Euclidean Relativistic Quantum Mechanics

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Motivation

- **Construct relativistic quantum models of systems with a finite number of degrees of freedom**

- **Desirable features:**

  - **Models motivated by field theory**

  - **Cluster properties**

Elements of relativistic quantum theory

- **Hilbert space $\mathcal{H}$ (requirement for a quantum theory)**
- **Unitary representation of the Poincaré group (requirement for relativistic invariance of quantum probabilities).**
- **Cluster properties (required for tests of relativity on isolated subsystems)**
- **Spectral condition (required for stability of theory)**

Input

Reflection-positive Euclidean Green function(s) or generating functional

Problem

Construct relativistic quantum mechanical models

(We want to avoid using analytic continuation!)

Field theory motivation

Euclidean generating functional or Green functions:

$$Z[f] := \frac{\int D_e[\phi] e^{-A[\phi]+i\phi(f)}}{\int D_e[\phi] e^{-A[\phi]}} = \sum_n \frac{(i)^n}{n!} S_n \underbrace{(f, \cdots, f)}_{\text{n times}}$$

$A[\phi] =$ **Action,** $D_e[\phi]$ **= Euclidean "path measure"**

$f(\tau, \mathbf{x}) =$ **Positive Euclidean-time support test functions**

$$\mathcal{S}_+ := \{f(\tau, \mathbf{x}) \in \mathcal{S} | f(\tau, \mathbf{x}) = 0, \quad \tau < 0\}$$

Euclidean time reflection

$$\theta f(\tau, \mathbf{x}) := f(-\tau, \mathbf{x})$$

Reconstruction of Quantum Mechanics
Osterwalder and Schrader - C.M.P. 31(1973)83;42(1975)281
Fröhlich - Helv. Phys. Acta. 47(1974)265

Vectors (dense set)

$$B[\phi] = \sum_{j=1}^{N_b} b_j e^{i\phi(f_j)} \qquad C[\phi] = \sum_{k=1}^{N_c} c_k e^{i\phi(g_k)}$$

$$b_j, c_k \in \mathbb{C} \qquad f_j, g_k \in \mathcal{S}_+ \qquad N_b, N_c < \infty$$

Hilbert space inner product

$$\boxed{\langle B|C\rangle = \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j^* c_k Z[g_k - \theta f_j]}$$

Remarks

$B[\phi]$ **"wave functionals"**

$$\langle B|C\rangle = \sum_{j=1}^{N_b}\sum_{k=1}^{N_c} b_j^* c_k Z[g_k - \theta f_j]$$

- **The inner product is the physical (Minkowski) inner product!**
- **The generating functional and test functions are Euclidean!**
- **All integrals are over Euclidean space-time variables!**
- **No analytic continuation is used to calculate the Minkowski scalar product!**

Reflection positivity

(Osterwalder-Schrader Positivity)

$$\boxed{\langle B|B\rangle \geq 0}$$

Property of $Z[f]$ **or** $\{S_n(\mathrm{x}_1, \cdots, \mathrm{x}_n)\}$

$$M_{ij} = Z[f_i - \theta f_j] \geq 0 \qquad \forall \qquad \{f_1, \cdots, f_N\} \in \mathcal{S}_+$$

Operator algebra

(for scattering asymptotic condition)

$$BC[\phi] = B[\phi]C[\phi] = \sum_{j=1}^{N_b}\sum_{k=1}^{N_c} b_j c_k e^{i\phi(g_k+f_j)} = \sum_{n=1}^{N_c} d_n e^{i\phi(h_n)}$$

$$N_d = N_b N_c \qquad h_n = g_k + f_j \qquad d_n = b_j c_k$$

Cluster properties

$$g_{\mathbf{a}}(\tau, \mathbf{x}) := g(\tau, \mathbf{x} - \mathbf{a})$$

$$\boxed{\lim_{|\mathbf{a}| \to \infty} \left(Z[f + g_{\mathbf{a}}] - Z[f] Z[g] \right) \to 0}$$

$$\lim_{|\mathbf{a}| \to \infty} S_{m+n}(f, \cdots, f, g_{\mathbf{a}}, \cdots, g_{\mathbf{a}}) =$$

$$S_m(f, \cdots f) S_n(g, \ldots, g)$$

Operators

$$[T(\beta, \mathbf{a}), B][\phi] := \sum_{j=1}^{N_b} b_j e^{i\phi(f_{j,\beta,\mathbf{a}})}$$

$$f_{n,\beta,\mathbf{a}}(\tau, \mathbf{x}) := f_n(\tau - \beta, \mathbf{x} - \mathbf{a}) \qquad \beta > 0$$

$$[U(R), B][\phi] := \sum_{j=1}^{N_b} b_j e^{i\phi(f_{j,R})}$$

$$f_{j,R}(\tau, \mathbf{x}) := f_j(\tau, R\mathbf{x}) \qquad f_j \in \mathcal{S}_+ > 0$$

$$[W(\hat{\mathbf{n}}, \psi), B][\phi] := \sum_j c_j e^{i\phi(f_{k,\psi,\hat{\mathbf{n}}})}$$

$$f_{j,\phi,\hat{\mathbf{n}}}(\tau, \mathbf{x}) := f_j(\tau', \mathbf{x}') \qquad f_f \in \mathcal{S}_{\chi,+}$$

$$\mathcal{S}_{\chi,+} = \{f \in \mathcal{S}_+ \,|\, f(\tau, \mathbf{x}) = 0 \quad \tan^{-1}(\frac{\tau}{|\mathbf{x}|}) \geq \chi\}$$

$$\psi < \pi/2 - \chi$$

$$\tau' = \tau\cos(\psi) - x_{\hat{\mathbf{n}}}\sin(\psi) \qquad x'_{\hat{\mathbf{n}}} = x_{\hat{\mathbf{n}}}\cos(\psi) + \tau\sin(\psi)$$

