Universality in Few- and Many-Body Systems

Lucas Platter
Institute for Nuclear Theory
University of Washington

Collaborators: Braaten, Hammer, Kang, Phillips, Ji

Ultracold Gases

- the scattering length ***a*** is a variable parameter in experiments with ultracold gases (Feshbach resonances)
- study BEC-BCS crossover
- system similar to dilute neutron matter (less scales $\Rightarrow$ cleaner)
- study few-body physics in systems with variable scattering length

Low Energy Physics

- for a finite range potential t-matrix can be written as

$$t(k) \sim \frac{1}{k \cot \delta - ik}$$

- for low energies k cot(δ) can be expanded in powers of k⇒ effective range expansion

$$k \cot \delta = -\frac{1}{a} + \frac{r}{2} k^2 + ...$$

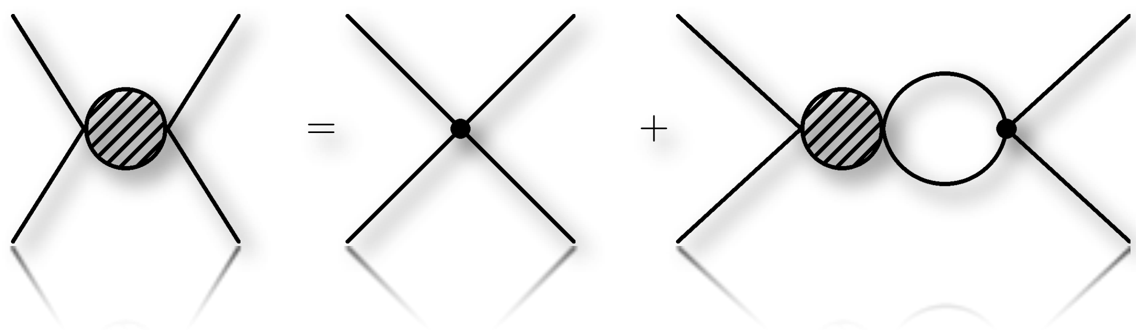
The LO Lagrangian

- the system is described by the Lagrangian

$$\mathcal{L} = \sum_\sigma \psi_\sigma^\dagger (i\partial_t - \frac{\nabla^2}{2m})\psi_\sigma - \frac{g(\Lambda)}{m}\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1$$

- the coupling $g(\Lambda)$ is determined by a
- 2-body amplitude is $\mathcal{A} = \frac{4\pi}{m}\frac{1}{-1/a + \sqrt{-mE}}$

➡ $g(\Lambda) = \frac{4\pi a}{1 - 2a\Lambda/\pi}$ from bubble sum



Continuus Scaling Symmetry

- ➡ rescale parameters w/ real parameter λ

$$a \rightarrow \lambda a \quad E \rightarrow \lambda^{-2} E$$

- ➡ observables scale with powers of λ suggested by NDA, e.g.

$$B_D(\lambda a) = \lambda^{-2} B_D(a)$$

- ➡ this is a conformal theory in the unitary limit, ie $a \rightarrow \infty$

The Bertsch Challenge

- What is the gs energy of a two component fermion system in the unitary limit (1/a=0 ⇒ no remaining scales)

$$E_{\rm gs} = \xi \cdot \frac{3}{5}\frac{k_F}{2m}N$$

- **analytical approaches**
  - ★ ε-expansion (Nishida & Son)
- **numerical approaches**
  - ★ Monte Carlo, Lattice (Carlson, Gezerlis, Lee, Bulgac, Drut, ...)

At finite *a*

- Shina Tan derived a number of relations for finite scattering length (Tan 2009)

Energy Relation:

$$E = \sum_\sigma \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{2m} \left(\rho_\sigma(\mathbf{k}) - \frac{C}{k^4} \right) + \frac{C}{4\pi m a} + \langle V \rangle$$

Adiabatic Relation:

$$\frac{d}{da} E = \frac{1}{4\pi m a^2} \, C$$

and others ....

What is C?

What is C?

- Tan Relations contain the extensive quantity C

$$C = \int \mathrm{d}^3 r \; \mathcal{C}(r)$$

What is C?

- Tan Relations contain the extensive quantity C

$$C = \int \mathrm{d}^3 r \; \mathcal{C}(r)$$

- Contact C can be defined via momentum density tail

$$\rho(k) \to \frac{C}{k^4}$$

Examples:

- The contact density is known in certain cases. Consider the unpolarized, T=0, Fermi gas:
- use adiabatic relation $$\frac{d}{da}E = \frac{1}{4\pi m a^2}\,C$$
- BEC limit:

$$\mathcal{E} \approx -\frac{1}{ma^2}\frac{n}{2} \longrightarrow \mathcal{C} \approx \frac{4\pi n}{a}$$

Examples:

- The contact density is known in certain cases. Consider the unpolarized, T=0, Fermi gas:

