Applications of AdS/CFT correspondence to cold atom physics

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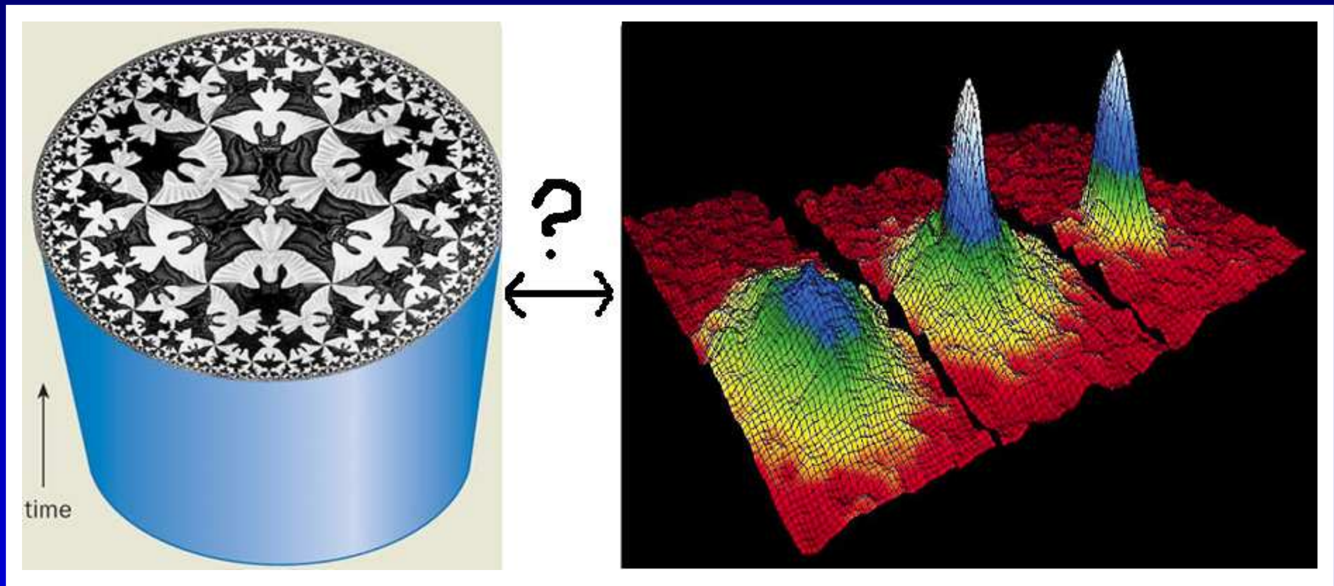

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Outline

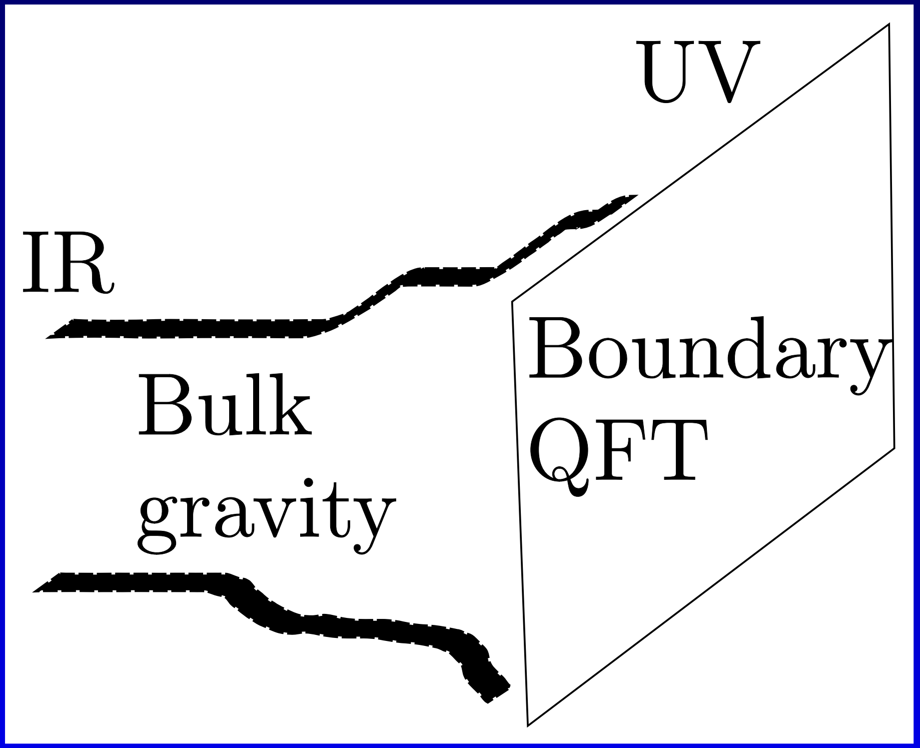
- Basics of AdS/CFT correspondence
- Schrödinger group and correlation functions
- Nonrelativistic AdS/CFT and cold atoms
- Holographic Efimov effect