Poincaré generators

$$[H, B][\phi] = -\frac{\partial}{\partial \beta} \left(T(\beta, \mathbf{0}) B \right) [\phi]_{\beta=0}$$

$$[\mathbf{P}, B][\phi] = -i \frac{\partial}{\partial \mathbf{a}} \left(T(0, \mathbf{a}) B \right) [\phi]_{\mathbf{a}=0}$$

$$[(\mathbf{K} \cdot \hat{\mathbf{n}}), B][\phi]) = -\frac{\partial}{\partial \psi} \left(W(\hat{\mathbf{n}}, \psi) B \right) [\phi]_{\psi=0}$$

$$[(\mathbf{J} \cdot \hat{\mathbf{n}}), B][\phi]) = -i \frac{\partial}{\partial \psi} \left(R(\hat{\mathbf{n}}, \psi) B \right) [\phi]_{\psi=0}$$

$$[M^2, B][\phi] := \left(\frac{\partial^2}{\partial \beta^2} + \frac{\partial^2}{\partial \mathbf{a}^2} \right) \left(T(\beta, \mathbf{a}) B \right) [\phi]_{\beta=0, \mathbf{a}=0}$$

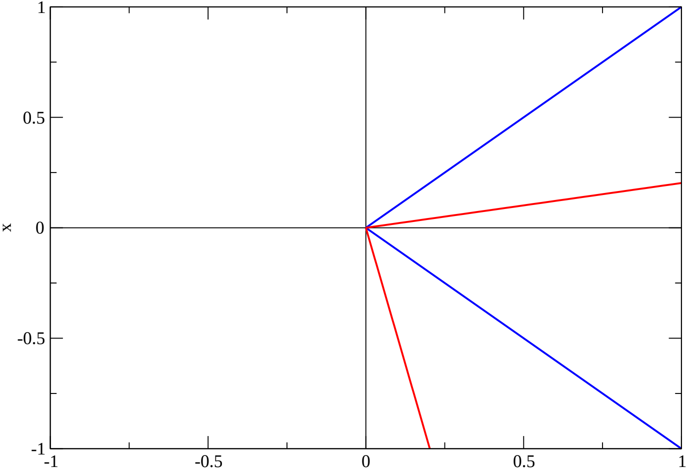
One parameter groups and semigroups

$$T(\beta, \mathbf{a}) = e^{-\beta H + i\mathbf{a}\cdot\mathbf{P}}$$

$$U(R(\hat{\mathbf{n}}, \psi)) = e^{i\mathbf{J}\cdot\hat{\mathbf{n}}\psi}$$

$$W(\hat{\mathbf{n}}, \psi) = e^{\mathbf{K}\cdot\hat{\mathrm{n}}\psi}$$

Domains for local symmetric semigroups (A. Klein, L. Landau, J.F.A. 44(1981)121; Frörlich, Osterwalder, Seiler, Ann. Math 118(1983)461)



$$\{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$$

Self-adjoint (on physical Hilbert space)

$H \geq 0$ **(Follows from reflection positivity)**

Satisfy Poincaré commutation relations

No analytic continuation used!

Given a reflection positive Euclidean Green function or generating function we have:

- **Hilbert space scalar product.**
- **A dense set of normalizable vectors.**
- **A representation of the Poincaré Lie algebra in terms of self-adjoint operators.**

Comments:

- **While analytic continuation is not used, reflection positivity ensures the existence of an analytic continuation.**
- **We can exploit the ability to calculate matrix elements of all operators in a dense set of normalizable states.**
- $e^{-\beta H}$ **and** H **have the same eigenstates.**

Particles: mass eigenstates

Dense set + Gram-Schmidt

$$\Downarrow$$

Orthonormal basis of "wave functionals"

$$B_n[\phi] \qquad \langle B_n | B_m \rangle = \delta_{mn}$$

Solve for eigenstates in point spectrum of M^2

$$(M^2 B_\lambda)[\phi] = \lambda^2 B_\lambda[\phi]$$

$$B_\lambda[\phi] = \sum_n b_n B_n[\phi]$$

$$\sum_n \langle B_m | M^2 | B_n \rangle b_n = \lambda^2 b_m$$

Particles: mass-momentum eigenstates (use translations and Fourier transforms)

$$B_\lambda[\phi] = \sum_n b_n e^{i\phi(f_n)}$$ **(mass eigenfunctional)**

$$B_\lambda(\mathbf{p})[\phi] = \int \frac{d^3 a}{(2\pi)^{3/2}} e^{-i\mathbf{p}\cdot\mathbf{a}} [T(0,\mathbf{a}), B_\lambda][\phi]$$

$$\langle C | B_\lambda(\mathbf{p}) \rangle := \int \frac{d^3 a}{(2\pi)^{3/2}} e^{-i\mathbf{p}\cdot\mathbf{a}} \sum_{j=1}^{N_c} \sum_n b_n c_j^* Z[f_{n,\mathbf{a}} - \theta g_j]$$

Particles: Mass-momentum-spin eigenstates (project on $SU(2)$ irreducible representations)