- use adiabatic relation $$\frac{d}{da}E = \frac{1}{4\pi m a^2}\,C$$

- unitary limit:

$$\mathcal{E} \approx \frac{k_F^5}{10\pi^2 m}\left(\xi - \frac{\zeta}{k_f a}\right) \longrightarrow \mathcal{C} \approx \frac{2\zeta}{5\pi}\left(3\pi^2 n\right)^{4/3}$$

Examples:

- The contact density is known in certain cases. Consider the unpolarized, T=0, Fermi gas:

- use adiabatic relation $$\frac{d}{da}E = \frac{1}{4\pi m a^2} C$$

- BCS limit:

$$\mathcal{E} \approx \frac{k_F^5}{10\pi^2 m} \quad 1 + \frac{10}{9\pi} k_F a \quad \longrightarrow \mathcal{C} \approx 4\pi^2 n^2 a^2$$

The OPE for Cold Atoms

- Rederive Tan relations with the operator product expansion (Braaten & Platter 2009)
- express non-local operator through sum of local operators multiplied with Wilson coefficients (Wilson 1964 & 1969, Kadanoff 1969, formal proofs: Brandt 1967, Zimmerman 1972)

$$\psi_\sigma^\dagger(\mathbf{R} - \tfrac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R} + \tfrac{1}{2}\mathbf{r}) = \sum_n C_{\sigma,n}(\mathbf{r})\mathcal{O}_n(\mathbf{R})$$

- operators at equal time

- **The OPE is an operator identity**
  - ➡ we can match the Wilson coefficients in any state
  - ➡ determine the Wilson coefficients of an n-body operator in an n-body state
  - ➡ calculate momentum distribution **exactly** for the two-body system

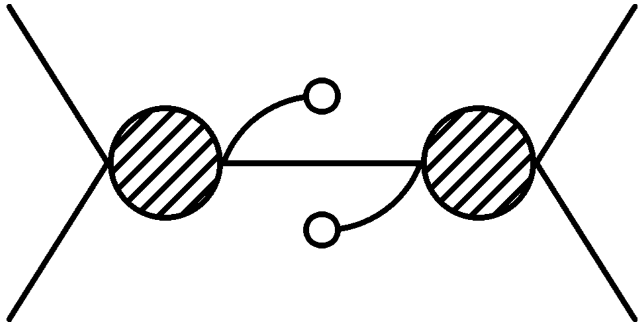
$$\langle X|\mathcal{O}_A(\mathbf{R}+\tfrac{1}{2}\mathbf{r},T+\tfrac{1}{2}t)\ \mathcal{O}_B(\mathbf{R}-\tfrac{1}{2}\mathbf{r},T-\tfrac{1}{2}t)|X\rangle = \\ = \langle X|\sum_C W_C(\mathbf{r},t)\mathcal{O}_C(\mathbf{R},T)|X\rangle$$

Consider 2→2 Scattering

- Elastic scattering with rel. momentum p

$$\langle \psi_\sigma^\dagger(-\tfrac{1}{2}\mathbf{r})\psi_\sigma(+\tfrac{1}{2}\mathbf{r})\rangle_{\pm\mathbf{p}} = \frac{im^2 e^{ipr}}{8\pi p}\mathcal{A}^2(\frac{p^2}{m}) + \dots$$

- this involves evaluating diagrams such as

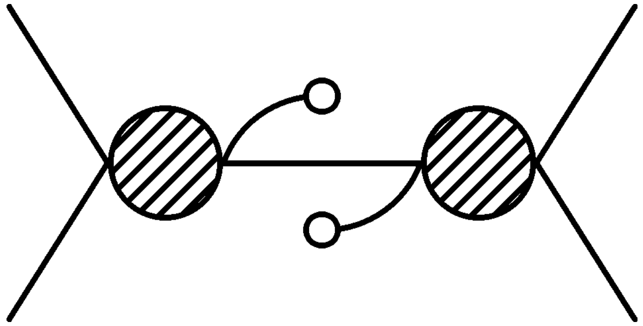


Consider 2→2 Scattering

- Elastic scattering with rel. momentum p

$$\langle \psi_\sigma^\dagger(-\tfrac{1}{2}\mathbf{r})\psi_\sigma(+\tfrac{1}{2}\mathbf{r})\rangle_{\pm\mathbf{p}} = \frac{im^2 e^{ipr}}{8\pi p}\mathcal{A}^2(\frac{p^2}{m}) + \ldots$$

- this involves evaluating diagrams such as



- expand the exponential

$$\longrightarrow \quad \frac{im^2}{8\pi p}\mathcal{A}(\frac{p^2}{m})\,(1 + ipr + \ldots)$$

