



*Unitary Fermi Gas:  
Quarky Methods*

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# Outline

- ❖ Fermion Lagrangian
- ❖ Monte Carlo calculation of  $T_c$
- ❖ Superfluid EFT
- ❖ Random matrix theory





*Fermion L*

# Dilute Fermi gas, 2 spins

Hamiltonian

$$\mathcal{H} = -\frac{1}{2m} \sum_{\sigma} \bar{\psi}_{\sigma} \nabla^2 \psi_{\sigma} - \frac{g^2}{2} (\bar{\psi}_1 \bar{\psi}_2 \psi_2 \psi_1)$$

Lagrangian, real time

$$\mathcal{L}_t = \bar{\psi} i \partial_t \psi + \frac{1}{2m} \bar{\psi} \nabla^2 \psi + \frac{g^2}{2} (\bar{\psi} \psi)^2$$

Galilean invariance  $t \rightarrow t, \quad \vec{x} \rightarrow \vec{x} + \vec{v}t \Rightarrow$

$$\psi(t, \vec{x}) \rightarrow \psi'(t, \vec{x}) = e^{im\vec{v} \cdot \vec{x}} \psi(t, \vec{x} - \vec{v}t)$$

SU(2) spin symmetry  $\psi \rightarrow e^{i\vec{\beta} \cdot \vec{\sigma}} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{-i\vec{\beta} \cdot \vec{\sigma}}$

U(1) fermion number  $\psi \rightarrow e^{i\alpha} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{-i\alpha}$

# Transformations

Hubbard-Stratonovich, as in Chen & Kaplan, PRL (2003).

$$\mathcal{L}_t = \bar{\psi} i \partial_t \psi + \frac{1}{2m} \bar{\psi} \nabla^2 \psi + g \bar{\psi} \phi \psi - \frac{1}{2} \phi^2$$

Imaginary time for finite temperature formulation

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\phi \exp \left( - \int_0^\beta d\tau \int d^3x \mathcal{L}_\tau \right)$$

with

$$\begin{aligned} \mathcal{L}_\tau &= \bar{\psi} \partial_\tau \psi - \frac{1}{2m} \bar{\psi} \nabla^2 \psi - g \bar{\psi} \phi \psi + \frac{1}{2} \phi^2 \\ &\equiv \bar{\psi} \mathbf{K} \psi + \frac{1}{2} \phi^2 \end{aligned}$$

# Gor'kov basis

$$\mathcal{L}_\tau = \frac{1}{2}(\psi^T, -\bar{\psi}(i\sigma_2)) \begin{pmatrix} 0 & K^\dagger(i\sigma_2) \\ K(i\sigma_2) & 0 \end{pmatrix} \begin{pmatrix} \psi \\ i\sigma_2\bar{\psi}^T \end{pmatrix} + \frac{1}{2}\phi^2$$

$$\equiv \frac{1}{2}\Psi^T \mathcal{K} \Psi + \frac{1}{2}\phi^2$$

SU(2) spin symmetry  $\Psi \rightarrow V\Psi = \Psi + i\vec{\beta} \cdot \overbrace{\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}}^{\gamma_k \gamma_5} \Psi + \dots$

U(1) fermion number  $\Psi \rightarrow U\Psi = \Psi + i\alpha \underbrace{\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}}_{\gamma_0} \Psi + \dots$

Chen & Kaplan, PRL (2003).

