When Perturbation Theory Fails...

Brian Tiburzi
(University of Maryland)

When Perturbation Theory Fails...

- SU(3) chiral perturbation theory?
- Charm quark in HQET, NRQCD?
- Extrapolations of lattice QCD data?

$$\frac{1}{4} \ll 1$$

$$\frac{1}{2} \ll 1$$

Solution? Not this talk...

Literal interpretation of title

Non-Perturbative Examples

- Toy model
- Hadrons in uniform electromagnetic fields
- Hyperons in SU(2) chiral perturbation theory

Toy model: $0 < x \ll 1$

$$F(x) = \int_0^\infty \frac{e^{-s}}{1 + s\,x} ds$$

Cannot series expand about 0

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$$F(x) = \int_0^\infty ds\, e^{-s} \left(\sum_{j=0}^{\infty} (-s\,x)^j \right) \overset{!}{=} \sum_{j=0}^{\infty} (-x)^j \left(\int_0^\infty ds\, s^j\, e^{-s} \right)$$

"Do it anyway"
(Physicist)

Suggests approximation

$$F_N(x) = \sum_{j=0}^{N} (-)^j j!\, x^j$$

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Suggests approximation

$$F_N(x) = \sum_{j=0}^{N} (-)^j j!\, x^j$$

$$\begin{aligned} |F(x) - F_N(x)| &= x^{N+1} \int_0^\infty \frac{s^{N+1} e^{-s}\, ds}{1 + s\,x} \\ &\leq x^{N+1} (N+1)! \end{aligned}$$

Toy model: $0 < x \ll 1$

$$F(x) = \int_0^\infty \frac{e^{-s}}{1 + s\,x} ds$$

$$F_N(x) = \sum_{j=0}^{N} (-)^j j!\, x^j$$

Minimize error for large N

$$|F(x) - F_N(x)| \quad \leq \quad x^{N+1}(N+1)! \approx \sqrt{2\pi N}(xN)^N e^{-N}$$

$x \sim 1/N$

$$\sim \sqrt{\frac{2\pi}{x}} e^{-\frac{1}{x}}$$

"Do it anyway"

(Physicist)

- Include more terms: limits to smaller x
- Make better for larger x: dropping terms

Asymptotic expansions: intuitively opposite

Toy model: $0 < x \ll 1$

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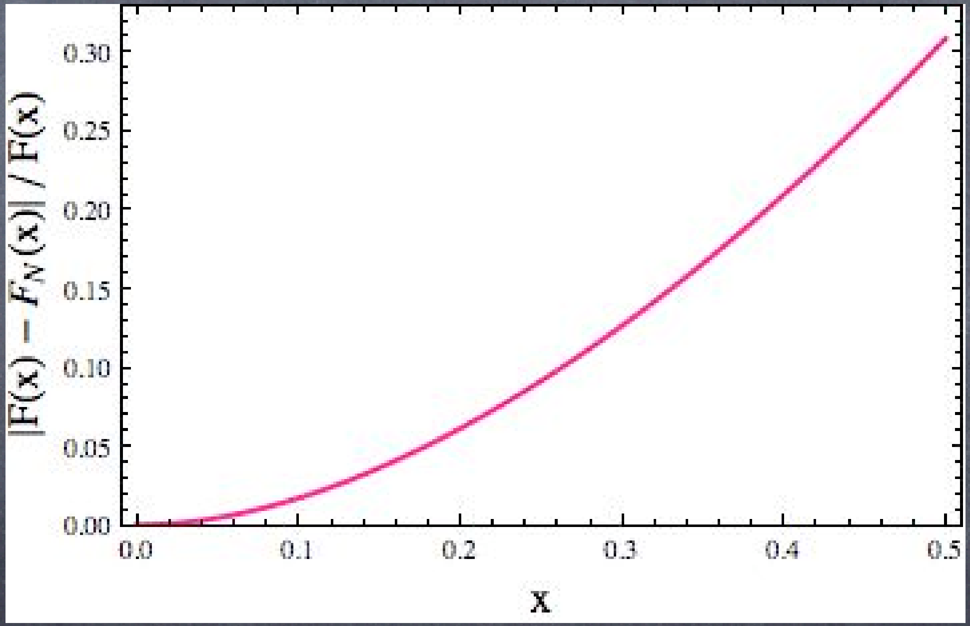

N=1

$x \sim 1/N$

Toy model: $0 < x \ll 1$

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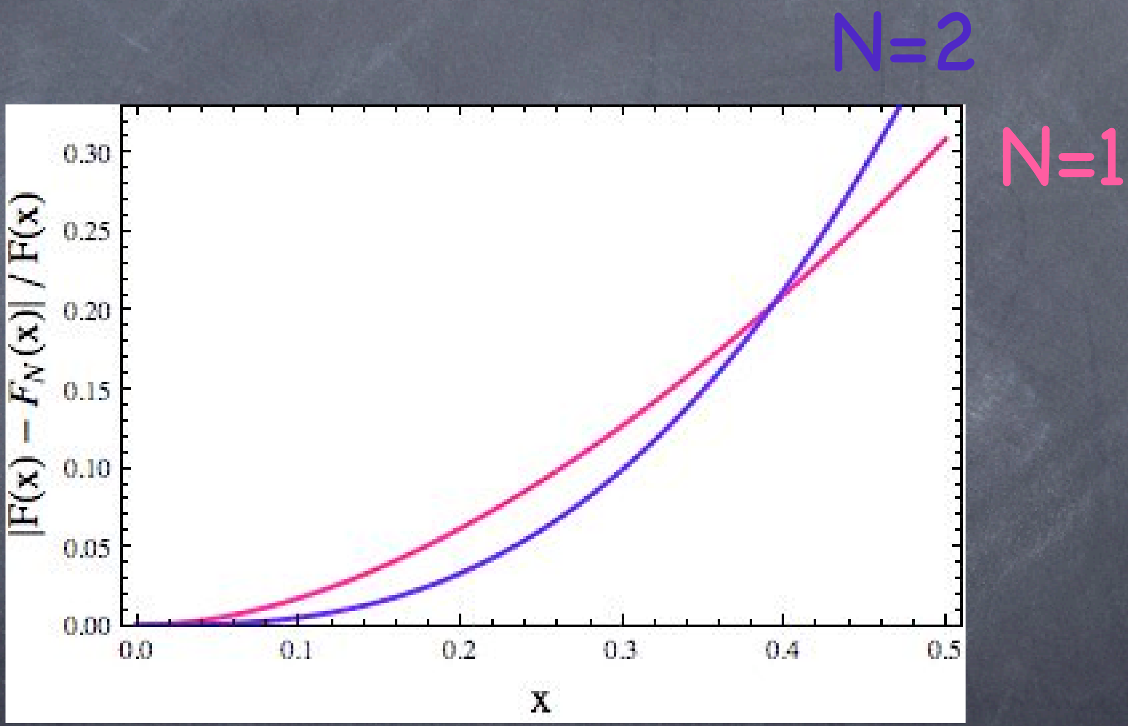

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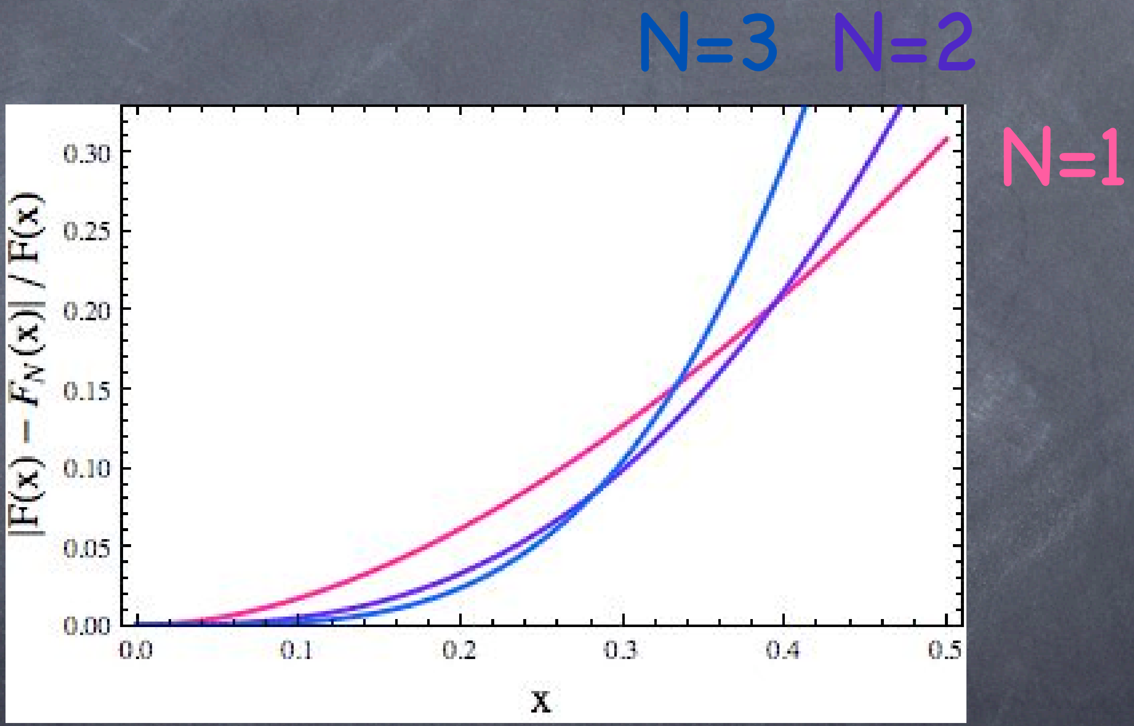