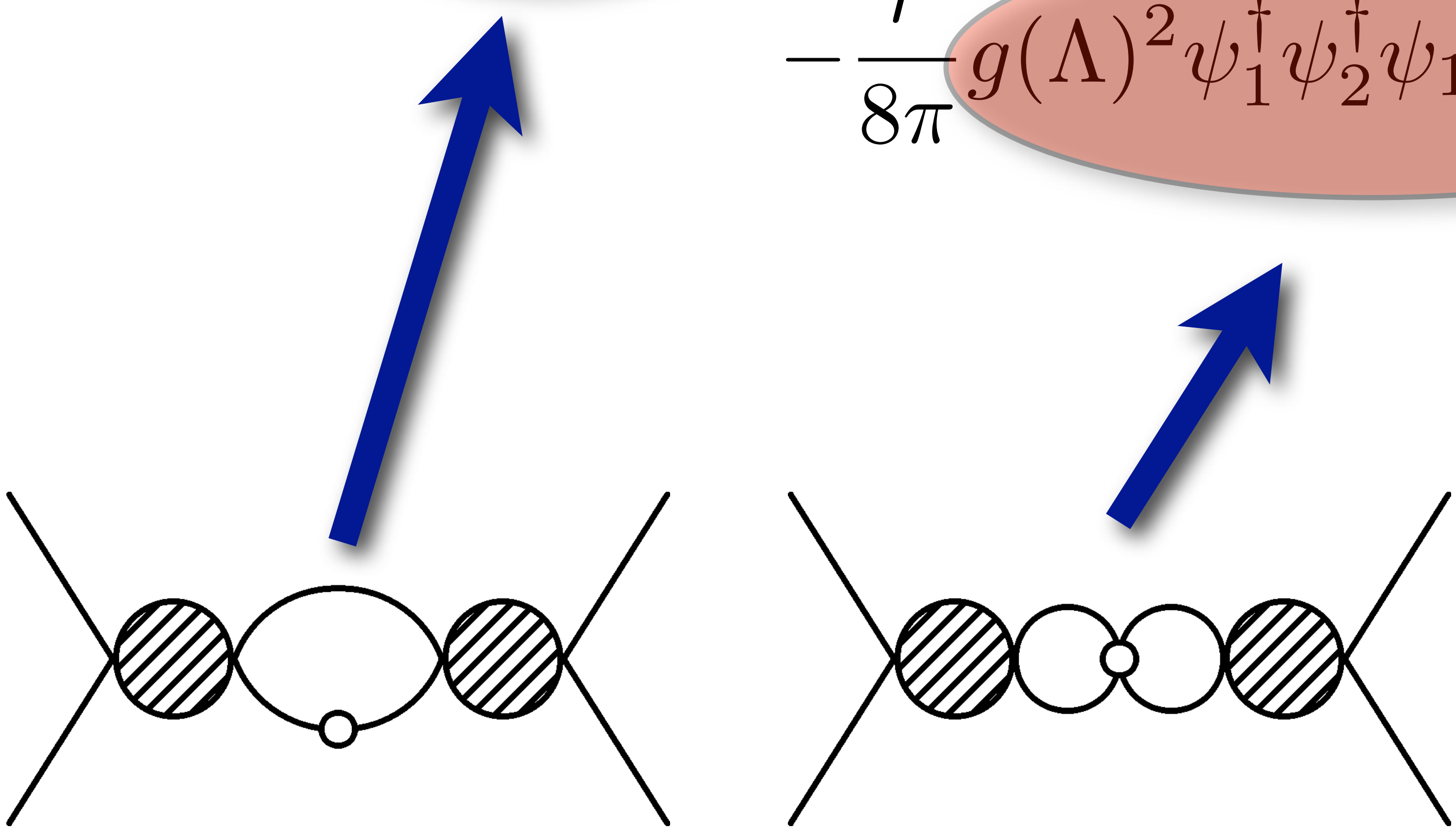
- Matching to the exact solution gives

$$
\begin{aligned}
\psi_\sigma^\dagger(\mathbf{R} - \tfrac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R} + \tfrac{1}{2}\mathbf{r}) \quad &= \\
&= \psi_\sigma^\dagger\psi_\sigma(\mathbf{R}) + \tfrac{1}{2}\mathbf{r}\cdot\left[\psi_\sigma^\dagger\nabla\psi_\sigma(\mathbf{R}) - \nabla\psi_\sigma^\dagger\psi_\sigma(\mathbf{R})\right] \\
&\qquad -\frac{r}{8\pi}g(\Lambda)^2\psi_1^\dagger\psi_2^\dagger\psi_1\psi_2^{(\Lambda)}(\mathbf{R}) + \ldots
\end{aligned}
$$

Scaling dimension of operator \int d^4 x <O(x) O(0)> -->p^{2d-5}

- Matching to the exact solution gives

$$\begin{aligned}\psi^\dagger_\sigma(\mathbf{R}-\tfrac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R}+\tfrac{1}{2}\mathbf{r}) \quad &= \\ &= \psi^\dagger_\sigma\psi_\sigma(\mathbf{R}) + \tfrac{1}{2}\mathbf{r}\cdot\left[\psi^\dagger_\sigma\nabla\psi_\sigma(\mathbf{R}) - \nabla\psi^\dagger_\sigma\psi_\sigma(\mathbf{R})\right] \\ &\quad - \frac{r}{8\pi} g(\Lambda)^2\psi^\dagger_1\psi^\dagger_2\psi_1\psi_2^{(\Lambda)}(\mathbf{R}) + \dots\end{aligned}$$