# Pairing source

$$+\frac{1}{2} (J\psi^T \sigma_2 \psi + J^* \bar{\psi} \sigma_2 \bar{\psi}^T)$$

$$\mathcal{J} = \begin{pmatrix} J\sigma_2 & 0 \\ 0 & J^*\sigma_2 \end{pmatrix}$$

$$\mathcal{A} = \mathcal{K} + \mathcal{J}$$

$$\{\gamma_0, \mathcal{K}\} = 0, \quad \{\gamma_0, \mathcal{J}\} \neq 0, \quad \{\gamma_k \gamma_5, \mathcal{A}\} = 0$$

conserves U(1)

breaks U(1)

conserves SU(2)

# Reduced Gor'kov basis

$$\begin{aligned}\mathcal{L} &= (\psi_1, \bar{\psi}_2) \begin{pmatrix} -iJ & \tilde{K}^\dagger \\ \tilde{K} & -iJ^* \end{pmatrix} \begin{pmatrix} \psi_2 \\ -\bar{\psi}_1 \end{pmatrix} + \frac{1}{2}\phi^2 \\ &\equiv \bar{\eta} \tilde{\mathcal{A}} \eta + \frac{1}{2}\phi^2\end{aligned}$$

$$\tilde{\mathcal{A}} = \tilde{\mathcal{K}} + \tilde{\mathcal{J}}$$

$$\{\sigma_3, \tilde{\mathcal{K}}\} = 0, \quad \{\sigma_3, \tilde{\mathcal{J}}\} \neq 0$$

conserves U(1)

breaks U(1)

$$\mathcal{Z} = \int \mathcal{D}\phi \det \tilde{\mathcal{A}} e^{-S_\phi}$$



# Pseudofermions

$$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\zeta \exp \left[ - \left( \int \zeta^\dagger \tilde{\mathcal{A}}^{-1} \zeta + \frac{1}{2} \phi^2 \right) \right]$$

- ❖ HMC: Molecular dynamics + accept/reject step
- ❖ Requires inversion of sparse matrix
- ❖ But matrix becomes singular in physical limit

# Banks-Casher for superfluidity

Order parameter (negl. signs, limits...)

$$\begin{aligned} \Sigma &\equiv \left. \frac{\partial \log Z}{\partial J} \right|_{J=0} = \frac{1}{2} \langle \psi^T \sigma_2 \psi + \bar{\psi} \sigma_2 \bar{\psi}^T \rangle \\ &= \frac{1}{2} \langle \Psi^T \sigma_{31} \Psi \rangle \\ &= \langle \bar{\eta} \eta \rangle \end{aligned}$$

$$\tilde{\mathcal{K}} \eta_n = \lambda_n \eta_n \quad \text{e.v. real, paired if nonzero} \quad \det \tilde{\mathcal{A}} = \prod_n (\lambda_n - iJ)$$

$$\Sigma = \lim_{J \rightarrow 0} \lim_{V \rightarrow \infty} \left\langle \frac{1}{V} \sum_n \frac{1}{\lambda_n - iJ} \right\rangle$$