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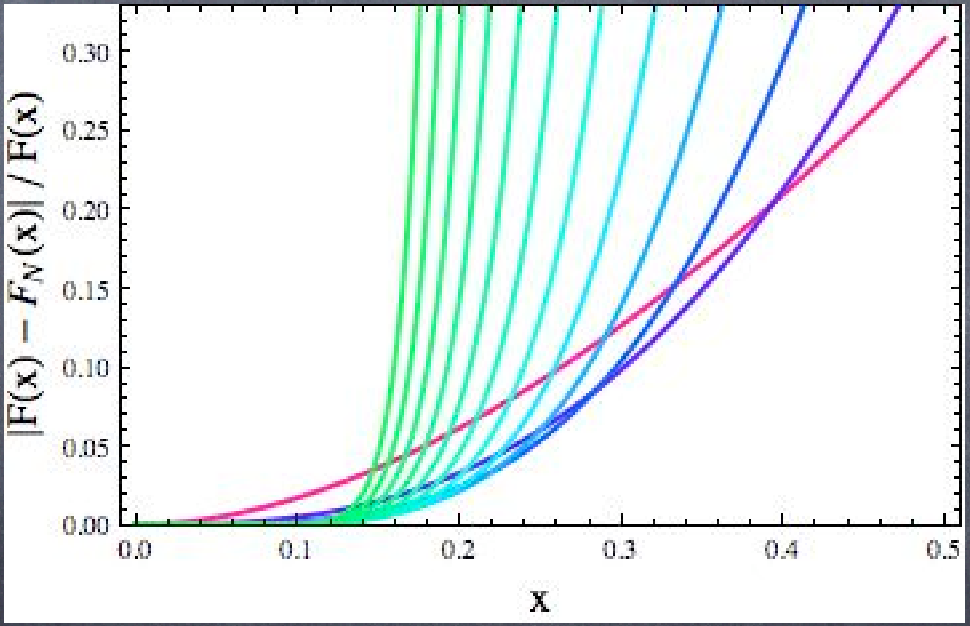
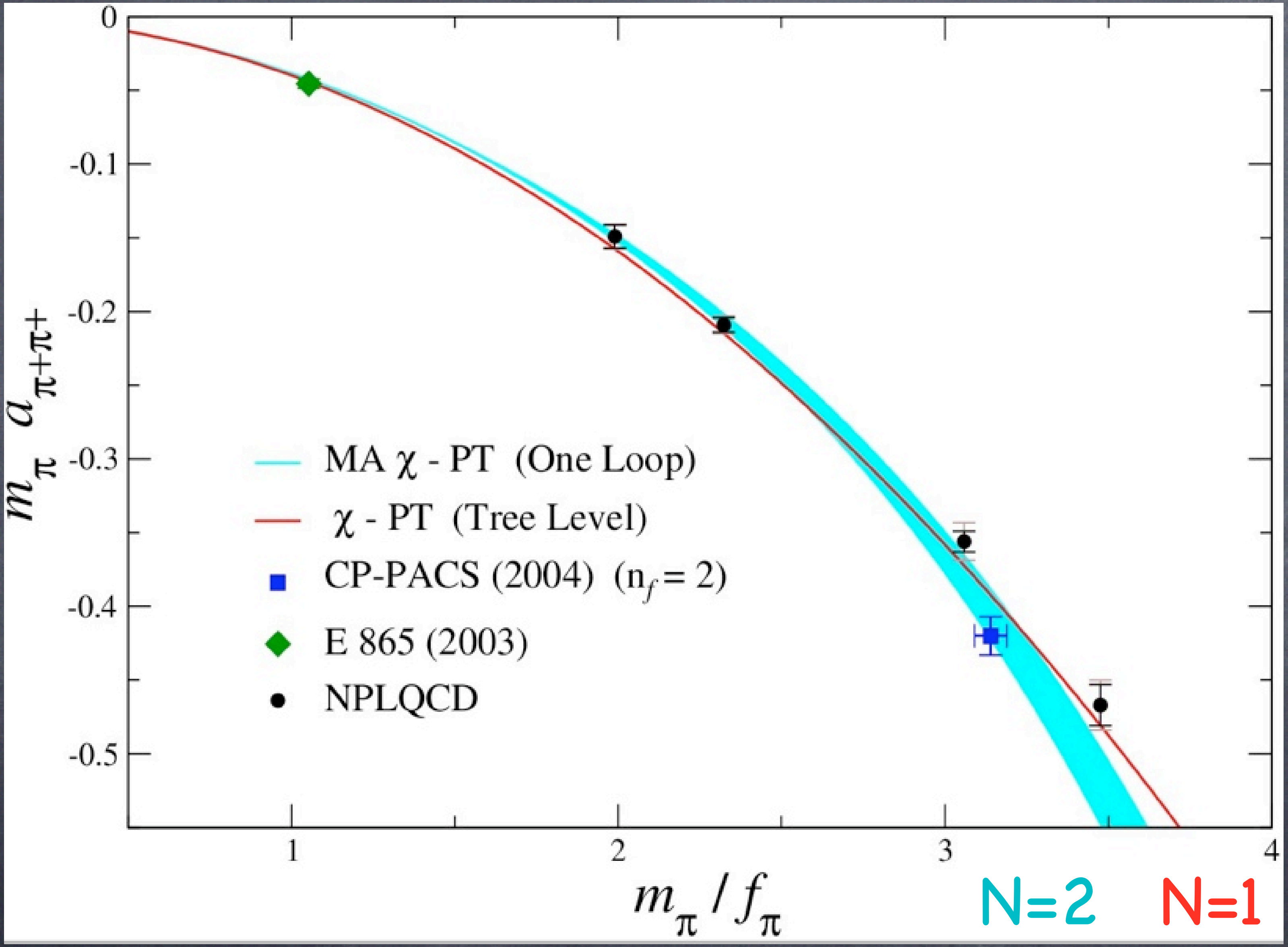


$x \sim 1/N$

Asymptotic expansions:
zero radius of convergence = intuitively opposite

Where have I seen this behavior?



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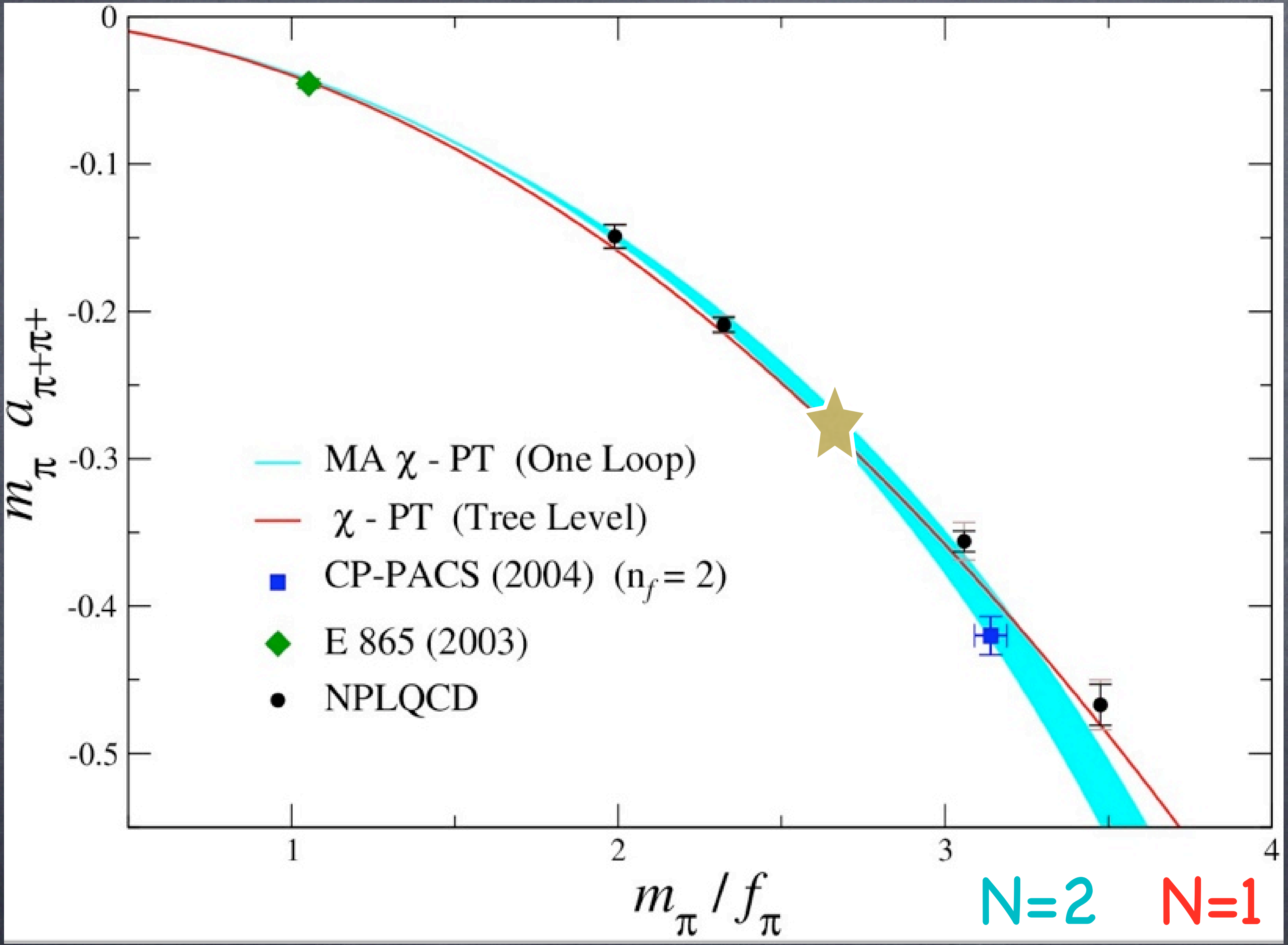