Normalize $B_\lambda(\mathbf{p})$

$$B_\lambda(\mathbf{p})[\phi] \qquad \langle B_\lambda(\mathbf{p}\,')|B_\lambda(\mathbf{p})\rangle = \delta(\mathbf{p}' - \mathbf{p})$$

$$B_{\lambda,j}(\mathbf{p},\mu)[\phi] := \int_{SU(2)} dR[U(R), B_\lambda(R^{-1}\mathbf{p})][\phi] D^{j*}_{\mu j}(R)$$

$$\langle C|B_{\lambda,j}(\mathbf{p},\mu)\rangle :=$$

$$\int \frac{d^3 a dR}{(2\pi)^{3/2}} e^{-iR^{-1}\mathbf{p}\cdot\mathbf{a}} \sum_{j=1}^{N_c} \sum_n c_j^* b_n Z[f_{n,\mathbf{a},R} - \theta g_j] D^{j*}_{\mu j}(R)$$

$$f_{n,\mathbf{a},R}(\tau,\mathbf{x}) = f_n(\tau, R\mathbf{x} - a)$$

Finite Poincaré transformations of one-particle states $(\lambda \in \sigma_{pp})$

One-particle subspaces are irreducible subspaces with respect to the Poincaré group

$$\Downarrow$$

$$\langle C|U[\Lambda, a]|B_{\lambda,j}(\mathbf{p}, \mu)\rangle =$$

$$\sum_{\mu'=-j}^{j} \int d\mathbf{p}' \langle C|B_{\lambda,j}(\mathbf{p}', \mu')\rangle \mathcal{D}^{\lambda,j}_{\mathbf{p}'\mu';\mathbf{p},\mu}[\Lambda, a]$$

$$\mathcal{D}^{\lambda,j}_{\mathbf{p}'\mu';\mathbf{p},\mu}[\Lambda, a] =$$

$$\delta(\mathbf{\Lambda} p - \mathbf{p}')\sqrt{\frac{\omega_\lambda(\mathbf{p}')}{\omega_\lambda(\mathbf{p})}} e^{-i\omega_\lambda(\mathbf{p}')a^0 - i\mathbf{p}'\cdot\mathbf{a}} D^j_{\mu'\mu}[\Lambda_c^{-1}(\frac{\mathbf{p}'}{\lambda})\Lambda\Lambda_c(\frac{\mathbf{p}}{\lambda})]$$

$$\omega_\lambda(\mathbf{p}) = \sqrt{\lambda^2 + \mathbf{p}\cdot\mathbf{p}} \qquad (\mathbf{p}')^j = \Lambda^j{}_0\omega_\lambda(\mathbf{p}) + \Lambda^j{}_k\mathbf{p}^k$$

Here $\Lambda_c(\frac{\mathbf{p}}{\lambda})$ is a rotationless Lorentz boost.

Scattering

- Time-dependent scattering has been used successfully to treat few-body problems (Kröger, Phys. Reports, 210(1992)46) in non-relativistic quantum mechanics.
- Calculations use wave packets and normalizable states.
- Haag-Ruelle scattering is a natural field theoretic generalization of non-relativistic time-dependent scattering theory (unlike LSZ it uses strong limits).

Haag-Ruelle Scattering (P.R. 112(1958),668, Helv. Phys. Acta. 35(1962),147.)

Construct

$$J : \otimes \mathcal{H}_{\lambda_i , j_i} = \mathcal{H}_f \to \mathcal{H} \qquad U_f[\Lambda, a] = \otimes U_{\lambda_i , j_i}[\Lambda, a]$$

The strong limit exists

$$|\Psi_{\pm}\rangle = \lim_{t\to\pm\infty} e^{iHt} J e^{-iH_f t} |\Psi_{f\pm}\rangle = \Omega_{\pm} |\Psi_{f\pm}\rangle$$

The wave operators satisfy (Ruelle H.P.A. 35(1962)34)

$$U[\Lambda, a]\Omega_{\pm} = \Omega_{\pm} U_f[\Lambda, a]$$

Structure of J

$$B_{\lambda,j}(\mathbf{p},\mu)|0\rangle = |(\lambda,j)\mathbf{p},\mu\rangle$$

Creates one-particle state out of the vacuum

$$\int J_i(\mathbf{p}_i,\mu_i)f(\mathbf{p},\mu)d\mathbf{p} =$$

$$\int (-i\omega_\lambda(\mathbf{p})B_{\lambda,j}(\mathbf{p},\mu) - i[H, B_{\lambda,j}(\mathbf{p},\mu)])f(\mathbf{p},\mu)d\mathbf{p}$$

$(\cdots)$ selects "creation part" of $B_{\lambda,j}(\mathbf{p},\mu)$

$$J(\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_n,\mu_n) = \prod J_i(\mathbf{p}_i,\mu_i)|0\rangle$$