Scaling dimension of operator \int d^4 x <O(x) O(0)> -->p^{2d-5}

- Matching to the exact solution gives

$$\begin{aligned}\psi_\sigma^\dagger(\mathbf{R}-\tfrac{1}{2}\mathbf{r})\psi_\sigma(\mathbf{R}+\tfrac{1}{2}\mathbf{r}) &= \\ &= \psi_\sigma^\dagger\psi_\sigma(\mathbf{R}) + \tfrac{1}{2}\mathbf{r}\cdot\left[\psi_\sigma^\dagger\nabla\psi_\sigma(\mathbf{R}) - \nabla\psi_\sigma^\dagger\psi_\sigma(\mathbf{R})\right] \\ &\quad -\frac{r}{8\pi}g(\Lambda)^2\psi_1^\dagger\psi_2^\dagger\psi_1\psi_2^{(\Lambda)}(\mathbf{R}) + \ldots\end{aligned}$$

- For a state X the contact C is therefore

$$C = \int \mathrm{d}^3 r \langle X | g(\Lambda)^2 \psi_1^\dagger\psi_2^\dagger\psi_1\psi_2(\mathbf{r}) | X\rangle$$

Scaling dimension of operator \int d^4 x <O(x) O(0)> -->p^{2d-5}

Rederive Tan Relations

Energy Relation:

- Hamiltonian density can be rewritten as

$$\mathcal{H} = \Big(\sum_{\sigma} \frac{1}{2m} \nabla \psi_\sigma^\dagger \cdot \nabla \psi_\sigma^{(\Lambda)} - \frac{\Lambda}{2\pi^2 m} g^2 \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2 \Big)$$

$$+ \frac{1}{4\pi m a} g^2 \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2 + \mathcal{V}$$

- integrate over R use the definition of C

$$E = \sum_{\sigma} \int^{\Lambda} \frac{d^3 k}{(2\pi)^3} \frac{k^2}{2m} \left(\rho_\sigma(\mathbf{k}) - \frac{C}{k^4} \right) + \frac{C}{4\pi m a} + \int d^3 R \, \langle \mathcal{V} \rangle$$

Adiabatic Relation:

- Change in energy E= ⟨H⟩ due to small change in scattering length *a*

➡ Use Feynman-Hellman

$$\partial \mathcal{H}/\partial a = \frac{g^2}{4\pi m a^2} \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2$$

➡ integrate over space and obtain

$$\frac{d}{da} E = \frac{1}{4\pi m a^2}\ C$$

RF Spectroscopy

- use RF pulse to transfer atoms of type 2 to atoms to type 3
- the Hamiltonian contains

$$H_{\mathrm{RF}}(t) = \lambda\theta(t)\sin(\omega t)\int d^3r\,\left(\psi_3^\dagger\psi_2(\mathbf{r},t) + \psi_2^\dagger\psi_3(\mathbf{r},t)\right)$$

- for the inclusive rate $\Gamma(\omega)$ we need to calculate

$$\langle \psi_2^\dagger\psi_3(\mathbf{r},t)\,\psi_3^\dagger\psi_2(\mathbf{r}',t')\rangle$$

Use Short-Time OPE

Braaten, Kang & LP 2010

- expand operators non-local in time & space

$$\mathcal{O}_A(\mathbf{R}+\tfrac{1}{2}\mathbf{r}, T+\tfrac{1}{2}t)\ \mathcal{O}_B(\mathbf{R}-\tfrac{1}{2}\mathbf{r}, T-\tfrac{1}{2}t) = \\ = \sum_C W_C(\mathbf{r},t)\mathcal{O}_C(\mathbf{R},T)$$

- match n-body Wilson coefficient to n-body matrix element

$$\int dt\ e^{i\omega t} \int d^3r\ \psi_2^\dagger \psi_3(\mathbf{R}+\tfrac{1}{2}\mathbf{r}, t)\ \psi_3^\dagger \psi_2(\mathbf{R}-\tfrac{1}{2}\mathbf{r}, 0) \\ = (i/\omega)\ \psi_2^\dagger\psi_2(\mathbf{R}) + iW_{12}(\omega)\ g_{12}^2 \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1(\mathbf{R})$$

New Information

\Omega is Rabi Frequency: strength coupling between light and transition

- large ω tail of Γ(ω)

$$\Gamma(\omega) \longrightarrow \frac{\Omega^2 (a_{12}^{-1} - a_{13}^{-1})^2}{4\pi\sqrt{m}\omega^{3/2}(a_{13}^{-2} + m\omega)} \, C_{12}.$$

- can be used to calculate sum rules

$$\int_{-\infty}^{\infty} d\omega \frac{\gamma/\pi}{(\omega - \omega_0)^2 + \gamma^2} \Gamma(\omega) = \frac{\Omega^2 \gamma}{\omega_\gamma^2} N_2$$
$$+ \frac{\Omega^2 \left[(\omega_0^2 - \gamma^2) b_+ + 2\omega_0 \gamma b_- - 2\omega_0 \gamma a_{12}^{-1}\right]}{4\pi m \omega_\gamma^4} C_{12} + \ldots$$

$$b_\pm = \sqrt{m(\omega_\gamma \pm \omega_0)/2}$$

Compare to Randeria, Zwerger, ...