# Banks-Casher for superfluidity

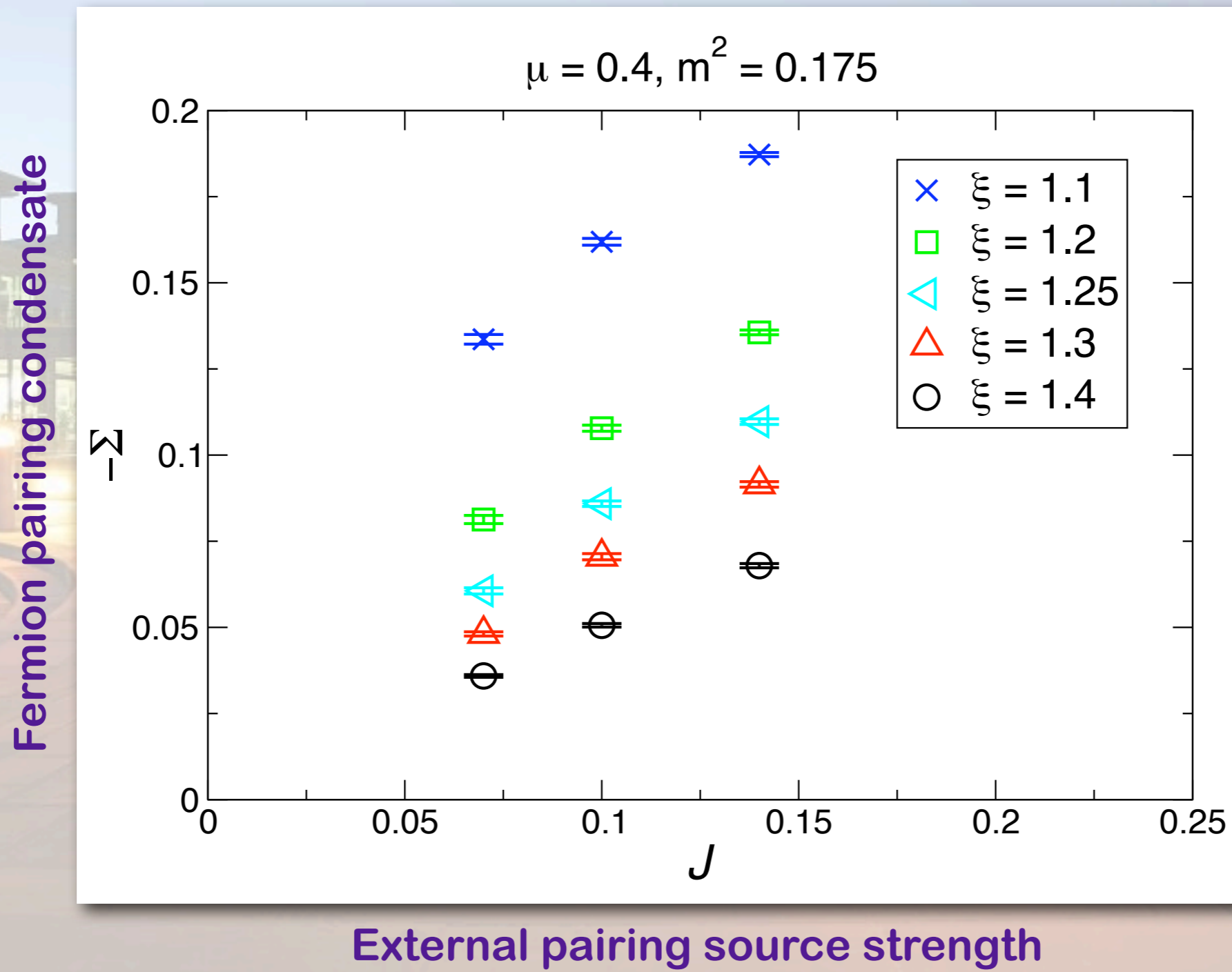
$$\begin{aligned}\Sigma &= \lim_{J \rightarrow 0} \lim_{V \rightarrow \infty} \left\langle \frac{1}{V} \sum_n \frac{1}{\lambda_n - iJ} \right\rangle \\ &= \lim_{J \rightarrow 0} \int_0^\infty \frac{2iJ \rho(\lambda)}{\lambda^2 + J^2} d\lambda \\ &= 2i\pi \rho(0)\end{aligned}$$

- ❖ Non-vanishing condensate implies accumulation of zero-modes in the thermodynamic limit
- ❖ Interesting, but makes inversions for HMC increasingly difficult