★ When Perturbation Theory Fails...

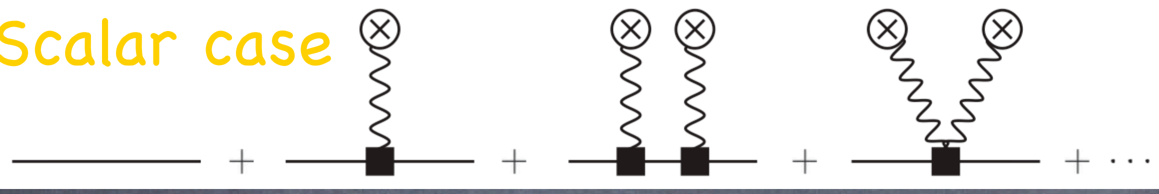
Hadrons in uniform electromagnetic fields

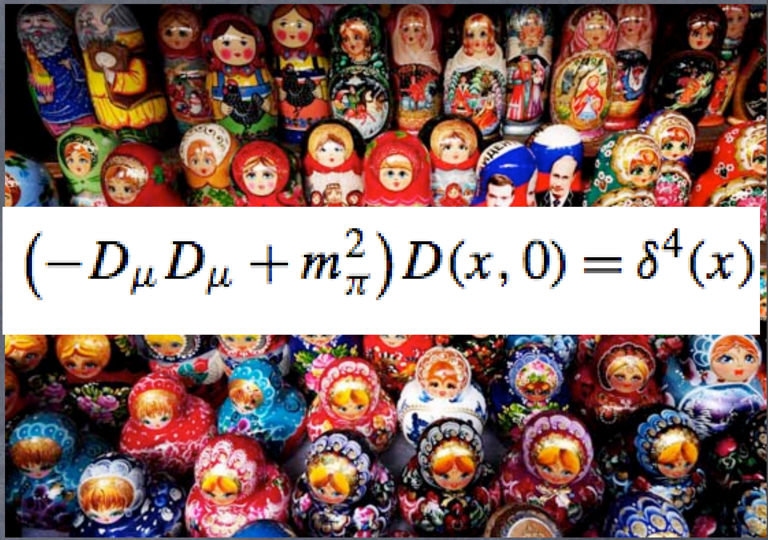
Schwinger (1951), ..., Tiburzi (2008)

- Green's functions exist in closed form for uniform EM fields

- Use to study non-perturbative effects

Scalar case





$$\left(-D_\mu D_\mu + m_\pi^2\right) D(x,0) = \delta^4(x)$$

Magnetic field:

$$A_\mu = (-Bx_2, 0, 0, 0)$$

Solution

$$D(x',x) = \frac{1}{2}\int_0^\infty ds \int \frac{d\tilde{k}}{(2\pi)^3} e^{i\tilde{k}\cdot(x'-x)} \left\langle x_2' - \frac{k_1}{eB}, s \,\middle|\, x_2 - \frac{k_1}{eB}, 0 \right\rangle e^{-sE_\perp^2/2}$$

Harmonic oscillator propagator

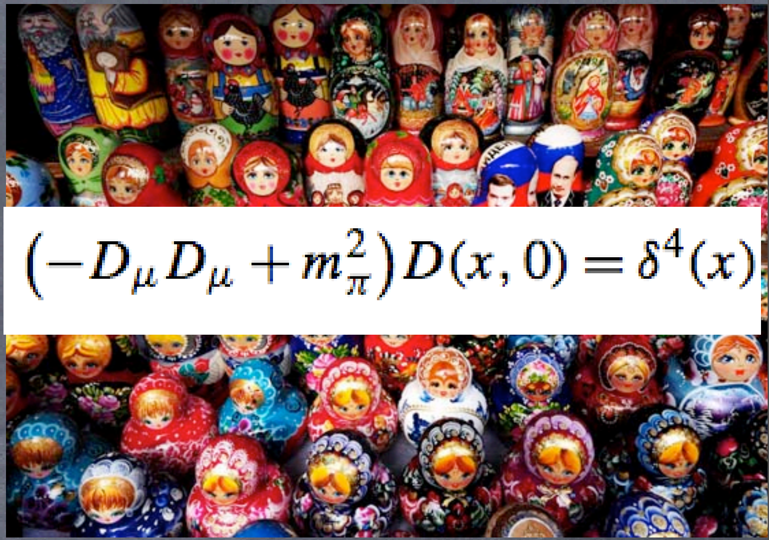
$$\langle X', s | X, 0\rangle = \sqrt{\frac{eB}{2\pi \sinh eBs}} \exp\left\{-\frac{eB}{2\sinh eBs}\left[(X'^2 + X^2)\cosh eBs - 2X'X\right]\right\}$$

Hadrons in uniform electromagnetic fields

- Chiral perturbation theory in strong QED

$$\mathcal{L} = \frac{f^2}{8}\langle D_\mu \Sigma^\dagger D_\mu \Sigma \rangle - \frac{\lambda}{2}\langle m_Q (\Sigma^\dagger + \Sigma)\rangle$$

$$D_\mu \Sigma = \partial_\mu \Sigma + ieA_\mu [\mathcal{Q}, \Sigma]$$



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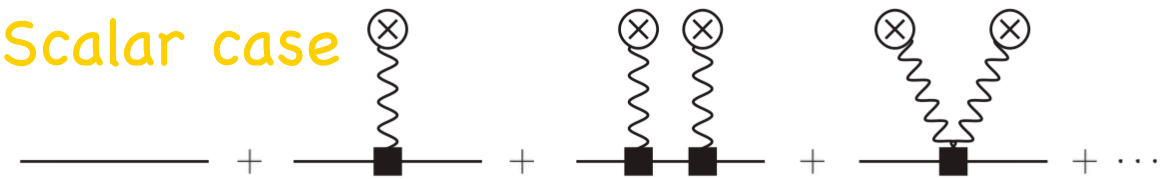
Scalar case

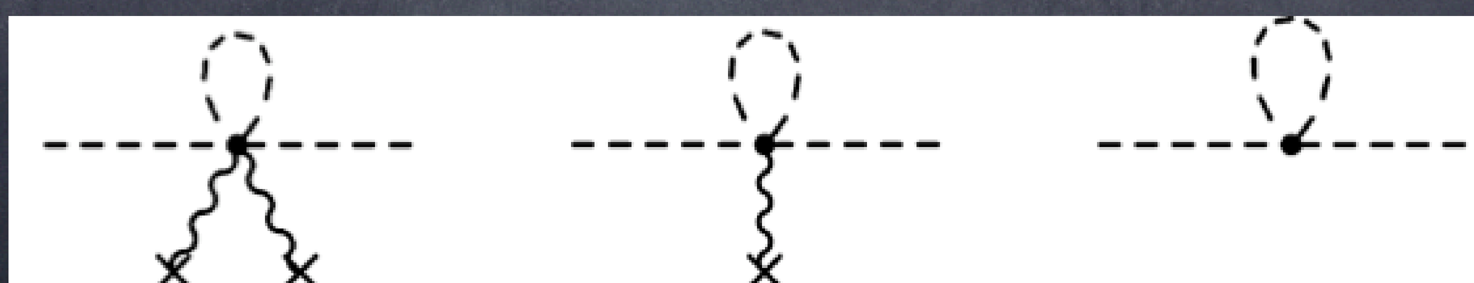


- Power counting

$$\frac{k^2}{\Lambda_\chi^2} \sim \frac{m_\pi^2}{\Lambda_\chi^2} \sim \frac{eF_{\mu\nu}}{\Lambda_\chi^2} \sim \varepsilon^2$$

$$eB/m_\pi^2 \sim 1$$

- Calculate the neutral pion energy in magnetic field



Hadrons in uniform electromagnetic fields

- Chiral perturbation theory in strong QED

Neutral pion energy

$$m^2_{\mathrm{eff},\pi^0} = m^2_\pi \left[1 + \frac{2e|B|}{(4\pi f)^2} \mathcal{I}\left(\frac{m^2_\pi}{e|B|} \right) \right]$$