Active field of research

Theoretical:

- Punk & Zwerger 2008
- Schneider & Randeria 2010
- Werner & Castin 2010
- Son & Thompson 2010

Experimental:

- Hu et al. 2010
- Gaebler, Stewart & Jin 2010

Let’s switch gears

- in few-body systems we can do calculations exactly
- what are the implications of a large scattering length in such systems
- what types of universality exist here

The 3-Boson Problem

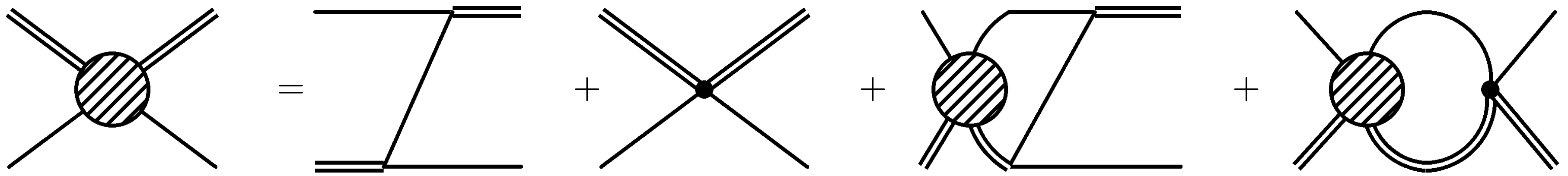
- 3-Boson Problem different from 3-Fermion problem $\longrightarrow$ Efimov effect
- Lagrangian is now

$$\mathcal{L} = \psi^\dagger (i\partial_t - \frac{\nabla^2}{2m})\psi - \frac{g(\Lambda)}{m}(\psi^\dagger\psi)^2 - \frac{h(\Lambda)}{m}(\psi^\dagger\psi)^3$$

- **3-body force is needed for renormalization in the 3-body sector**

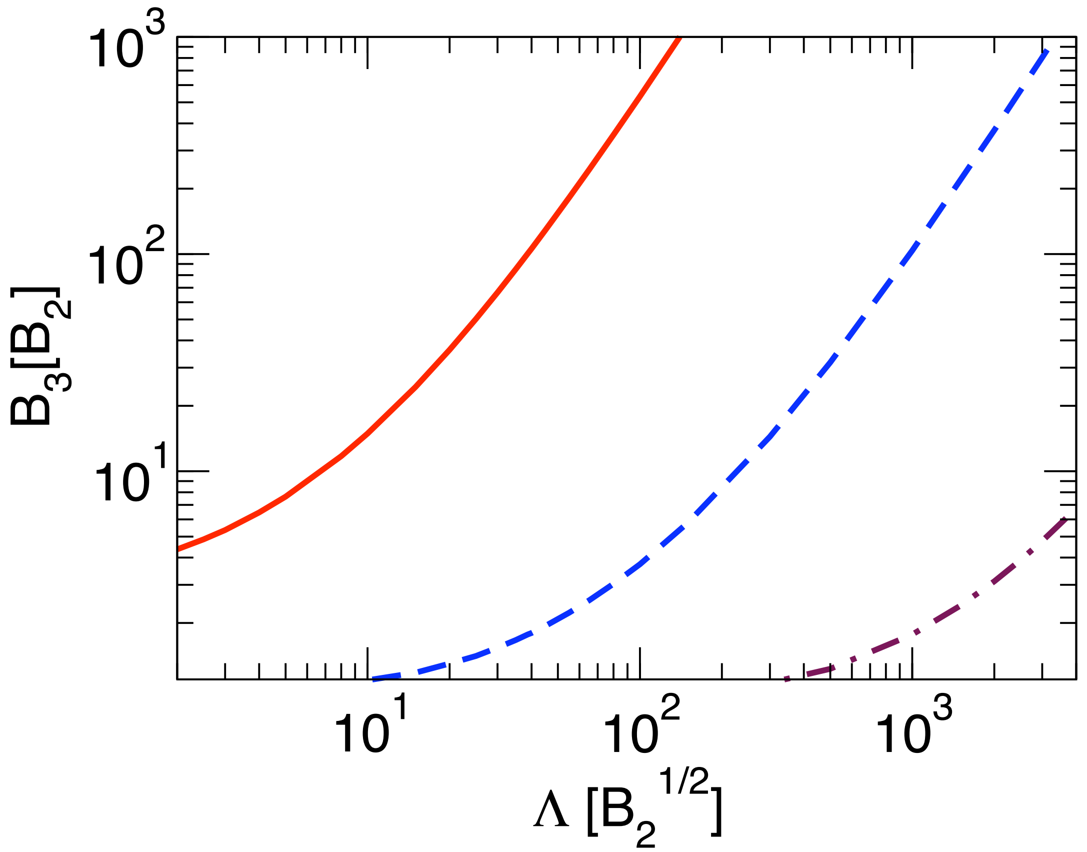
How is this shown?