Holographic principle

't Hooft 93, Susskind 95

- We have some evidence that

Quantum field theory in d spacetime dimensions $\leftrightsquigarrow$ Quantum gravity in higher dimensions



- Holographic radial direction – RG scale

AdS spacetime and its relation to CFT

- Einstein-Hilbert gravity action

$$S = \frac{1}{2\kappa} \int dz d^d x \sqrt{-g} \left(R + \frac{d(d-1)}{L^2} \right)$$

- Most symmetric solution is Anti-de Sitter spacetime

$$ds^2 = L^2 \frac{dz^2 + \eta_{\mu\nu} x^\mu x^\nu}{z^2}$$

- Isometry group of AdS_{d+1} is $SO(d,2)$
- Conformal group in $Mink_d$ is also $SO(d,2)$!

AdS/CFT correspondence

Maldacena 97

$\mathcal{N} = 4,\ SU(N_c)$ in $d = 4$ Minkowski $\leftrightsquigarrow$ IIB string theory in $AdS_5 \times S_5$

- Parameters on gauge side: g_{YM} and N_c
  Parameters on gravity side: L, l_{st} and κ
- Mapping

$$\frac{L}{l_{st}} \sim g_{YM}^2 N_c \qquad \frac{L^8}{\kappa} \sim N_c^2$$

- It is a weak/strong duality
- Classical supergravity is valid if $N_c^2, g_{YM}^2 N_c \gg 1$

Holographic dictionary

Boundary QFT	Bulk gravity
operator $\mathcal{O}$	dynamical field ϕ
scaling dimension $\Delta_{\mathcal{O}}$	mass m_ϕ
global symmetry	gauge symmetry
finite T	Hawking T of a black hole
entropy	Hawking entropy
chemical potential	U(1) gauge field

- Correlation functions can be calculated from

$$\langle e^{\int \phi_0 \mathcal{O}} \rangle_{CFT} = Z_{gravity}[\phi \to \phi_0]$$

Schrödinger group

- Spacetime symmetries of free Schrödinger/diffusion equation form Schrödinger group $Sch(D)$ Niederer 72, Hagen 72
- Schrödinger group comprises
  - translations P_i and H
  - spatial rotations M_{ij}
  - Galilean boosts K_i
  - number operator N
  - scale transformation $D \Rightarrow$ dynamical exponent $z = 2$
  - special conformal transformation C
- Theory symmetric under $Sch(D)$ is called NRCFT
- Example: two particles with conformal $1/r^2$ potential

Schrödinger group

- Nonrelativistic primary operators $\mathscr{O}$ have well-defined scaling dimension $\Delta_{\mathscr{O}}$ and particle number $N_{\mathscr{O}}$ Nishida& Son 08

$$[D, \mathscr{O}] = i\Delta_{\mathscr{O}} \qquad [N, \mathscr{O}] = N_{\mathscr{O}}$$

  and commute with Galilean boosts K_i and special conformal transformation C

$$[K_i, \mathscr{O}] = 0 \qquad [C, \mathscr{O}] = 0$$

  Descendants are formed by commutators with P_i and H

- Kinematic invariants of $Sch(D)$ group Volovich&Wen 09
  - mixed invariants $v_{ijn} = \frac{(\vec{x}_{in}t_{jn} - \vec{x}_{jn}t_{in})^2}{2t_{ij}t_{in}t_{jn}} \quad i < j < n$
  - time cross-ratios $\frac{t_{ij}t_{kl}}{t_{ik}t_{jl}}$