Wave functional representation

$$B_{\lambda,j}(\mathbf{p},\mu) \to B_{\lambda,j}(\mathbf{p},\mu)[\phi]$$

$$\int J_i(\mathbf{p}_i,\mu_i)[\phi] f(\mathbf{p},\mu) d\mathbf{p} =$$

$$\int \left(-i\omega_\lambda(\mathbf{p}) B_{\lambda,j}(\mathbf{p},\mu)[\phi] - i[H, B_{\lambda,j}(\mathbf{p},\mu)][\phi]\right) f(\mathbf{p},\mu) d\mathbf{p}$$

$$J[\phi] = \prod J_i(\mathbf{p}_i,\mu_i)[\phi]$$

Two-space Scattering

$$\lim_{t\to\pm\infty} \|(e^{-iHt}|\Psi_\pm\rangle - Je^{-iH_f t}|\Psi_{f\pm}\rangle)\| = 0$$

$$|\Psi_\pm\rangle = \lim_{t\to\pm\infty} e^{iHt} J e^{-iH_f t}|\Psi_{f\pm}\rangle = \Omega_\pm(H, J, H_f)|\Psi_f\rangle$$

$$S_{fi} = \langle\Psi_+|\Psi_-\rangle = \langle\Psi_{f+}|\Omega_+^\dagger(H, J, H_f)\Omega_-(H, J, H_f)|\Psi_{f-}\rangle$$

$$= \lim_{t\to\infty}\langle\Psi_{f+}|e^{iH_f t} J^\dagger e^{-2iHt} J e^{iH_f t}|\Psi_{f-}\rangle$$

Kato-Birman invariance principle

$$\lim_{t\to\pm\infty} \|(e^{-iHt}|\Psi_\pm\rangle - Je^{-iH_f t}|\Psi_{f\pm}\rangle)\| = 0$$

$$\Updownarrow$$

$$\lim_{n\to\pm\infty} \|(e^{ine^{-\beta H}}|\Psi_\pm\rangle - Je^{ine^{-\beta H_f}}|\Psi_{f\pm}\rangle)\| = 0$$

Provides a possible computational strategy

$$\langle\Psi_{f+}|S|\Psi_{f-}\rangle \approx \langle\Psi_{f+}|e^{-ine^{-\beta H_f}} J^\dagger e^{2ine^{-\beta H}} Je^{-ine^{-\beta H_f}}|\Psi_{f-}\rangle$$

$$e^{2inx} \approx \sum_{m=0}^{N(n)} c_m(n)x^m \quad \to \quad e^{2ine^{-\beta H}} \approx \sum_{m=0}^{N(n)} c_m(n)e^{-m\beta H}$$

Convergence is uniform for each fixed n!

Matrix elements

$$\boxed{\langle B|e^{-m\beta H}|C\rangle = \sum_{j=1}^{N_b}\sum_{k=1}^{N_c} b_j^* c_k Z[g_{k,m\beta,\mathbf{0}} - \theta f_j]}$$

$$C[\phi] = Je^{-ine^{-\beta H_f}}|\Psi_{f-}\rangle[\phi]$$

$$B[\phi] = Je^{ine^{-\beta H_f}}|\Psi_{f-}\rangle[\phi]$$

Computable by quadrature in terms of $Z[f]$ or $\{S_n\}$

Scattering in Euclidean space

- The results are standard Minkowski space results expressed in a representation where the Minkowski scalar product is evaluated in terms of a Euclidean generating functional.

- The time limits in the scattering theory are strong limits (compared with the weak limits used in LSZ scattering)

- The Haag-Ruelle scattering theory does not distinguish elementary and composite asymptotic states.

- Explicit representations of the Poincaré group exist for the bound and scattering states.

$$U[\Lambda, a]\Omega_{\pm} = \Omega_{\pm} U_f[\Lambda, a]$$

Maiani-Testa No-Go Theorem (P.L.B. 245(1990)585)

$$\langle 0|\phi_\pi(p_1)\phi_\pi(p_2)J(0)|0\rangle$$

- **Use LSZ interpolating fields for pions. Field creates more than 1-pion states from vacuum.**
- **Uses** $\beta_1 \gg \beta_2 \gg 0$
- **Uses Euclidean correlation functions.**

This approach

- **Uses Haag-Ruelle fields. Fields create only 1 pion states from vacuum - products approach exact scattering states in strong limit.**
- β **is a fixed adjustable parameter (**$H \leftrightarrow e^{-\beta H}$**).**
- **Uses Minkowski scalar product.**
- **Calculations require wave packets, one-body solutions; no singularities, no analytic continuation.**

Summary of formal results
Given a reflection positive generating functional

$$\Downarrow$$

- **Hilbert-space scalar product,** $\{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$
- **Single-particle states**
- **Scattering states,** S**-matrix elements.**
- **Finite Poincaré transformations on single-particle states and scattering states.**

Comments

- Constructing a reflection positive Euclidean invariant generating functional is almost equivalent to constructing a non-trivial field theory (this must be relaxed for model applications).

- Practical calculations use a weakened form of reflection positivity (limited permutation symmetry).

- Full permutation symmetry = locality.

- Osterwalder-Schrader reconstruction of relativistic quantum mechanics does not require locality.

Green function approach - limited reflection positivity

$$Z[f] = \sum_n \frac{i^n}{n!} S_n(f, \cdots, f)$$

$$\mathsf{x} := (\tau, \mathbf{x}) \qquad \theta\mathsf{x} = (-\tau, \mathbf{x}) \qquad f(\mathsf{x}_1, \cdots, \mathsf{x}_n) \in \mathcal{S}_+$$

$$\int d^4\mathsf{x}_1 \cdots d^4\mathsf{x}_4 f_2^*(\theta\mathsf{x}_2, \theta\mathsf{x}_1) S_4(\mathsf{x}_1, \mathsf{x}_2; \mathsf{x}_3, \mathsf{x}_4) f_2(\mathsf{x}_3, \mathsf{x}_4) \geq 0$$

$$\int d^4\mathsf{x}_1 d^4\mathsf{x}_2 f_1^*(\theta\mathsf{x}_1) S_2(\mathsf{x}_1; \mathsf{x}_2) f_1(\mathsf{x}_2) \geq 0.$$

$$S_4(\mathsf{x}_1, \mathsf{x}_2; \mathsf{x}_3, \mathsf{x}_4) = S_4(\mathsf{x}_2, \mathsf{x}_1; \mathsf{x}_3, \mathsf{x}_4) = S_4(\mathsf{x}_1, \mathsf{x}_2; \mathsf{x}_4, \mathsf{x}_3)$$

