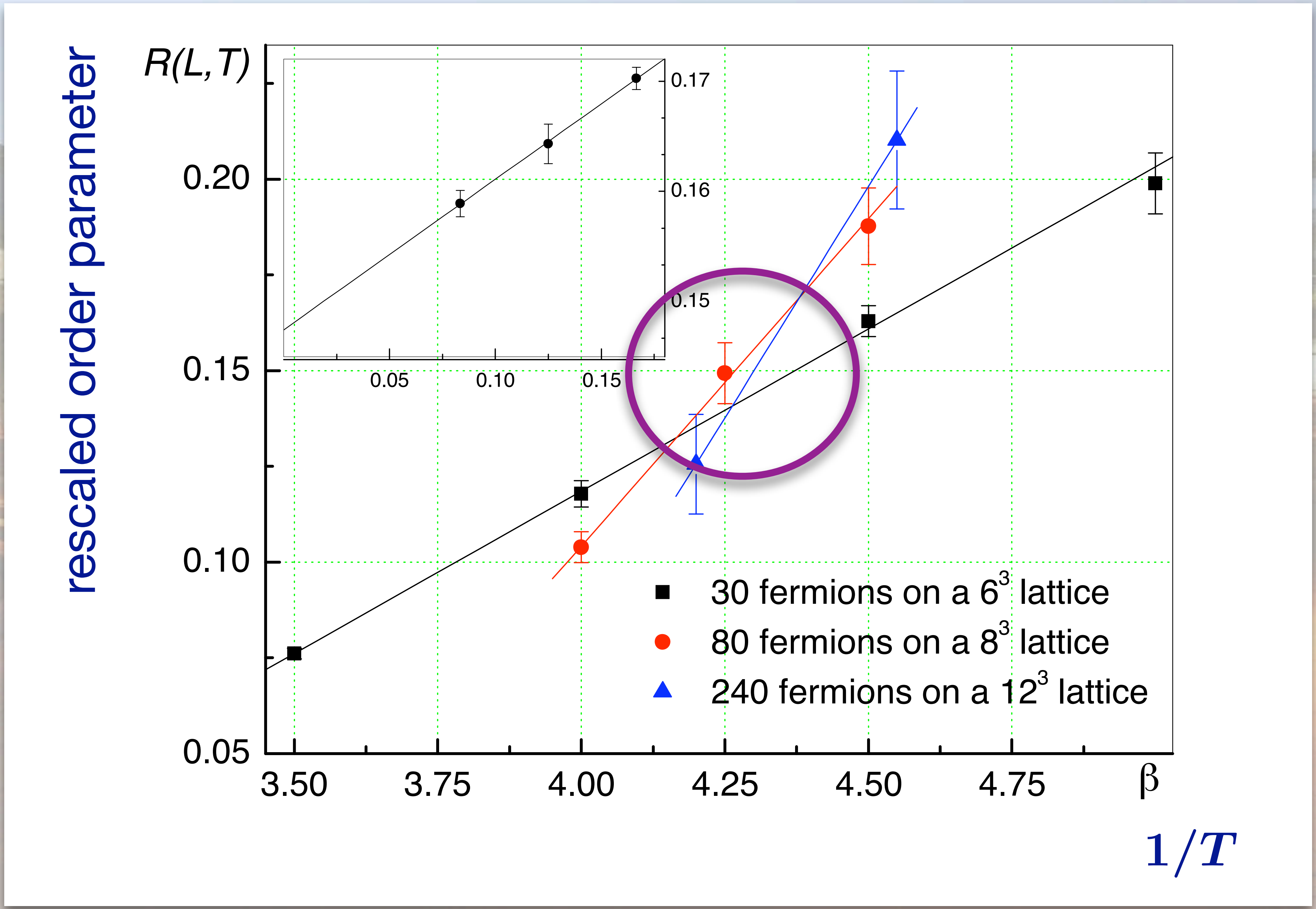
Diagrammatic Determinant Monte Carlo

Burovski, Prokof'ev, Svistunov, Troyer, New J. Phys 8, 153 (2003)