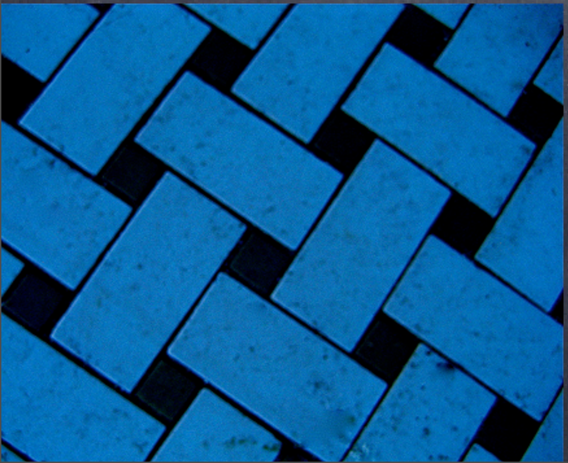
$eB/m^2_\pi \sim 1$

Closed form

$$\mathcal{I}(x) = \int_0^\infty \frac{ds}{s^2} e^{-xs} \left(\frac{s}{\sinh s} - 1 \right) = x\left(1 - \log \frac{x}{2}\right) + 2 \log \Gamma\left(\frac{1+x}{2} \right) - \log 2\pi$$

- Why? Background field lattice QCD computations in this regime

$$m_{\mathrm{eff}}(B) = m - \frac{1}{2} 4\pi \beta_M B^2 + \mathcal{O}(B^4)$$



Closed torus can leak no flux

$$\exp(ieBA)$$

$$e\,B = \frac{2\pi n}{L^2}$$

Hadrons in uniform electromagnetic fields

- Chiral perturbation theory in strong QED

Neutral pion energy

$$m^2_{\mathrm{eff},\pi^0} = m^2_\pi \left[1 + \frac{2e|B|}{(4\pi f)^2} \mathcal{I}\left(\frac{m^2_\pi}{e|B|} \right) \right]$$

$eB \ll m_\pi^2$

$$E_{\pi^0}(\boldsymbol{p}=\boldsymbol{0}) = m_\pi + \frac{m_\pi^3}{(4\pi f)^2} \left[-\frac{1}{6}\left(\frac{eB}{m_\pi^2}\right)^2 + \frac{7}{180}\left(\frac{eB}{m_\pi^2}\right)^4 - \frac{31}{630}\left(\frac{eB}{m_\pi^2}\right)^6 + \cdots \right]$$

"Do it anyway"
(Physicist)

$$m_{\mathrm{eff}}(B) = m - \frac{1}{2} 4\pi \beta_M B^2 + \mathcal{O}(B^4)$$

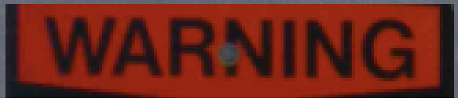
$$\beta_M^{\pi^0} = \frac{\alpha_{\mathrm{f.s.}}}{3(4\pi f)^2 m_\pi}$$

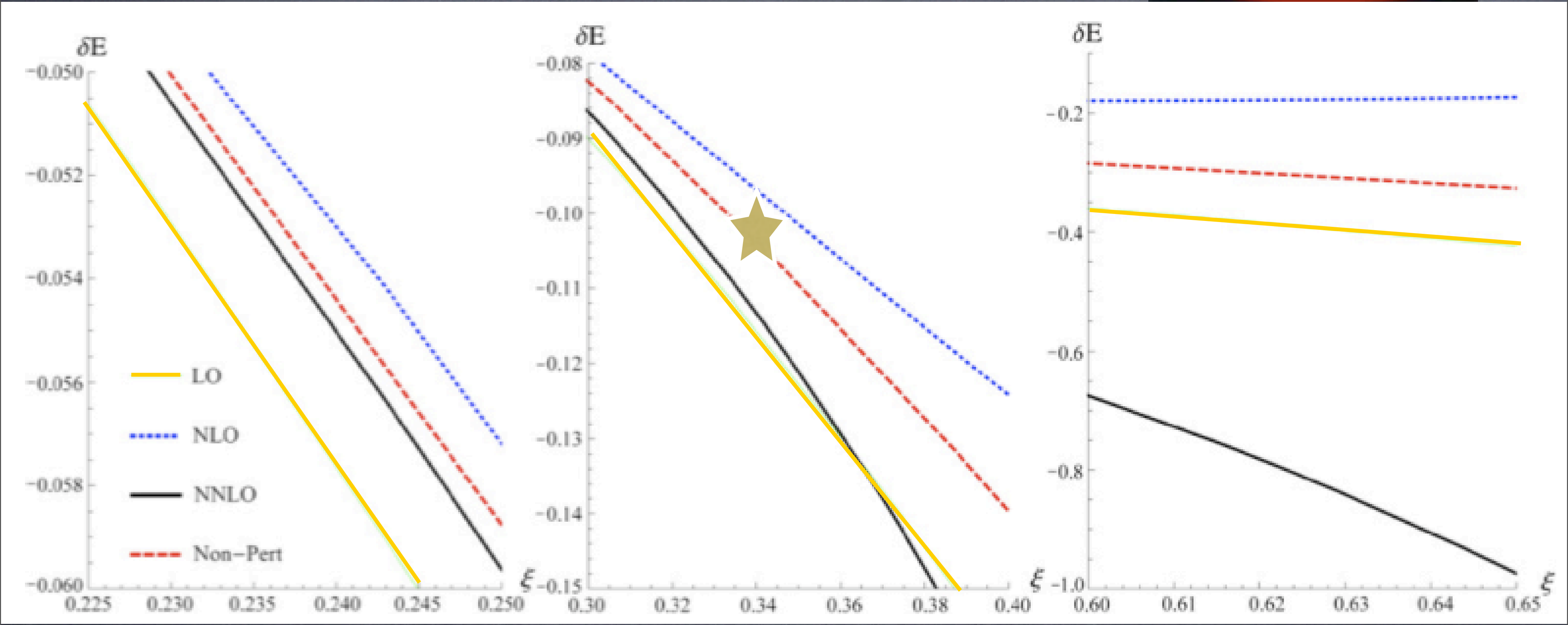
Hadrons in uniform electromagnetic fields