Green function representation

$$Z \to S = \begin{pmatrix} S_2(\mathrm{x};\mathrm{y}) & S_3(\mathrm{x};\mathrm{y}_1,\mathrm{y}_2) & \cdots \\ S_3(\mathrm{x}_1,\mathrm{x}_2;\mathrm{y}) & S_4(\mathrm{x}_1,\mathrm{x}_2;\mathrm{y}_1,\mathrm{y}_2) & \vdots \\ \vdots & \cdots & \ddots \end{pmatrix}$$

$$B[\phi] \to \begin{pmatrix} f_1(\mathrm{x}_{11}) \\ f_2(\mathrm{x}_{21},\mathrm{x}_{22}) \\ \vdots \end{pmatrix}$$

$$\langle B|C\rangle = (\theta f_B, Sf_C)_e$$

$$\langle \mathsf{x}|H|f\rangle :=$$
$$\{0, \frac{\partial}{\partial \mathsf{x}_{11}^{0}} f_1(\mathsf{x}_{11}), \left(\frac{\partial}{\partial \mathsf{x}_{21}^{0}} + \frac{\partial}{\partial \mathsf{x}_{22}^{0}}\right) f_2(\mathsf{x}_{21}, \mathsf{x}_{22}), \cdots\}$$

$$\langle \mathsf{x}|\mathbf{P}|f\rangle :=$$
$$\{0, -i\frac{\partial}{\partial \vec{\mathsf{x}_{11}}} f_1(\mathsf{x}_{11}), -i\left(\frac{\partial}{\partial \vec{\mathsf{x}_{21}}} + \frac{\partial}{\partial \vec{\mathsf{x}_{22}}}\right) f_2(\mathsf{x}_{21}, \mathsf{x}_{22}), \cdots\}$$

$$\langle \mathsf{x}|\mathbf{J}|f\rangle :=$$
$$\{0, -i\vec{\mathsf{x}}_{11} \times \frac{\partial}{\partial \vec{\mathsf{x}}_{11}} f_1(\mathsf{x}_{11}), -i\left(\vec{\mathsf{x}}_{21} \times \frac{\partial}{\partial \vec{\mathsf{x}}_{21}} + \vec{\mathsf{x}}_{22} \times \frac{\partial}{\partial \vec{\mathsf{x}}_{22}}\right) f_2(\mathsf{x}_{21}, \mathsf{x}_{22}), \cdots\}$$

$$\langle \mathsf{x}|\mathbf{K}|f\rangle :=$$
$$\{0, \left(\vec{\mathsf{x}}_{11}\frac{\partial}{\partial \mathsf{x}_{11}^{0}} - \mathsf{x}_{11}^{0}\frac{\partial}{\partial \vec{\mathsf{x}}_{11}}\right) f_1(\mathsf{x}_{11}),$$
$$\left(\vec{\mathsf{x}}_{21}\frac{\partial}{\partial \mathsf{x}_{21}^{0}} - \mathsf{x}_{21}^{0}\frac{\partial}{\partial \vec{\mathsf{x}}_{21}} + \vec{\mathsf{x}}_{22}\frac{\partial}{\partial \mathsf{x}_{22}^{0}} - \mathsf{x}_{22}^{0}\frac{\partial}{\partial \vec{\mathsf{x}}_{22}}\right) f_2(\mathsf{x}_{21}, \mathsf{x}_{22}), \cdots\}.$$

Modifications for spin

$$\mathbf{J}: \qquad \left(-i\vec{\mathsf{x}}_{11} \times \frac{\partial}{\partial \vec{\mathsf{x}}_{11}}\right) \to \left(-i\vec{\mathsf{x}}_{11} \times \frac{\partial}{\partial \vec{\mathsf{x}}_{11}} + \vec{\Sigma}\right)$$

$$\mathbf{K}: \qquad \left(\vec{\mathsf{x}}_{11}\frac{\partial}{\partial \mathsf{x}^0_{11}} - \mathsf{x}^0_{11}\frac{\partial}{\partial \vec{\mathsf{x}}_{11}}\right) \to \left(\vec{\mathsf{x}}_{11}\frac{\partial}{\partial \mathsf{x}^0_{11}} - \mathsf{x}^0_{11}\frac{\partial}{\partial \vec{\mathsf{x}}_{11}} + \vec{\mathcal{B}}\right)$$

where

$$\vec{\Sigma} = i\vec{\nabla}_\phi D(e^{\frac{-i}{2}\vec{\sigma}\cdot\vec{\phi}}, e^{\frac{i}{2}\vec{\sigma}^t\cdot\vec{\phi}})_{aa'}$$

and

$$\vec{\mathcal{B}} = \vec{\nabla}_\rho D(e^{\frac{-i}{2}\vec{\sigma}\cdot\vec{\rho}}, e^{\frac{-i}{2}\vec{\sigma}^t\cdot\vec{\rho}})_{aa'}$$

and $D(g_1, g_2)$ is a representation of $SU(2) \times SU(2)$.