- Write down integral equation for particle-dimer scattering



Skorniakov & Ter-Martirosian '56

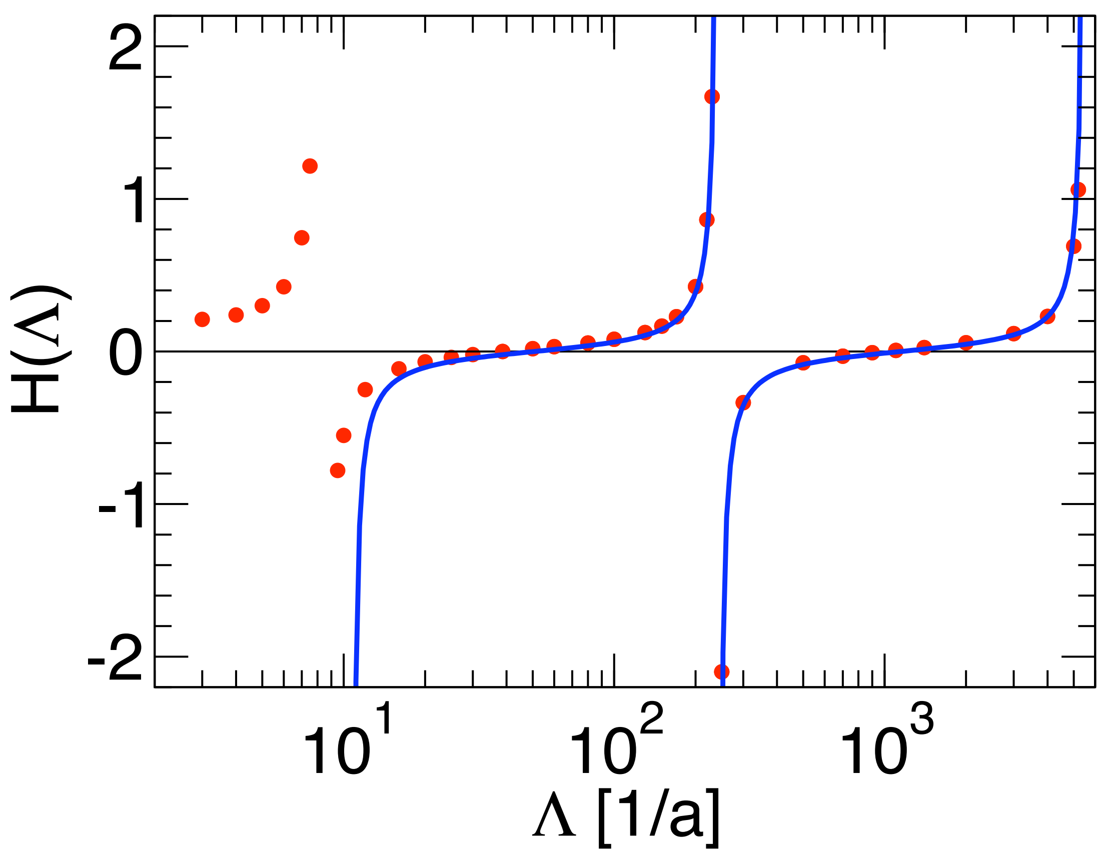
- Solve without 3-body force



- introduce 3-body force for renormalization (Bedaque, Hammer, van Kolck 1999)

Discrete Scale Invariance

- extra counterterm introduces new scale
- rg flow is a limit cycle (Bedaque, Hammer, van Kolck 1999)
- continuous scaling symmetry is broken down to discrete scaling symmetry (Efimov 1979)



➡ Consequences for observables

$$\frac{B_3^{(m)}}{B_3^{(m+1)}} \approx 515$$

The 4-Boson Problem

- **Question:** Do we need 4-body force for renormalization?

➡ **No new parameter in the 4-body system** (Hammer, Meissner, LP 2005)

➡ **New universal predictions** (Hammer & LP 2007, Stecher et al. 2008)