- Chiral perturbation theory in strong QED

$$\xi = \frac{e|B|}{\sqrt{6}m_\pi^2}$$

Neutral pion energy





intuitively obvious

asymptotically opposite

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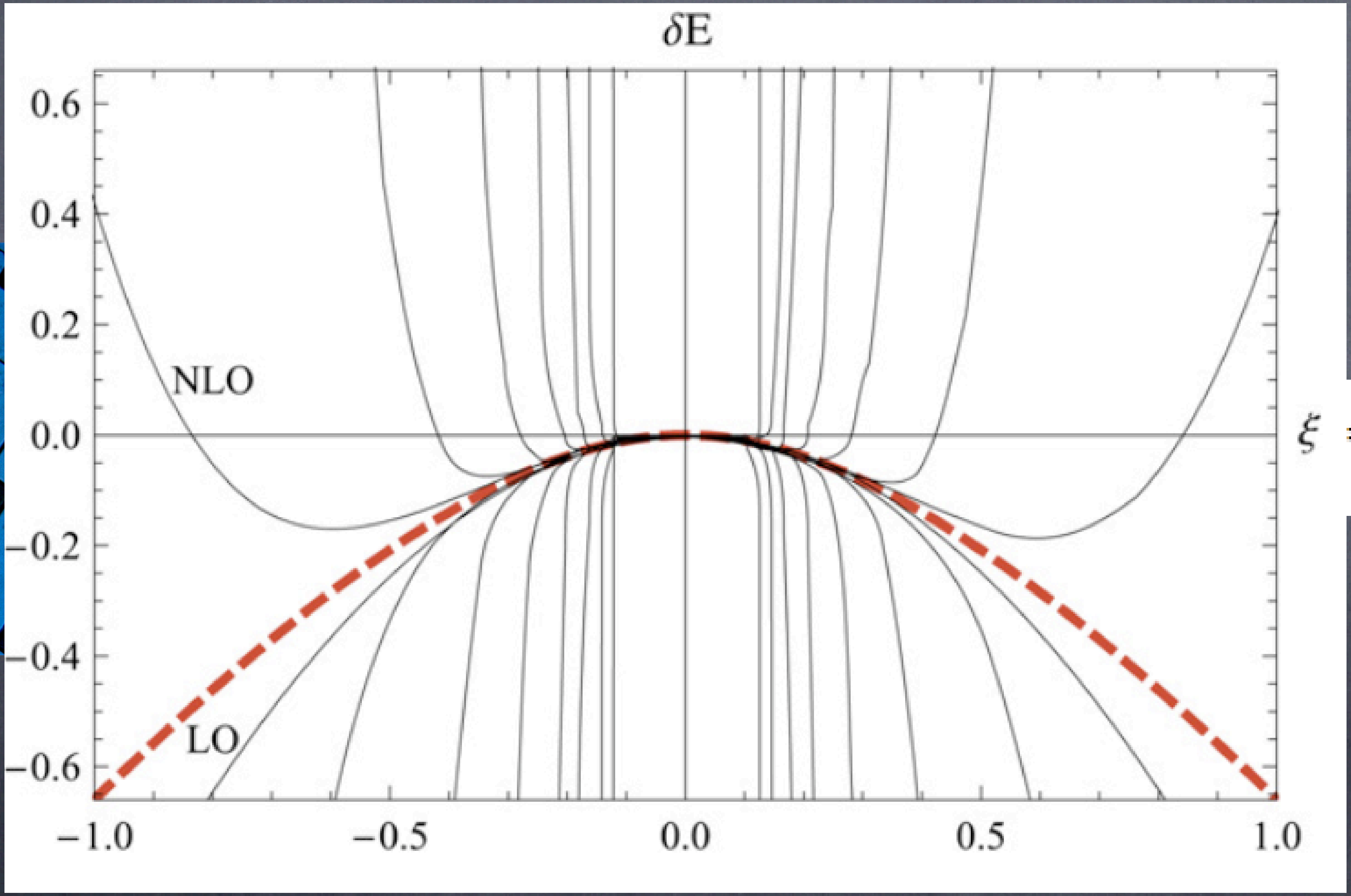
Hadrons in uniform electromagnetic fields

- Chiral perturbation theory in strong QED

Neutral pion energy



$$\xi = \frac{e|B|}{\sqrt{6}m_\pi^2}$$

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IDLE AMUSEMENT

- Gell-Mann / Oaks / Renner Relation

$$f_\pi^2 m_\pi^2 = 4\langle \bar{q} q \rangle m$$

$$f_\pi^2(B)\, m_{\pi,\text{eff}}^2(B) = 4\langle \bar{q} q \rangle_B m$$

- Electric Field: analytic continuation

$$\Gamma_{\pi^0} = -\Im\mathrm{m}(m_{\text{eff},\pi^0}^2)/m_\pi$$

Schwinger mechanism

$$\Gamma_n = \Gamma\left[\pi^0 \xrightarrow{E} \pi^0(\pi^+\pi^-)^n\right] = (-1)^{n+1}\frac{2e|E|m_\pi}{(4\pi f)^2}\exp\left(-\frac{n\pi m_\pi^2}{e|E|}\right)$$

- Charged pions

- Nucleon

In strong magnetic fields, proton beta decays to neutron

Hyperons in SU(2) Chiral Perturbation Theory

Tiburzi and Walker-Loud (2008)

Jiang, Tiburzi, and Walker-Loud (2009)

Motivation SU(3)

Heavy baryon chiral perturbation theory

$$\delta M_N(\mu = \Lambda_\chi)/M_N = -39\%$$

$$\delta M_\Lambda(\mu = \Lambda_\chi)/M_\Lambda = -67\%$$

$$\delta M_\Sigma(\mu = \Lambda_\chi)/M_\Sigma = -89\%$$

$$\delta M_\Xi(\mu = \Lambda_\chi)/M_\Xi = -98\%$$

"Do it anyway"
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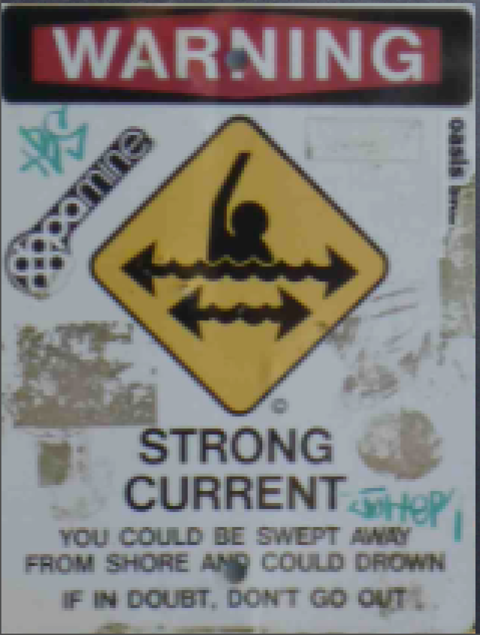
$$m_\eta/M_B \sim 1/2$$



Kaon, eta contributions large & increase with strangeness

$$m_s \sim \Lambda_{QCD} \text{ ?}$$

SU(3) expansion precarious

Hyperons in SU(2) Chiral Perturbation Theory

Schematic SU(3) Expansion of Sigma Mass:

$$M_\Sigma = M^{SU(3)} + a m_K^2 + b m_K^3 + \ldots$$

Large Kaon contributions

$$m_K^2 = \frac{1}{2} m_\pi^2 + \frac{1}{2} m_{\eta_s}^2 \qquad m_{\eta_s} = 672\,\texttt{MeV}$$

Reorganize! $\quad m_\pi^2 / m_{\eta_s}^2 = 0.04 \ll 1$

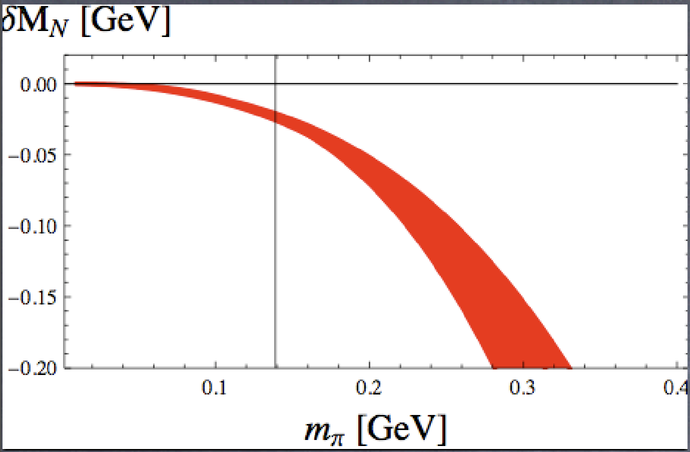
$$M_\Sigma = M^{SU(3)} + a' m_{\eta_s}^2 + a'' m_\pi^2 + b' m_{\eta_s}^3 + b'' m_{\eta_s} m_\pi^2 + b''' m_\pi^3 \left(\frac{m_\pi}{m_{\eta_s}} \right) + .$$

$$M_\Sigma = M_\Sigma^{SU(2)} + \alpha m_\pi^2 + \beta m_\pi^3 + \ldots$$

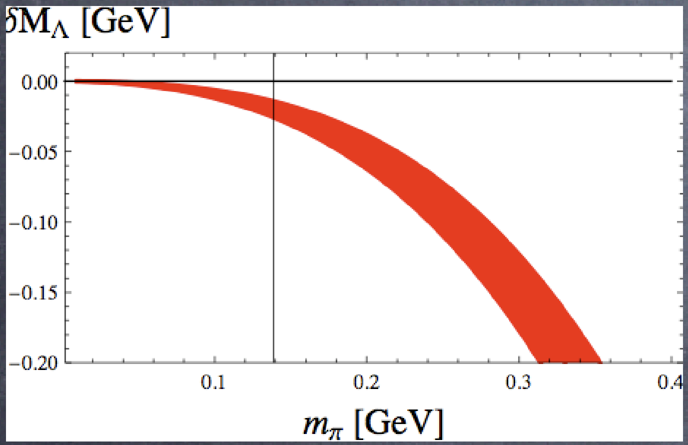
Expansion of Sigma Mass about the SU(2) chiral limit