Two- body scattering

$$S_2(\mathrm{x};\mathrm{y}) \qquad K(\mathrm{x}_1,\mathrm{x}_2;\mathrm{y}_2,\mathrm{y}_1)$$

$$S_0 = S_2(\mathrm{x}_1;\mathrm{y}_1)S_2(\mathrm{x}_2;\mathrm{y}_2)$$

$$S_4 = S_0 + S_0 K S_4$$

$$S = \begin{pmatrix} S_2(\mathrm{x};\mathrm{y}) & 0 \\ 0 & S_4(\mathrm{x}_1,\mathrm{x}_2;\mathrm{y}_1,\mathrm{y}_2) \end{pmatrix}$$

$$\langle C|B\rangle = (\theta g_1, S_2 f_1)_e + (\theta g_2, S_4 f_2)_e$$

Particles - scalar
(case of free S_2)

$$(f, \theta S_2 f)_e$$

$$= \frac{1}{(2\pi)^4} \int d^4x d^4y d^4p f(x) \frac{e^{ip\cdot(\theta x - y)}}{p^2 + m^2} f(y)$$

$$= \frac{1}{(2\pi)^4} \int d^4x d^4y d^4p f(x) \frac{e^{-ip_0\cdot(x_0+y_0)+i\vec{p}\cdot(\vec{x}-\vec{y})}}{(p^0 + i\omega_m(\vec{p}\,))(p^0 - i\omega_m(\vec{p}\,))} f(y)$$

$$= \int d^3p \frac{|g(\vec{p})|^2}{2\omega_m(\vec{p}\,)} \geq 0$$

where

$$g(\vec{p}) := \frac{1}{(2\pi)^{3/2}} \int d^4y f(y) e^{-\omega_m(\vec{p})y_0 - i\vec{p}\cdot\vec{y}}.$$

Particles - fermions
(Euclidean time reversal has a spinor component)

$$(f, \theta\gamma^0 S_2 f)_e$$

$$= \frac{1}{(2\pi)^4} \int d^4\mathrm{x} d^4\mathrm{y} d^4\mathrm{p} f(\mathrm{x})\, e^{i\mathrm{p}\cdot(\theta\mathrm{x}-\mathrm{y})} \gamma^0 \frac{m - p\cdot\gamma_e}{\mathrm{p}^2 + m^2} f(\mathrm{y})$$

$$= \int g^\dagger(\vec{\mathrm{p}}) \frac{\Lambda_+(p)}{(2\pi)^3} g(\vec{\mathrm{p}}) d^3\mathrm{p}$$

where

$$\Lambda_+(p) := \frac{\omega_m(\vec{p}) + \gamma^0\vec{\gamma}\cdot\vec{p} - m\gamma^0}{2\omega_m(\vec{p})}$$

Scattering in Euclidean space

Approximation 1: Use sharply peaked (in momentum) normalizable states to approximate plane-wave on-shell transition matrix elements.

$$\langle \Psi_{f+} | S | \Psi_{f-} \rangle = \langle \Psi_{f+} | \Psi_{f-} \rangle - 2\pi i \langle \Psi_{f+} | \delta(E_+ - E_-) T | \Psi_{f-} \rangle$$

$$\langle \mathbf{p}'_1, \mu'_1, \mathbf{p}'_2, \mu'_2 | T | \mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2 \rangle \approx \frac{\langle \Psi_{f+} | S | \Psi_{f-} \rangle - \delta_{ab} \langle \Psi_{f+} | \Psi_{f-} \rangle}{2\pi i \langle \Psi_{f+} | \delta(E_+ - E_-) | \Psi_{f-} \rangle}$$

Scattering injection operators (N=2)

Approximation 2: **Calculate** $\psi_\lambda(\mathbf{x}, \tau; \mathbf{p}, \mu)$

- $\psi_\lambda(\mathbf{x}, \tau; \mathbf{p}, \mu)$**: eigenstate of** $M^2, \mathbf{P}, \mathbf{j}^2, j_z$ **in one-body Hilbert space generated by** S_2**.**

$$J_i(\mathbf{x}, \tau; \mathbf{p}, \mu) := (-i\omega_\lambda(\mathbf{p}) + i\frac{\partial}{\partial\tau})\psi_\lambda(\mathbf{x}, \tau; \mathbf{p}, \mu)$$

$$J(\mathbf{x}_1, \tau_1, \mathbf{x}_2, \tau_2; \mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2) = J_1(\mathbf{x}_1, \tau_1; \mathbf{p}_1, \mu_1) J_2(\mathbf{x}_2, \tau_2; \mathbf{p}_2, \mu_2)$$

Scattering in Euclidean space

Use time-dependent scattering to calculate S matrix elements in normalizable states.

Use Kato-Birman invariance principle to express S in terms of $e^{-\beta H}$.

$$\langle \Psi_{f+} | S | \Psi_{f-} \rangle$$

$$= \lim_{t \to \infty} \langle \Psi_{f+} | e^{iH_f t} J^\dagger e^{-2iHt} J e^{iH_f t} | \Psi_{f-} \rangle$$

$$= \lim_{n \to \infty} \langle \Psi_{f+} | e^{-ine^{-\beta H_f}} J^\dagger e^{2ine^{-\beta H}} J e^{-ine^{-\beta H_f}} | \Psi_{f-} \rangle$$

Scattering in Euclidean space

Approximation 3: **Replace n by large fixed n.**

$$\langle \Psi_{f+} | S | \Psi_{f-} \rangle$$
$$\approx \langle \Psi_{f+} | e^{-ine^{-\beta H_f}} J^{\dagger} e^{2ine^{-\beta H}} J e^{-ine^{-\beta H_f}} | \Psi_{f-} \rangle$$