★ **1D Correlation between 3- and 4-body observables**

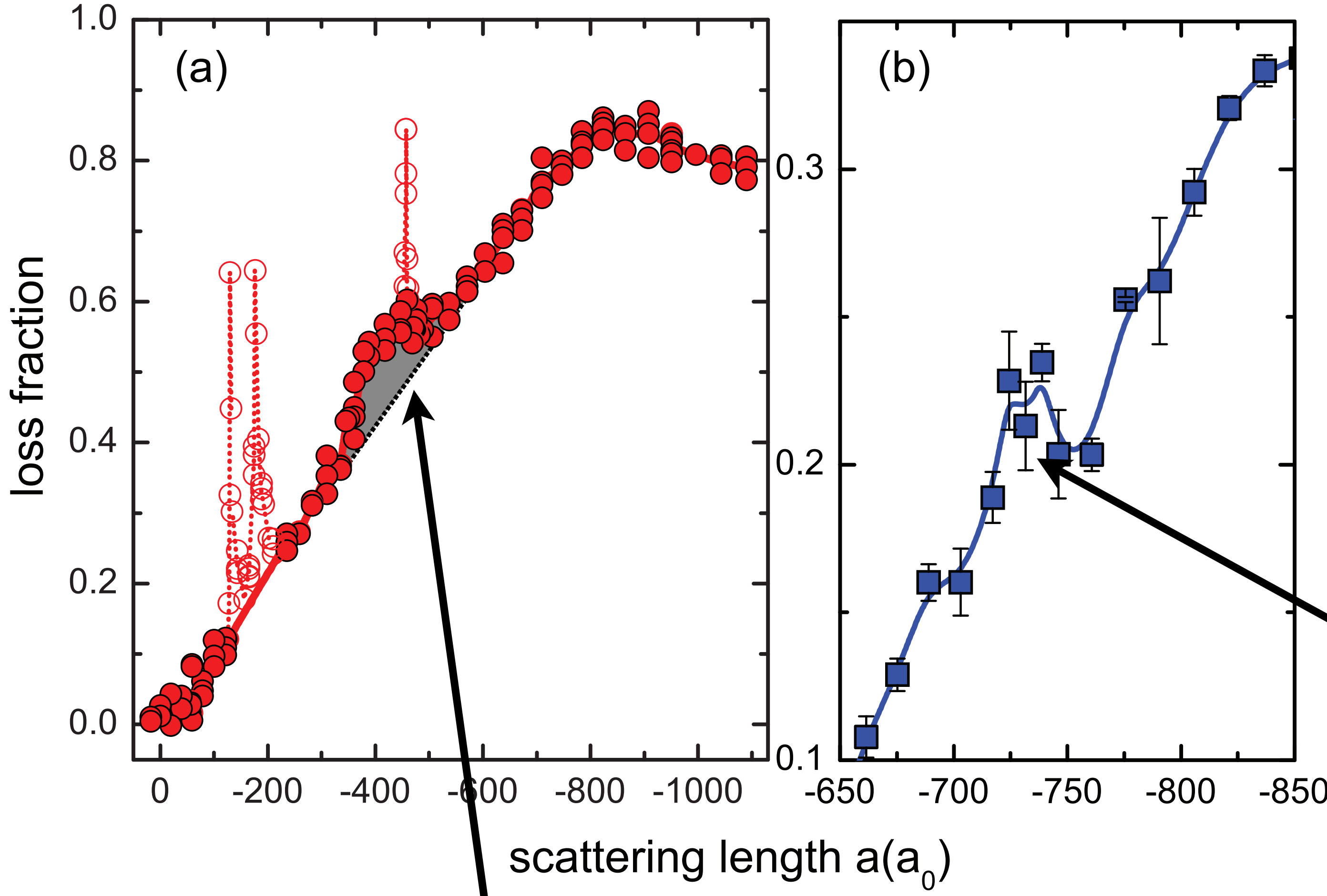
★ **2 4-body states tied to every**

Confirmed Experimentally

- Recombination features in AMO experiments display four-body features

Confirmed Experimentally

- Recombination features in AMO experiments display four-body features



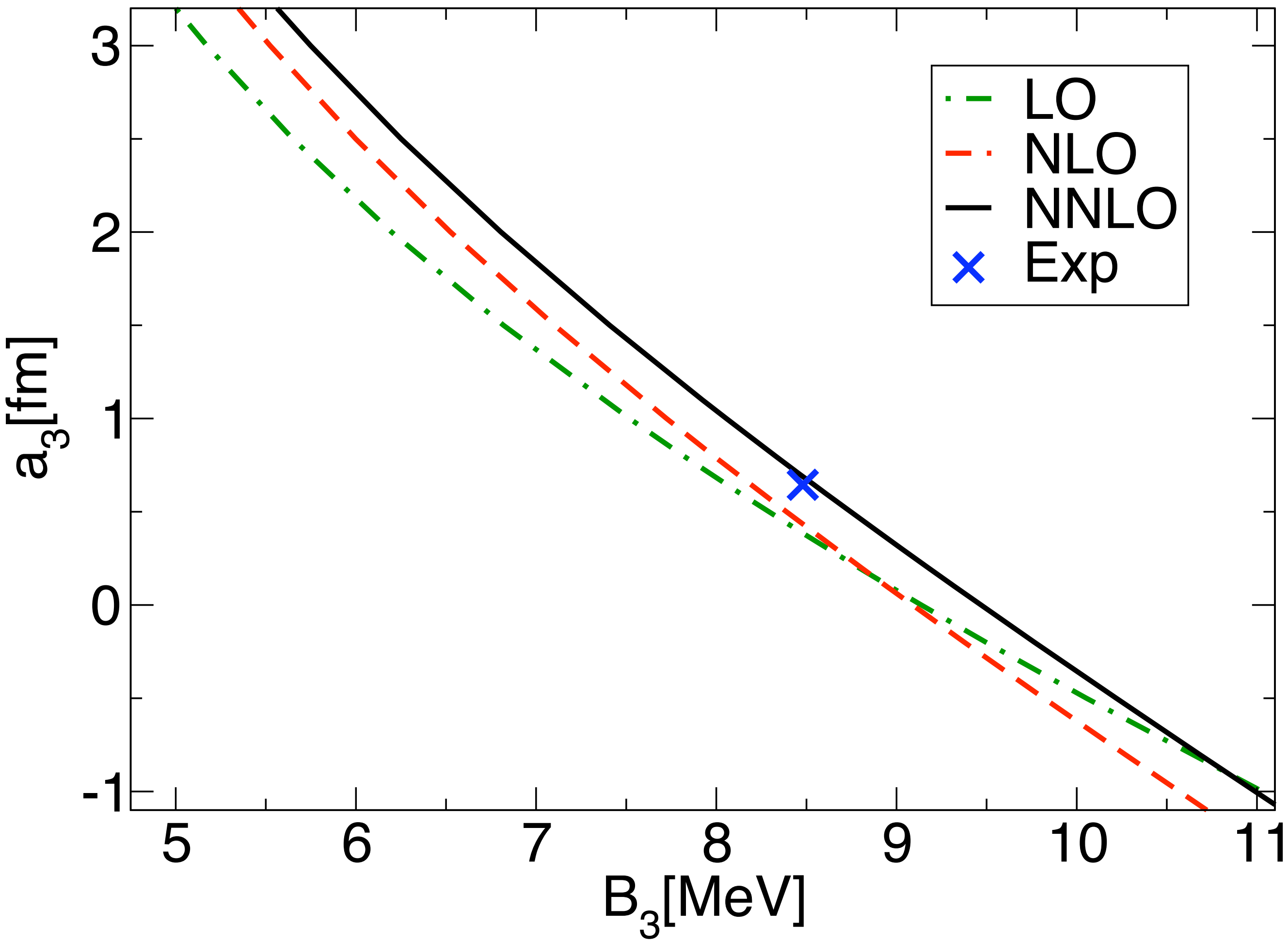
Tetramer 2

Tetramer 1

Ferlaino et al. 2009

Effective Range Corrections

- This is the LO of an Effective Field Theory
  - ➡ Include higher order corrections through operators with derivatives
  - ➡ important for nuclear physics



LP 2006

Range Corrections II

- Range corrections and power counting have been discussed mostly for fixed a
  - Hammer & Mehen 2001
  - Bedaque *et al.* 2003
  - Phillips & LP 2006
  - LP 2006
- What about variable scattering length?
  - ➡ Relevant for AMO and Lattice QCD exptrapolations

Linear Range Correction

Phillips, Ji &LP 2009 & *to be submitted 2010*