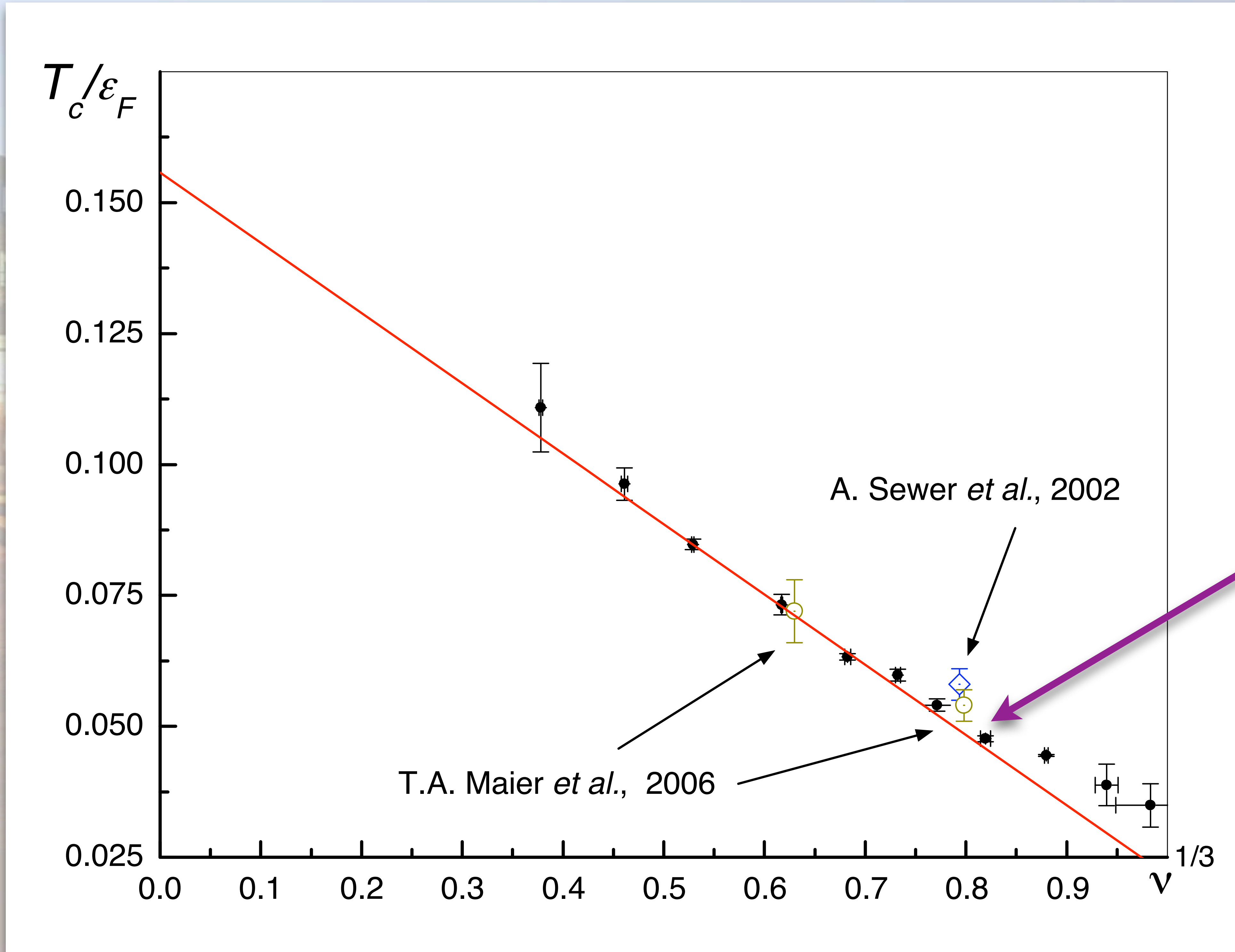
Critical Temperature

Burovski, Prokof'ev, Svistunov, Troyer, New J. Phys 8, 153 (2003)



Continuum limit

Burovski, Prokof'ev, Svistunov, Troyer, New J. Phys 8, 153 (2003)



HMC stuck here

Our investigations with DDMC

with Olga Goulko

- Implemented own version of code
- Found long autocorrelations in cases
- Update of result for T_c in balanced case
- Investigation of imbalanced gas, T_c vs. $\Delta n/n$, in continuum limit
- Final results later this spring

Lattice 2009 proceedings, arXiv:0910.3909

Trade-offs

- Can work directly with J=0
- Fine for finite volume determination of T_c
- Scales like V^2

Superfluid EFT

Phonon

$$\langle \psi^T \sigma_2 \psi \rangle \;=\; |\langle \psi^T \sigma_2 \psi \rangle|\, e^{2i\theta}$$

$$\chi(\tau, \vec{x}) \;\equiv\; e^{i\theta(\tau,\vec{x})}$$

$$X \;\equiv\; \chi\left(\partial_\tau - \frac{\overleftrightarrow{\nabla}^2}{2m}\right)\chi^\dagger \;=\; \left(-i\partial_\tau\,\theta \;+\; \frac{1}{2m}|\nabla\theta|^2\right)$$

Initial power counting: $(\partial\theta)^n \sim O(1), \;\; \partial^m(\partial\theta)^n \sim O(p^m)$

$$\mathcal{L}_{\text{eff}} \;=\; c_0\, m^{3/2} X^{5/2} \qquad c_0 = \frac{2^{5/2}}{15\pi^2\xi^{3/2}} \qquad \epsilon = \xi\,\frac{3}{5}\,\epsilon_F n.$$

NLO derived in D T Son & M W, Annals Phys (2006)