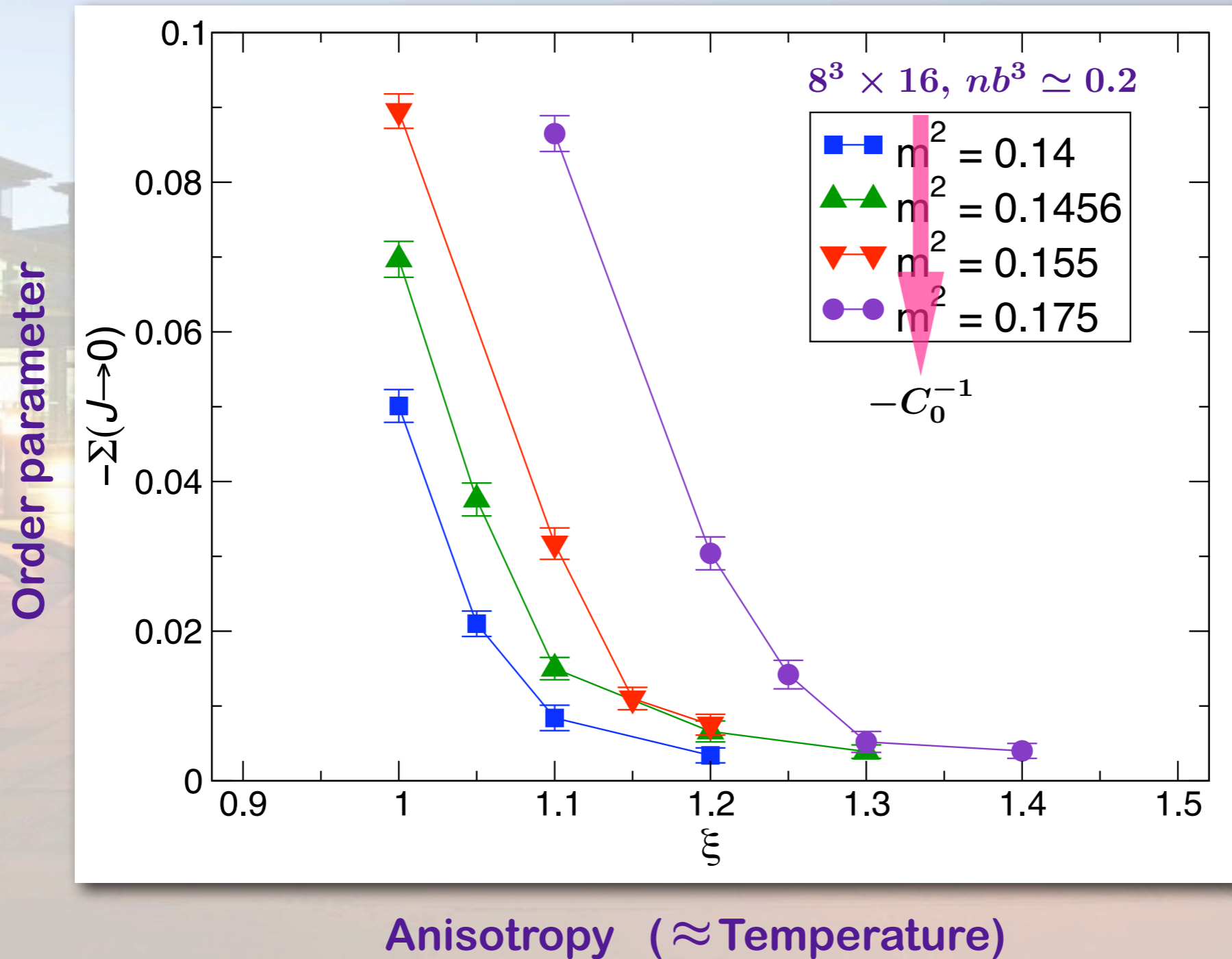
*T<sub>c</sub>*

# Extrapolation to zero external source





# Phase transition



MW, cond-mat/0502372

# Diagrammatic Determinant Monte Carlo

Burovski, Prokofev, Svistunov, Troyer, New J. Phys 8, 153 (2003)

$$\begin{aligned} Z &= 1 + \text{diagram 1} + \text{diagram 2} \\ &\quad - \text{diagram 3} - \text{diagram 4} + \text{diagram 5} + \dots \\ &= 1 + \text{diagram 6} + \text{diagram 7} + \dots \end{aligned}$$