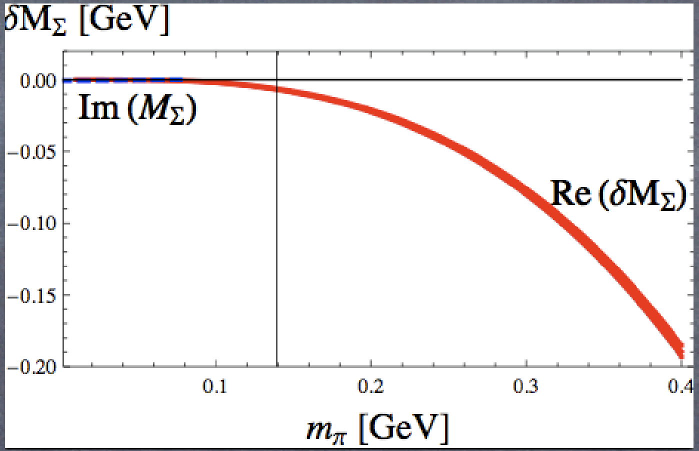
$$m_u, m_d \ll m_s \sim \Lambda_{QCD}$$

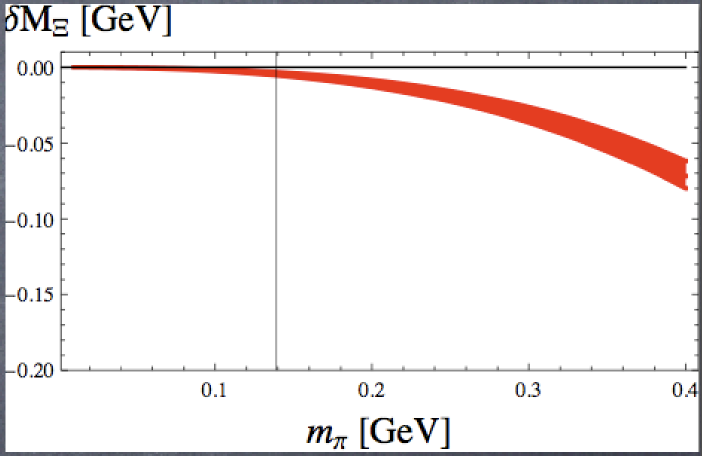
Hyperons in SU(2) Chiral Perturbation Theory



$$M = M^{SU(2)} + \alpha m_\pi^2 + \beta m_\pi^3 + \beta' F(m_\pi, \delta)$$







Trend opposite SU(3): greater strangeness, better convergence

$$g_A = 1.25, \;\; g_{\Sigma\Sigma} = 0.78, \;\; g_{\Xi\Xi} = 0.24$$

$$m_\pi/\Lambda_\chi \quad m_\pi/M_S$$

$$g_{\Delta N} = 1.48, \;\; g_{\Sigma^*\Sigma} = 0.76, \;\; g_{\Xi^*\Xi} = 0.69$$

Hyperons in SU(2) Chiral Perturbation Theory

SU(2) Perturbative Expansion can FAIL!

I) Perturbative expansion about SU(2) limit

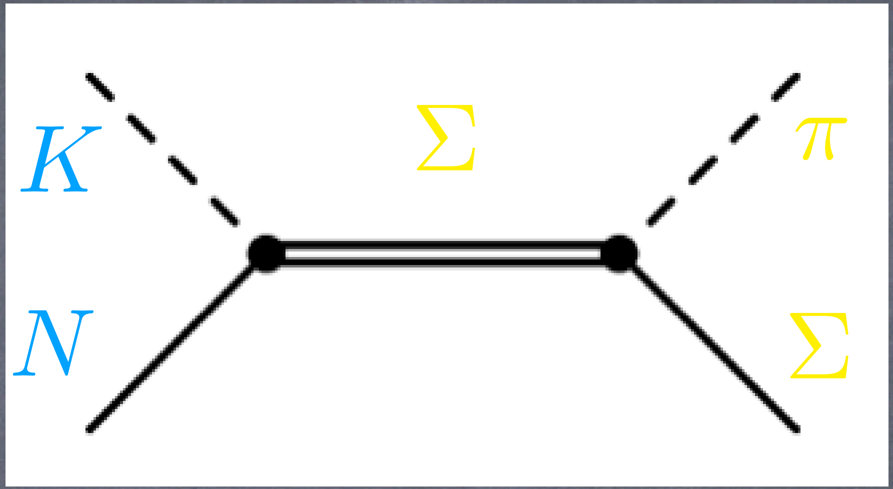
"Duh!" (Maryland Colleague) m_π/Λ_χ need lattice QCD

II) Perturbative SU(2) expansion of SU(3) !

Kaon thresholds... can study non-perturbatively

Hyperons in SU(2) Chiral Perturbation Theory

250 MeV



SU(2) Perturbative Expansion can FAIL!

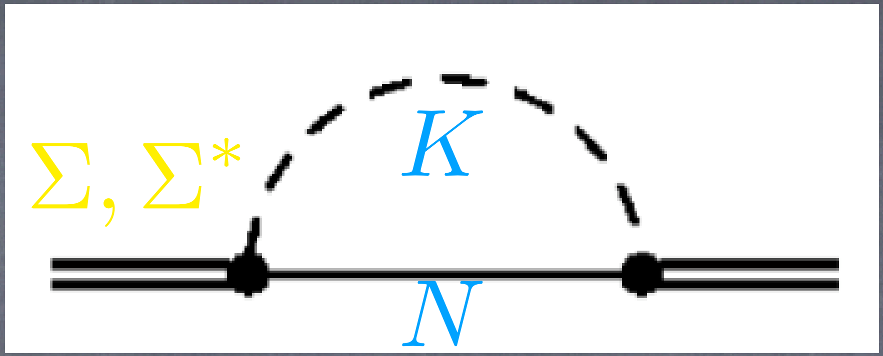
Kaon production cannot be described in SU(2)

$$m_u, m_d \ll m_s \ll \Lambda_{QCD}$$

II) Perturbative SU(2) expansion of SU(3) !

"Exact Solution" SU(3) is theory SU(2) is asymptotically describing

Hyperons in SU(2) Chiral Perturbation Theory



Do SU(2) expansions of hyperon masses break down because of KN thresholds?

$$m_K = 0.50\ \text{GeV}$$

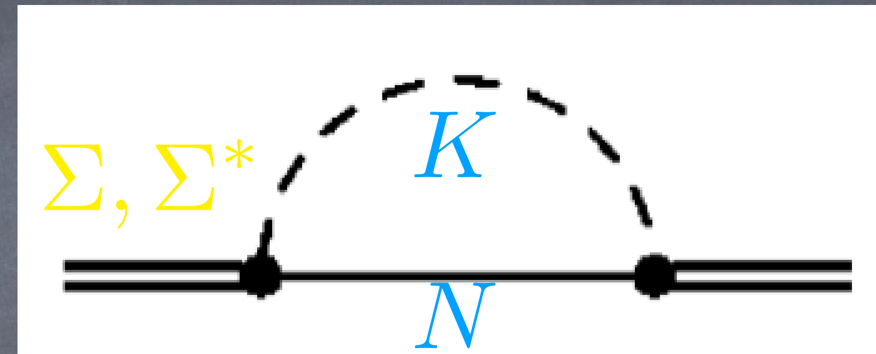
$$\delta_{N\Sigma} = 0.25\ \text{GeV}$$

$$\delta_{N\Sigma^*} = 0.45\ \text{GeV}$$

Only virtual, but analyticities near threshold

$$\mathcal{F}(m_K^2, -\delta_{BB'}, \mu) = (\delta_{BB'}^2 - m_K^2)^{3/2} \log\left(\frac{-\delta_{BB'} - \sqrt{\delta_{BB'}^2 - m_K^2 + i\epsilon}}{-\delta_{BB'} + \sqrt{\delta_{BB'}^2 - m_K^2 + i\epsilon}}\right)$$

Hyperons in SU(2) Chiral Perturbation Theory



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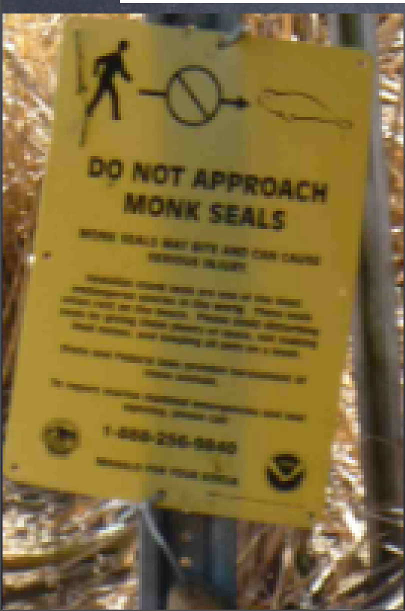

$$m_K^2 = \frac{1}{2}m_\pi^2 + \frac{1}{2}m_{\eta_s}^2$$

$$f\left(m_K^2 - \delta_{BB'}^2\right) = f\left(\frac{1}{2}m_{\eta_s}^2 - \delta_{BB'}^2\right) + \frac{1}{2}m_\pi^2\, f'\left(\frac{1}{2}m_{\eta_s}^2 - \delta_{BB'}^2\right) +$$

"Do it anyway"

(Physicist)

$$\varepsilon_{BB'} = \frac{\frac{1}{2}m_\pi^2}{\frac{1}{2}m_{\eta_s}^2 - \delta_{BB'}^2}$$