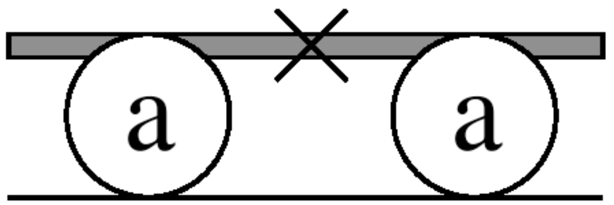
- Effect of linear range correction can be calculated exactly in the unitary limit (perturbatively & in coordinate space)

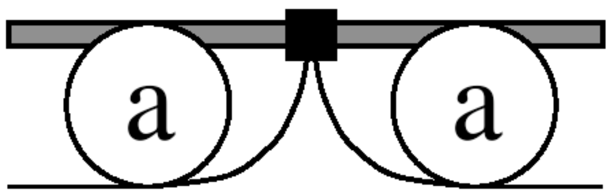
- Spectrum remains unchanged, i.e.

$$\Delta B_n^{(\mathrm{NLO})} = 0 \quad \text{for all} \quad n \quad \text{and} \quad 1/a = 0$$

Finite a

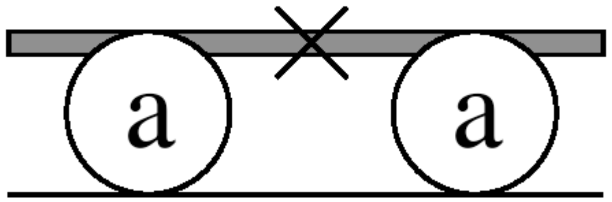
- Evaluate diagrams
- for finite a energy-independent 3-body force $H_1 = H_{10} + H_{11}/a$



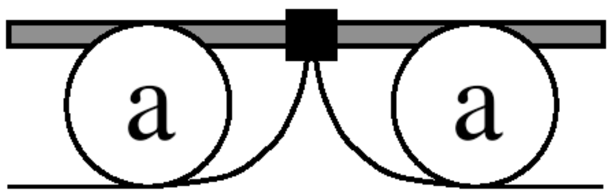


Finite a

- Evaluate diagrams



- for finite a energy-independent 3-body force $H_1 = H_{10} + H_{11}/a$



➡ no new input for fixed a

➡ new parameter required for variable a

➡ reminiscent of quark mass dependence in χPT

Summary

- Universality is important:
  - ➡ exact results for many-body systems
  - ➡ low-energy theorems for few-body systems
- Range corrections are important
  - ➡ first understand the physics then do the calculation