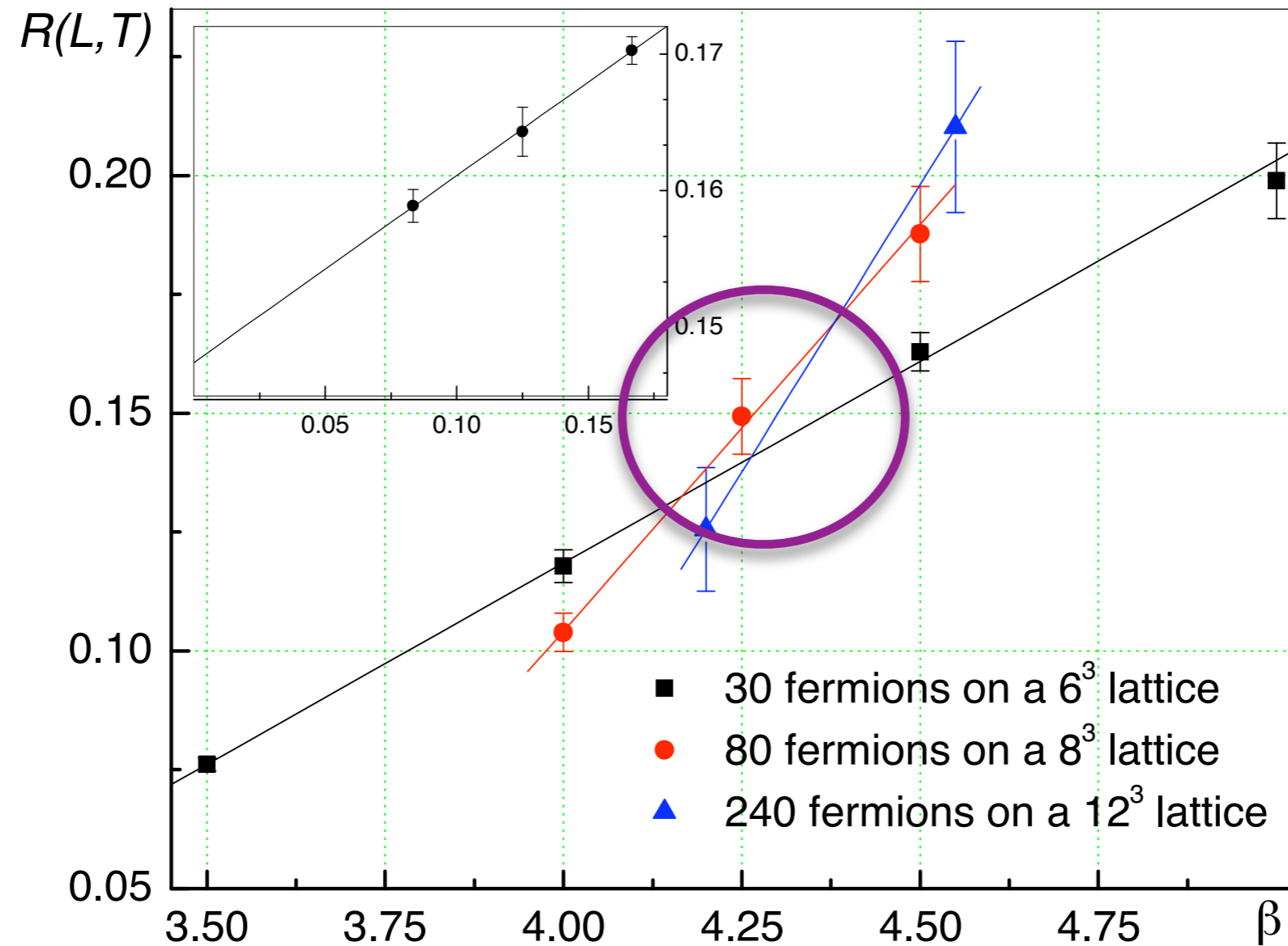
The diagrams are:

- Diagram 1: A single vertical figure-eight loop with a black dot at its center.
- Diagram 2: Two vertical figure-eight loops side-by-side, each with a black dot at its center.
- Diagram 3: A figure-eight loop with two black dots at the top and bottom vertices, connected by a horizontal line.
- Diagram 4: A figure-eight loop with two black dots at the left and right vertices, connected by a horizontal line.
- Diagram 5: A horizontal figure-eight loop with two black dots at the left and right vertices.
- Diagram 6: A black dot with four arrows pointing outwards in the cardinal directions.
- Diagram 7: Two black dots, each with four arrows pointing outwards in the cardinal directions.