Hyperons in SU(2) Chiral Perturbation Theory

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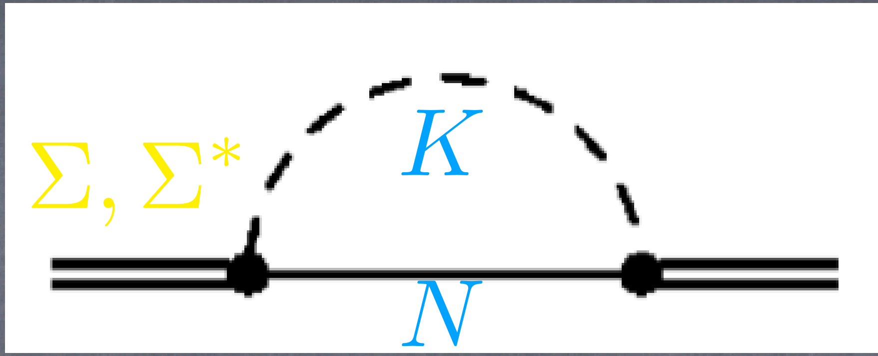
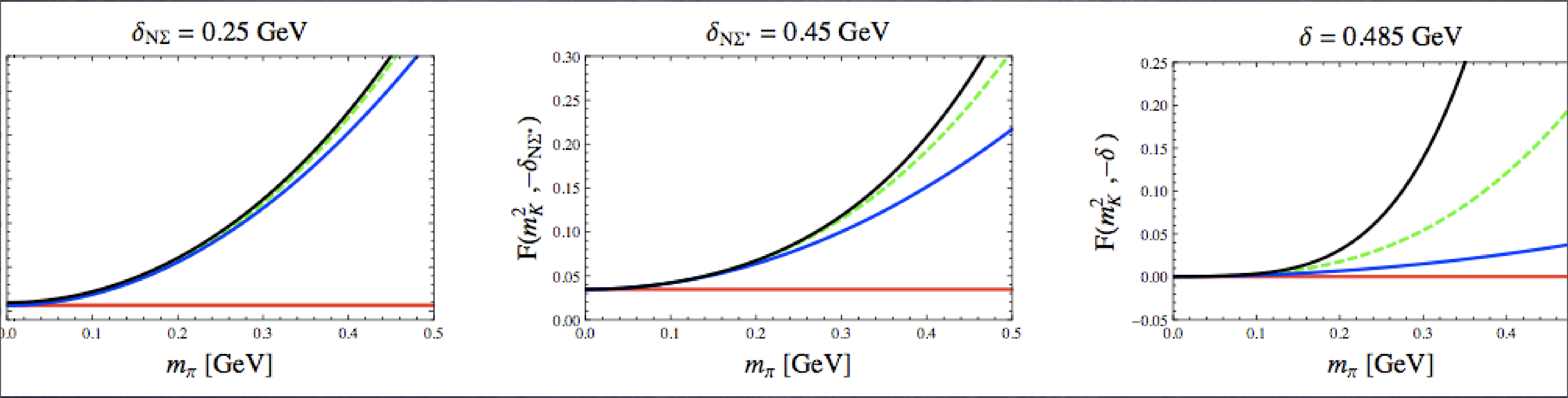


$$\mathcal{F}(m_K^2, -\delta_{BB'}) = \mathcal{F}^{(0)} + m_\pi^2\,\mathcal{F}^{(2)} + m_\pi^4\,\mathcal{F}^{(4)} + \ldots$$

$\varepsilon_{N\Sigma} = 0.05$ $\varepsilon_{N\Sigma^*} = 0.24$ $\varepsilon_{BB'} = 6.9$



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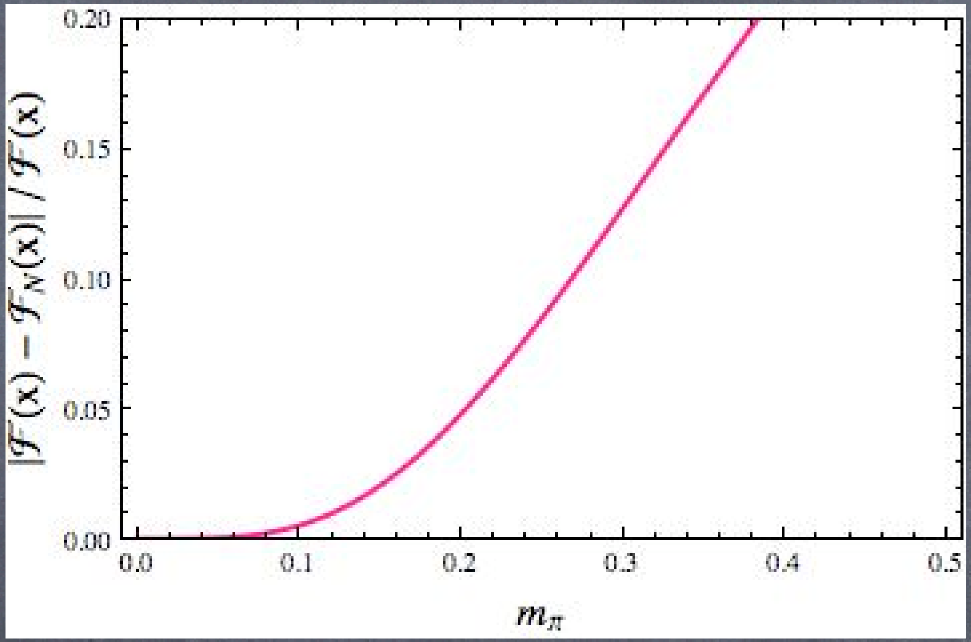
Hyperons in SU(2) Chiral Perturbation Theory

$$\delta_{N\Sigma^*} = 0.45\,\text{GeV}$$

$$\mathcal{F}(m_K^2, -\delta_{BB'}) = \mathcal{F}^{(0)} + m_\pi^2\,\mathcal{F}^{(2)} + m_\pi^4\,\mathcal{F}^{(4)} + \ldots$$

N=2



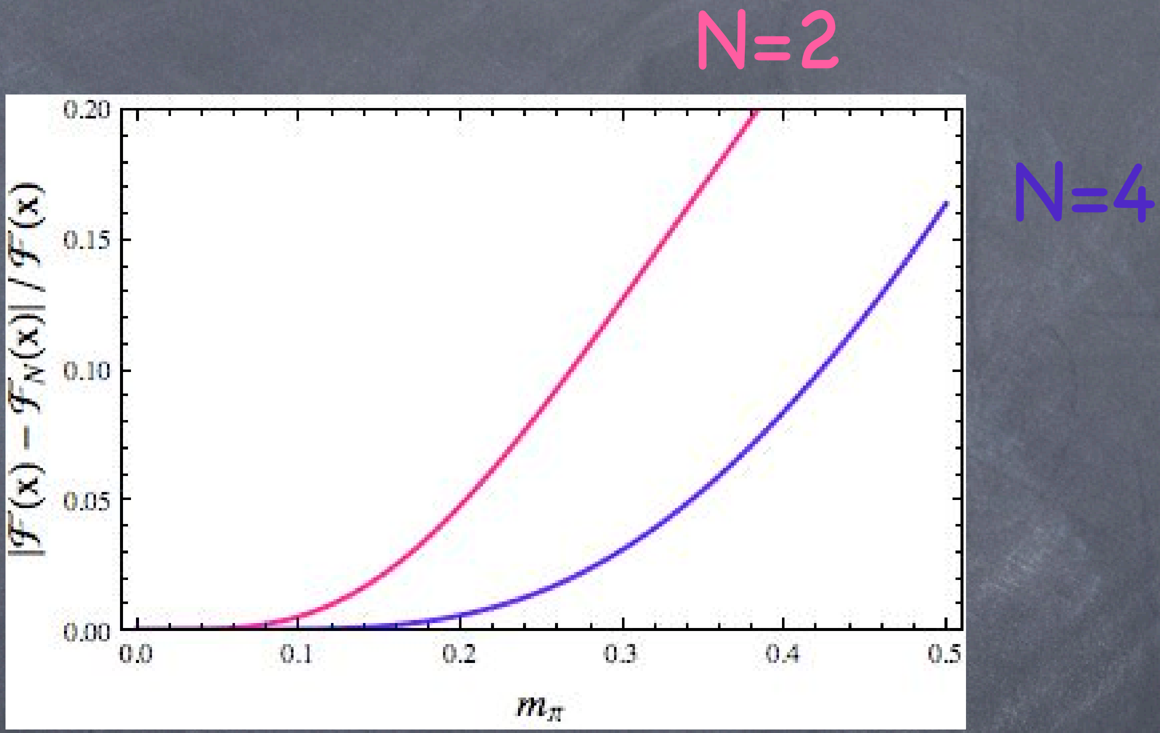
- Include more terms: limits to smaller pion mass
- Make better for larger pion mass: dropping terms