Approximation 4: Uniform polynomial approximation

$$e^{2ine^{-\beta H}} \approx \sum c_m(n)(e^{-\beta mH})$$

note $\sigma(e^{-\beta H}) \in [0,1]$ **(compact)**

$$e^{2inx} \approx \sum c_m(n)x^m \qquad x \rightarrow e^{-\beta H}$$

$$|e^{2inx} - \sum c_m(n)x^m| < \epsilon(n) \qquad \forall x \in [0,1]$$

$$\Downarrow$$

$$\|[e^{2ine^{-\beta H}} - \sum c_m(n)(e^{-\beta mH})]|\psi\rangle\| < \epsilon(n)\||\psi\rangle\|$$

$$f(x) \approx \frac{1}{2} c_0 T_0(x) + \sum_{k=1}^{N} c_k T_k(x)$$

$$c_j = \frac{2}{N+1} \sum_{k=1}^{N} f(\cos(\frac{2k-1}{N+1}\frac{\pi}{2})) \cos(j\frac{2k-1}{N+1}\frac{\pi}{2})$$

$$f(e^{-\beta H}) \approx \frac{1}{2} c_0 T_0(e^{-\beta H}) + \sum_{k=1}^{N} c_k T_k(e^{-\beta H})$$

$$f(x) = e^{2inx}$$

$$|e^{2inx} - P_N(x)| < 2\frac{n^{N+1}}{(N+1)!}$$

$$\langle \Psi_{f+} | S | \Psi_{f-} \rangle \approx$$

$$= \sum c_m(n) \langle \Psi_{f+} | e^{-ine^{-\beta H_f}} J^\dagger (e^{-\beta m H}) J e^{-ine^{-\beta H_f}} | \Psi_{f-} \rangle$$

Each approximation converges - the order of the approximations is important $(1) \rightarrow (2) \rightarrow (3) \rightarrow (4)$**.**

Three and four-body phenomenology and cluster properties

$$S^{-1}(123) =$$

$$S^{-1}(1)S^{-1}(2)S^{-1}(3) - K(12)S^{-1}(3) -$$

$$K(23)S^{-1}(1) - K(31)S^{-1}(2) - K(123)$$

$$S^{-1}(1234) =$$

$$S^{-1}(1)S^{-1}(2)S^{-1}(3)S^{-1}(4) + K(12)K(34) + K(13)K(24) +$$

$$K(14)K(23) - K(12)S^{-1}(3)S^{-1}(4) - K(13)S^{-1}(2)S^{-1}(4) +$$

$$-K(14)S^{-1}(2)S^{-1}(3) - K(23)S^{-1}(1)S^{-1}(4) +$$

$$-K(24)S^{-1}(1)S^{-1}(3) - K(34)S^{-1}(1)S^{-1}(2) + \cdots$$

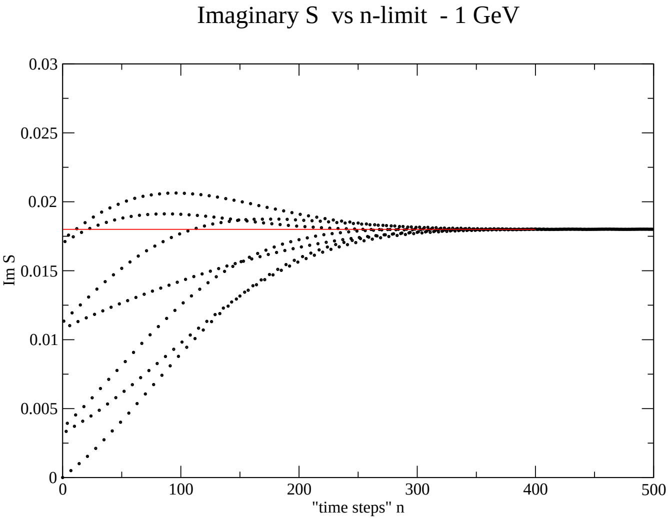
Test of method: non-relativistic separable potential (solvable so all approximations can be tested)

$$H = \frac{\mathbf{k}^2}{m} - |g\rangle\lambda\langle g|$$

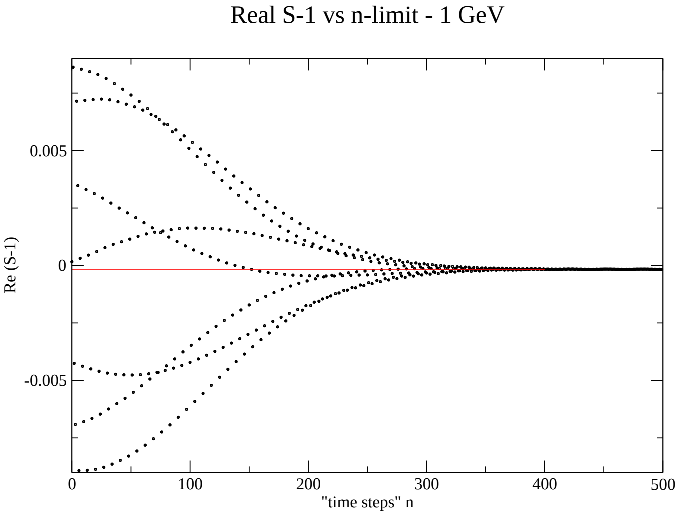
$$\langle \mathbf{k}|g\rangle = \frac{1}{m_\pi^2 + \mathbf{k}^2}$$

calculate $\langle \mathbf{k}'|T(k^+)|\mathbf{k}\rangle$ using matrix elements of $e^{-\beta H}$ in normalizable states.