# Critical Temperature

Burovski, Prokof'ev, Svistunov, Troyer, New J. Phys 8, 153 (2003)

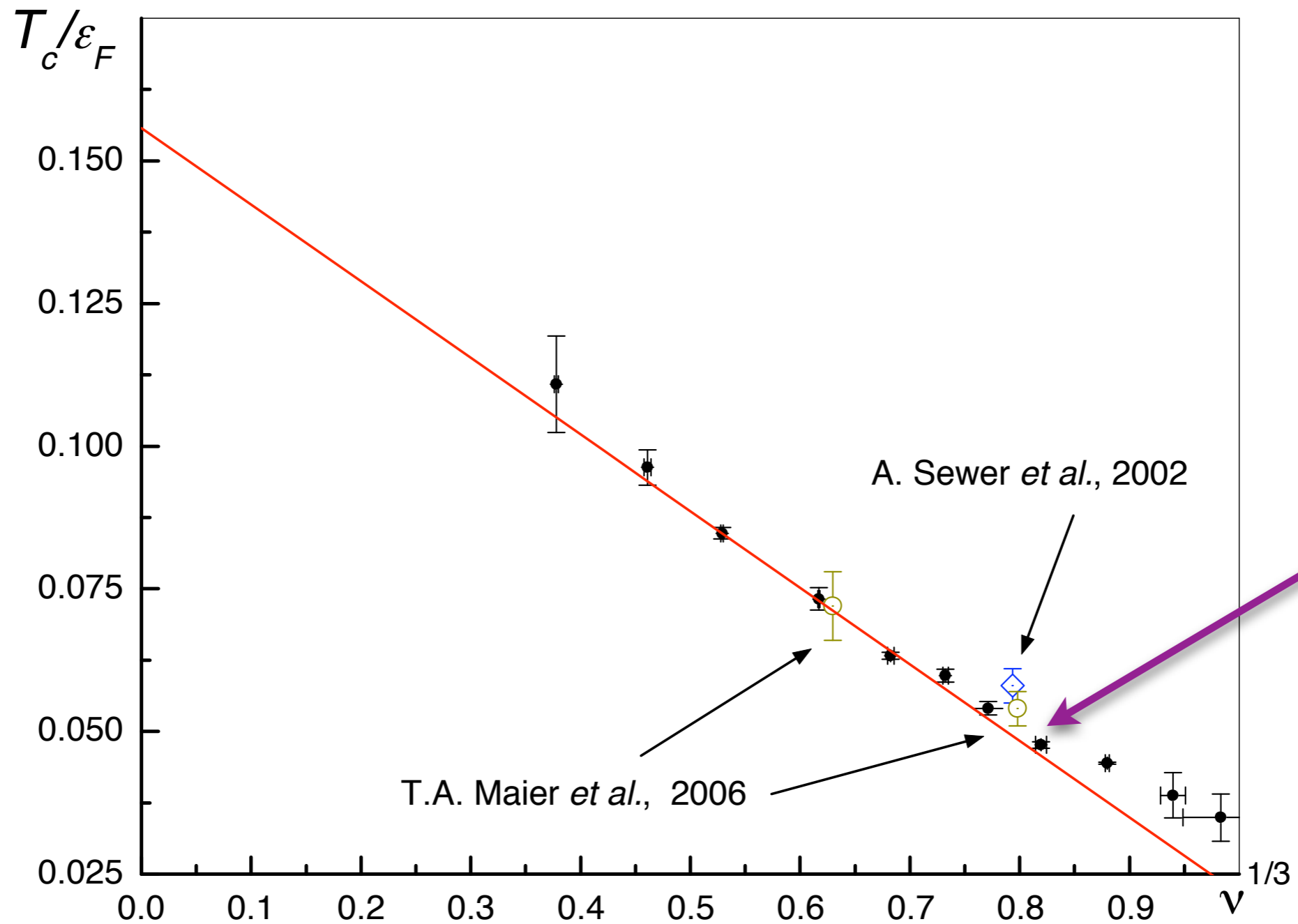
rescaled order parameter



$1/T$

# Continuum limit

Burovski, Prokof'ev, Svistunov, Troyer, New J. Phys 8, 153 (2003)