Expand about ground state

$$\varphi = \theta - i\mu\tau$$

$$X \; = \; -\,\mu \; + \; i\partial_\tau\varphi \; + \; \frac{|\nabla\varphi|^2}{2m}$$

$$\varphi(\tau,\vec{x}) \; \equiv \; \frac{\pi(\tau,\vec{x})}{F\sqrt{m}} \, .$$

$$\mathcal{L}_{\rm eff} = -c_0 m^{3/2}\Big[\mu^{5/2} \;\; - \;\; \frac{5\mu^{3/2}}{2F\sqrt{m}}\left(i\partial_\tau\,\pi + \frac{1}{2m^{3/2}F}|\nabla\pi|^2\right) \\ + \;\; \frac{15\mu^{1/2}}{4mF^2}\left(i\partial_\tau\,\pi + \frac{1}{2m^{3/2}F}|\nabla\pi|^2\right)^2 \; + \; \ldots\Big]$$

Include pairing source

work done with J-W Chen

$$\mathfrak{J} \to e^{-2i\alpha}\mathfrak{J}$$

$$Y = \mathfrak{J}(\chi^\dagger)^2 + \mathfrak{J}^*\chi^2$$

$$Y = 2J\cos 2\varphi = 2J(1 - 2\varphi^2 + \ldots)$$

$$\mathcal{L}_{\rm eff} = c_0 m^{3/2} X^{5/2} + d_0 m^{3/2} X^{3/2} Y$$

$$\mathcal{L}_{\rm eff} = -m^{3/2}\Big[\left(c_0 + \frac{2d_0 J}{\mu}\right)\mu^{5/2} - \frac{\mu^{3/2}}{F\sqrt{m}}\left(\frac{5c_0}{2} + \frac{3d_0 J}{\mu}\right)\left(i\,\partial_\tau\pi + \frac{1}{2m^{3/2}F}|\nabla\pi|^2\right) + \frac{\mu^{1/2}}{4mF^2}\left(\frac{15c_0}{2} + \frac{3d_0 J}{\mu}\right)\left(i\,\partial_\tau\pi + \frac{1}{2m^{3/2}F}|\nabla\pi|^2\right)^2 - \frac{4d_0 J\mu^{3/2}}{mF^2}\pi^2\Big]$$

Phonon mass

$$B = 2d_0 m^{1/2}\mu^{3/2} \propto \Sigma$$

LO phonon mass

$$M_0^2 = \frac{4B}{F^2 c_s^4} J$$

Correlation length

$$\xi = \frac{1}{\sqrt{2mMc_s^2}}$$

Must use power counting:

$$(\partial^n\theta) \sim O(p^n), \quad M^k \sim O(p^k)$$

Pressure

Constant part of L

$$P_0(\mu) = (c_0\mu^{5/2} + 2d_0 J\mu^{3/2})m^{3/2}$$

$$n = P_0'(\mu) = (\frac{5}{2}c_0\mu^{3/2} + 3d_0 J\mu^{1/2})m^{3/2}$$

Canonical norm'n for kinetic term

$$\mathcal{L}_{\text{eff}} = -P_0(\mu) + \frac{in}{F\sqrt{m}}\partial_\tau\pi + \frac{1}{2}(\partial_\tau\pi)^2 + \frac{c_s^2}{2}|\nabla\pi|^2 + \frac{1}{2}M_0^2\pi^2 + \mathcal{L}_{\text{int}}$$

Speed of sound

$$c_s^2 = \frac{1}{m}\frac{n(\mu)}{n'(\mu)} = \frac{2}{3}\frac{\mu}{m}\left(1 + \frac{4}{5}\frac{d_0}{c_0}\frac{J}{\mu} + \mathcal{O}\left(\frac{J^2}{\mu^2}\right)\right)$$

Finite volume effects

- *p* regime: small effects, e^{-ML}

$$\frac{1}{F^2} \ll \xi \ll L$$

- ε regime: phonon zero mode dominates, must be resummed.

$$\frac{1}{F^2} \ll L \ll \xi$$

- δ regime: symmetry restored

$$L \ll \frac{1}{F^2} \ll \xi$$

Random Matrix Theory

Random matrix theory

- Low energy EFT is universal given symmetries and symmetry breaking pattern
- In ε regime uniform phonon zero mode dominates over fluctuating nonzero modes
- Replace Gor'kov operator with random matrix with appropriate global symmetries (and no spacetime dependence)
- Spectral quantities on scale of average level spacing are thought to be universal
- Compare to Monte Carlo
- Guidance for algorithmic improvements

Random Matrix Model

Random matrix model

- Going beyond universal results...
- Use RMT as toy model (with same low energy EFT as Fermi gas)
- Explore larger population imbalances analytically
- Interesting phase diagram?

Summary

- Numerical calculation of critical temperature
- Need to improve Monte Carlo calculations (HMC)
- EFT study of SSB in finite volume
- Importance of low-lying eigenmodes
- Random matrix theory for unitary Fermi gas


Photo: Stefan Meinel