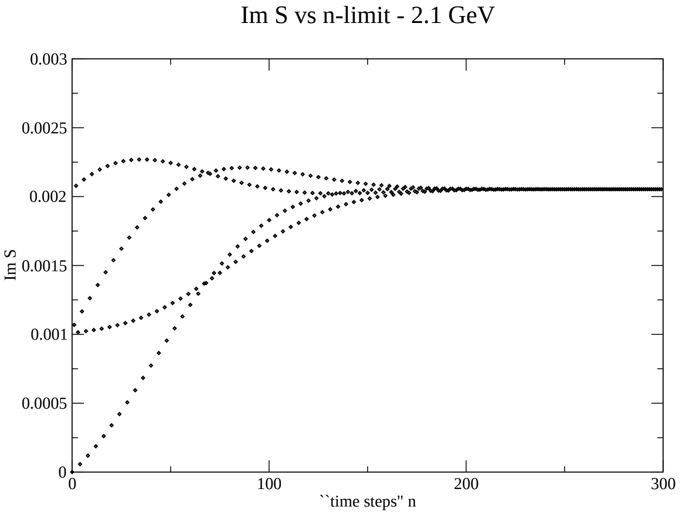
Approximation 3:



Approximation 3:



Approximation 3:

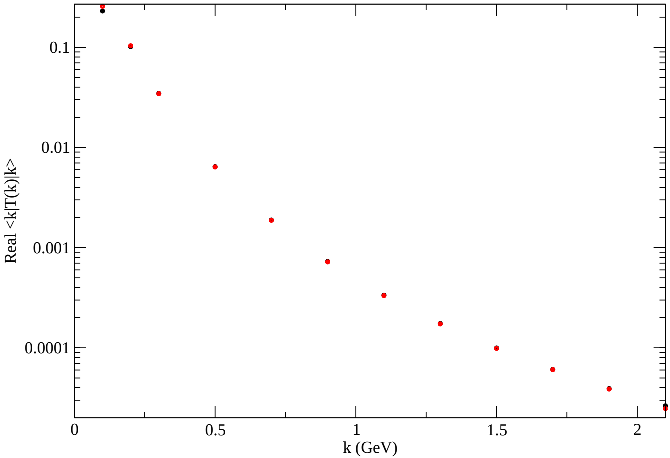


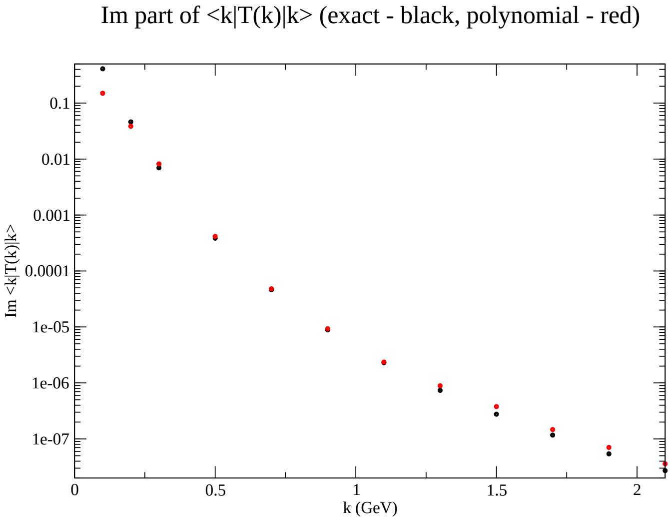
Approximation 4:

Degree 300 polynomial compared to e^{-inx}, $n = 220$

x	$\Delta \cos(nx)$	$\Delta \sin(nx)$
0	4.44089×10^{-16}	8.32667×10^{-15}
0.1	2.35367×10^{-14}	1.46966×10^{-14}
0.2	5.55112×10^{-16}	3.6797×10^{-14}
0.3	3.84137×10^{-14}	1.80689×10^{-14}
0.4	1.72085×10^{-14}	1.32672×10^{-14}
0.5	2.77556×10^{-15}	2.93793×10^{-14}
0.6	6.66134×10^{-16}	3.33344×10^{-14}
0.7	8.54872×10^{-15}	2.50355×10^{-14}
0.8	1.02141×10^{-14}	1.35447×10^{-14}
0.9	1.22125×10^{-15}	2.72282×10^{-14}
1	4.88498×10^{-15}	6.61415×10^{-14}

Real part of <k|T(k)|k> (exact - black, polynomial - red)





Im part of <k|T(k)|k> (exact - black, polynomial - red)
Im <k|T(k)|k>
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
0
0.5
1
1.5
2
k (GeV)

Conclusions - Outlook

- Phenomenology based on model reflection-positive Euclidean Green functions can be used to formulate a relativistic quantum theory.
- Analytic continuation is not necessary.
- The Poincaré invariant S-matrix. Cluster properties are easily satisfied for fixed N.
- Models can be motivated by field-theory based phenomenology.
- A test using an exactly solvable model suggests that GeV scale scattering calculations are possible in this framework.

Future directions

- **Euclidean BS free S_2.**
- **Euclidean BS S_2 with continuous Lehmann weight.**
- **Nakanishi representation and reflection positivity.**
- **Current matrix elements.**

Thanks!

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