HMC stuck here



# Our investigations with DDMC

with Olga Goulko

- ❖ Implemented own version of code
- ❖ Found long autocorrelations in cases
- ❖ Update of result for  $T_c$  in balanced case
- ❖ Investigation of imbalanced gas,  $T_c$  vs.  $\Delta n/n$ , in continuum limit
- ❖ Final results later this spring

Lattice 2009 proceedings, arXiv:0910.3909



# Trade-offs

- ♣ Can work directly with  $J=0$
- ♣ Fine for finite volume determination of  $T_c$
- ♣ Scales like  $V^2$



# *Superfluid EFT*

# Phonon

$$\langle \psi^T \sigma_2 \psi \rangle = |\langle \psi^T \sigma_2 \psi \rangle| e^{2i\theta}$$

$$\chi(\tau, \vec{x}) \equiv e^{i\theta(\tau, \vec{x})}$$

$$X \equiv \chi \left( \partial_\tau - \frac{\nabla^2}{2m} \right) \chi^\dagger = \left( -i\partial_\tau \theta + \frac{1}{2m} |\nabla \theta|^2 \right)$$

Initial power counting:  $(\partial\theta)^n \sim O(1)$ ,  $\partial^m (\partial\theta)^n \sim O(p^m)$

$$\mathcal{L}_{\text{eff}} = c_0 m^{3/2} X^{5/2} \quad c_0 = \frac{2^{5/2}}{15\pi^2 \xi^{3/2}} \quad \epsilon = \xi \frac{3}{5} \epsilon_F n.$$

NLO derived in D T Son & M W, Annals Phys (2006)

# Expand about ground state

$$\varphi = \theta - i\mu\tau$$

$$X = -\mu + i\partial_\tau\varphi + \frac{|\nabla\varphi|^2}{2m}$$

$$\varphi(\tau, \vec{x}) \equiv \frac{\pi(\tau, \vec{x})}{F\sqrt{m}}.$$

$$\mathcal{L}_{\text{eff}} = -c_0 m^{3/2} \left[ \mu^{5/2} - \frac{5\mu^{3/2}}{2F\sqrt{m}} \left( i\partial_\tau \pi + \frac{1}{2m^{3/2}F} |\nabla\pi|^2 \right) + \frac{15\mu^{1/2}}{4mF^2} \left( i\partial_\tau \pi + \frac{1}{2m^{3/2}F} |\nabla\pi|^2 \right)^2 + \dots \right]$$



# Include pairing source

work done with J-W Chen

$$\tilde{\mathcal{J}} \rightarrow e^{-2i\alpha} \tilde{\mathcal{J}}$$

$$Y = \tilde{\mathcal{J}}(\chi^\dagger)^2 + \tilde{\mathcal{J}}^* \chi^2$$

$$Y = 2J \cos 2\varphi = 2J(1 - 2\varphi^2 + \dots)$$

$$\mathcal{L}_{\text{eff}} = c_0 m^{3/2} X^{5/2} + d_0 m^{3/2} X^{3/2} Y$$

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & -m^{3/2} \left[ \left( c_0 + \frac{2d_0 J}{\mu} \right) \mu^{5/2} - \frac{\mu^{3/2}}{F\sqrt{m}} \left( \frac{5c_0}{2} + \frac{3d_0 J}{\mu} \right) \left( i\partial_\tau \pi + \frac{1}{2m^{3/2}F} |\nabla \pi|^2 \right) \right. \\ & \left. + \frac{\mu^{1/2}}{4mF^2} \left( \frac{15c_0}{2} + \frac{3d_0 J}{\mu} \right) \left( i\partial_\tau \pi + \frac{1}{2m^{3/2}F} |\nabla \pi|^2 \right)^2 - \frac{4d_0 J \mu^{3/2}}{mF^2} \pi^2 \right] \end{aligned}$$