Asymptotic expansions: intuitively opposite

Hyperons in SU(2)
Chiral Perturbation Theory

$$\delta_{N\Sigma^*} = 0.45\,\text{GeV}$$

$$\mathcal{F}(m_K^2, -\delta_{BB'}) = \mathcal{F}^{(0)} + m_\pi^2\,\mathcal{F}^{(2)} + m_\pi^4\,\mathcal{F}^{(4)} + \ldots$$



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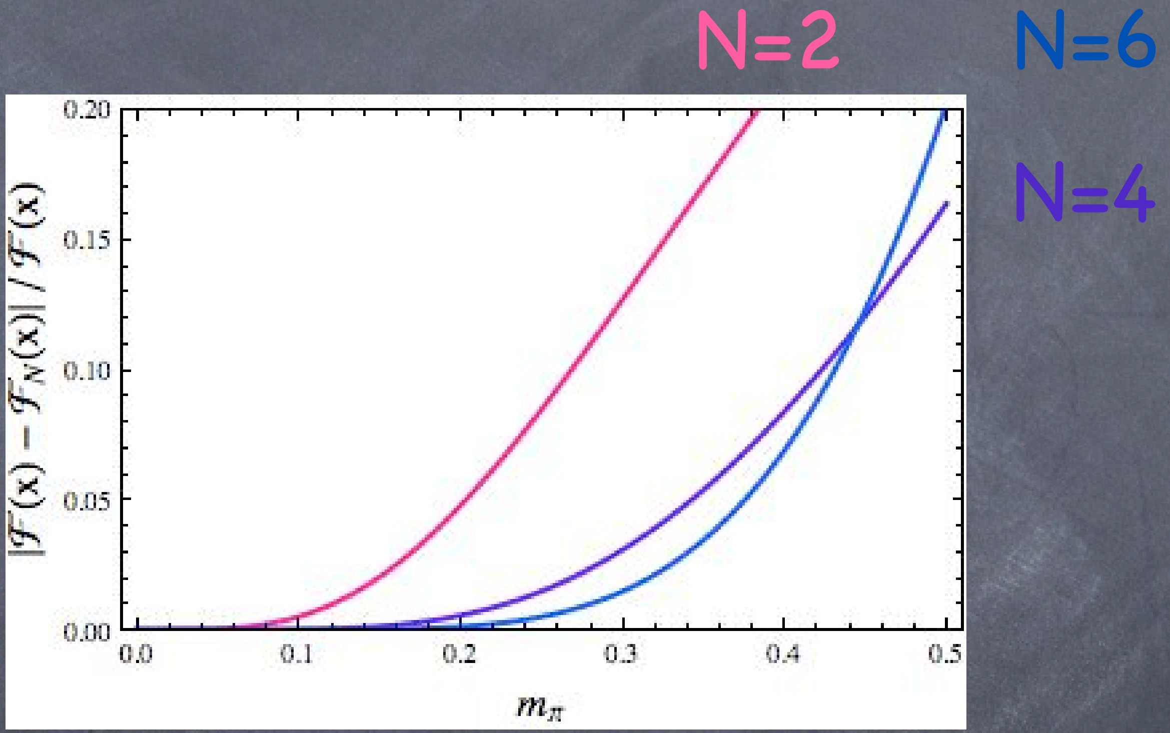

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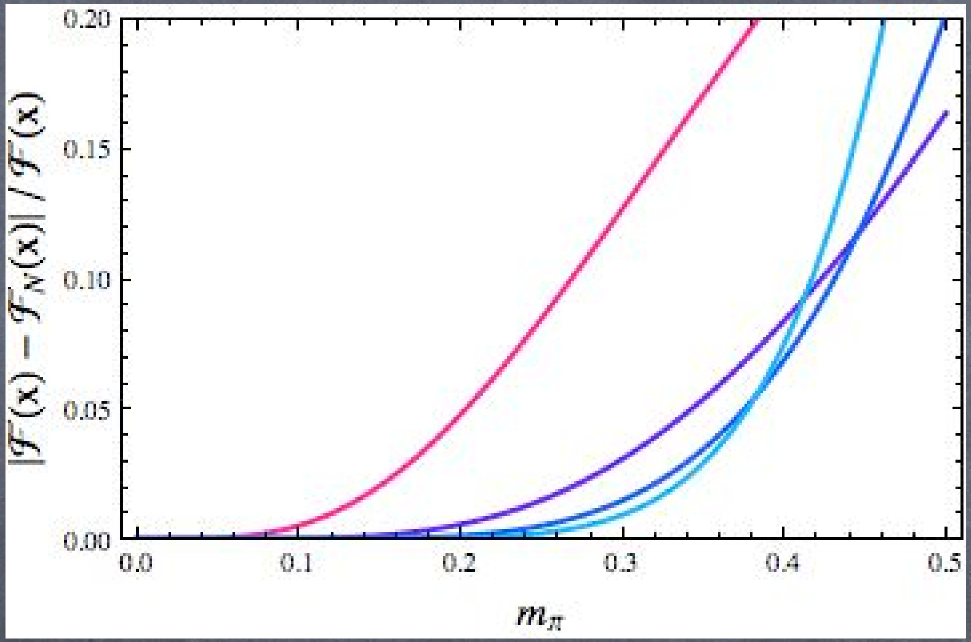
N=2 N=6 N=4



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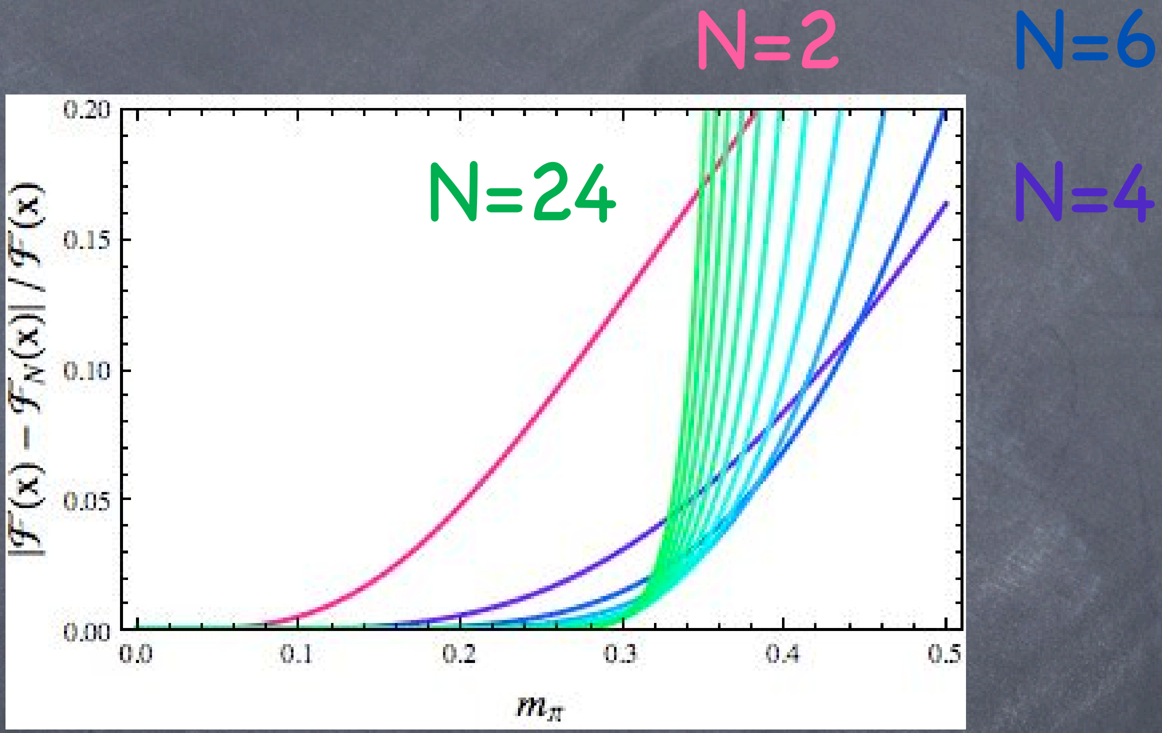
$$\delta_{N\Sigma^*} = 0.45\,\text{GeV}$$

$$\mathcal{F}(m_K^2, -\delta_{BB'}) = \mathcal{F}^{(0)} + m_\pi^2\,\mathcal{F}^{(2)} + m_\pi^4\,\mathcal{F}^{(4)} + \ldots$$



- Include more terms: limits to smaller pion mass
- Make better for larger pion mass: dropping terms

Asymptotic expansions: intuitively opposite

When perturbation theory fails, we probably try to use it anyway

Asymptotic expansions: intuitively opposite

Including more terms limits to smaller range

Make better for parameters by dropping terms (limited control)

$$eB/m_\pi^2 \sim 1$$

$$\varepsilon_{BB'} = \frac{\frac{1}{2}m_\pi^2}{\frac{1}{2}m_{\eta_s}^2 - \delta_{BB'}^2}$$

Neutral pion in electric field: B -> iE