# Phonon mass

$$B = 2d_0 m^{1/2} \mu^{3/2} \propto \Sigma$$

LO phonon mass

$$M_0^2 = \frac{4B}{F^2 c_s^4} J$$

Correlation length

$$\xi = \frac{1}{\sqrt{2mMc_s^2}}$$

Must use power counting:

$$(\partial^n \theta) \sim O(p^n), \quad M^k \sim O(p^k)$$

# Pressure

Constant part of L

$$P_0(\mu) = (c_0 \mu^{5/2} + 2d_0 J \mu^{3/2}) m^{3/2}$$

$$n = P'_0(\mu) = \left( \frac{5}{2} c_0 \mu^{3/2} + 3d_0 J \mu^{1/2} \right) m^{3/2}$$

Canonical norm'n for kinetic term

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & -P_0(\mu) + \frac{i n}{F \sqrt{m}} \partial_\tau \pi \\ & + \frac{1}{2} (\partial_\tau \pi)^2 + \frac{c_s^2}{2} |\nabla \pi|^2 + \frac{1}{2} M_0^2 \pi^2 + \mathcal{L}_{\text{int}} \end{aligned}$$

Speed of sound

$$c_s^2 = \frac{1}{m} \frac{n(\mu)}{n'(\mu)} = \frac{2}{3} \frac{\mu}{m} \left( 1 + \frac{4 d_0 J}{5 c_0 \mu} + \mathcal{O}\left(\frac{J^2}{\mu^2}\right) \right)$$

# Finite volume effects

♣  $p$  regime: small effects,  $e^{-ML}$

$$\frac{1}{F^2} \ll \xi \ll L$$

♣  $\varepsilon$  regime: phonon zero mode dominates, must be resummed.

$$\frac{1}{F^2} \ll L \ll \xi$$

♣  $\delta$  regime: symmetry restored

$$L \ll \frac{1}{F^2} \ll \xi$$

A photograph of a modern building at dusk, with the title "Random Matrix Theory" overlaid in a large, italicized font. The building features multiple levels with large windows and balconies, and is illuminated by warm lights. The sky is a soft, pale blue. The foreground shows a paved plaza with several vertical light posts.

# *Random Matrix Theory*



# Random matrix theory

- ❖ Low energy EFT is universal given symmetries and symmetry breaking pattern
- ❖ In  $\varepsilon$  regime uniform phonon zero mode dominates over fluctuating nonzero modes
- ❖ Replace Gor'kov operator with random matrix with appropriate global symmetries (and no spacetime dependence)
- ❖ Spectral quantities on scale of average level spacing are thought to be universal
- ❖ Compare to Monte Carlo
- ❖ Guidance for algorithmic improvements

A photograph of a modern building at dusk, with the title "Random Matrix Model" overlaid in a large, italicized font. The building features multiple levels with balconies and is illuminated from within, creating a warm glow against the twilight sky. The foreground shows a paved plaza with several vertical light poles.

# *Random Matrix Model*



# Random matrix model

- ♣ Going beyond universal results...
- ♣ Use RMT as toy model (with same low energy EFT as Fermi gas)
- ♣ Explore larger population imbalances analytically
- ♣ Interesting phase diagram?

# Summary

- ❖ Numerical calculation of critical temperature
- ❖ Need to improve Monte Carlo calculations (HMC)
- ❖ EFT study of SSB in finite volume
- ❖ Importance of low-lying eigenmodes
- ❖ Random matrix theory for unitary Fermi gas





**Photo: Stefan Meinel**