Correlators in Euclidean NRCFT

- Schrödinger symmetry and causality impose constraints on the correlators of the primary fields — Henkel 94
- 2-point function is fixed up to a constant ($\bar{x} = (\vec{x}, t)$)

$$G_2(\bar{x}_1, \bar{x}_2) = C\delta_{\Delta_1,\Delta_2}\delta_{M_1,M_2}\theta(t_{12})t_{12}^{-\Delta_1} \exp\left[-\frac{M_1}{2}\frac{\vec{x}_{12}^2}{t_{12}}\right]$$

- 3-point function is determined up to a function $\Psi(v_{123})$

$$G_3(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \delta_{M_1+M_2,M_3}\theta(t_{13})\theta(t_{23})\prod_{i<j} t_{ij}^{-\Delta_{ij,n}/2} \times \exp\left[-\frac{M_1}{2}\frac{\vec{x}_{13}^2}{t_{13}} - \frac{M_2}{2}\frac{\vec{x}_{23}^2}{t_{23}}\right]\Psi(v_{123})$$

- 4-point function is determined up to a non-universal function $\Psi(\frac{t_{12}t_{34}}{t_{14}t_{32}}, v_{124}, v_{134}, v_{234})$ — Volovich&Wen 09

Two-component fermions at unitarity

- Vacuum theory is defined by action

$$S[\psi,\phi] = \int dt d^D x \ \Big[\sum_{i=1}^{2} \psi_i^* (i\partial_t + \frac{\Delta}{2m})\psi_i - \frac{1}{c_0}\phi^*\phi + (\phi^*\psi_1\psi_2 + \phi\psi_2^*\psi_1^*)\Big]$$

- Bare parameter c_0 is related to scattering length a and cut-off Λ
- Unitarity regime $a^{-1} \to 0 \Rightarrow$ no intrinsic length scale in theory
- Exact propagators and scaling dimensions of ψ and ϕ can be determined analytically
- Fermions at unitarity are believed to be symmetric under the full Schrödinger group Mehen et al. 2000
- It is believed to be NRCFT (primary operators, operator/state correspondence, conformal Ward identities) Nishida&Son 08

Nonrelativistic AdS/CFT

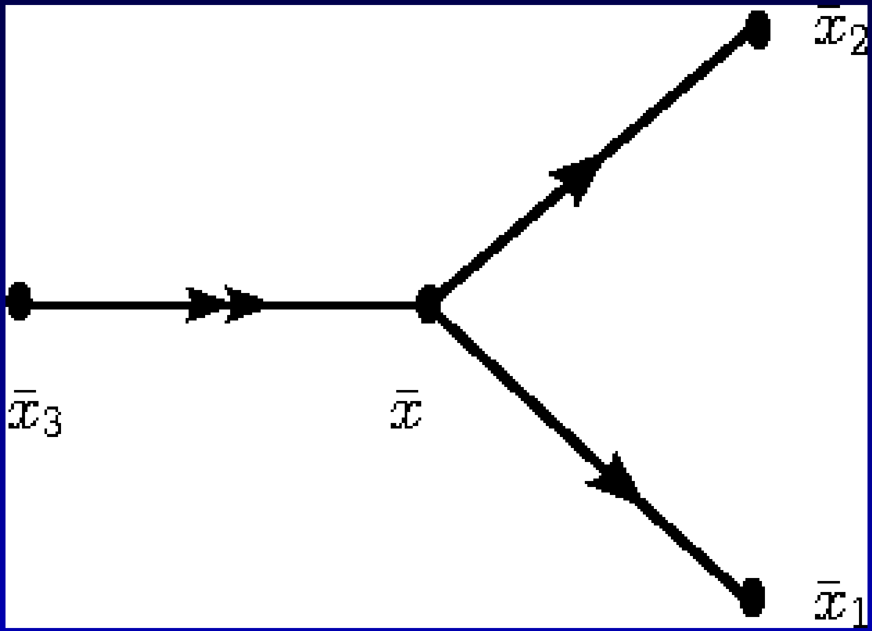
- NRCFT in D dimensions $\Leftrightarrow$ gravity theory in higher dimensions?
- Schrödinger algebra $Sch(D)$ is a subalgebra of the conformal algebra $so(D+2,2)$ in $Mink_{D+2}$ that commutes with the light-cone momentum P^+
- Deformation of AdS_{D+3} in light-cone coordinates leads to the new Sch_{D+3} metric Son 08, Balasubramanian&McGreevy 08

$$ds^2 = -\frac{dt^2}{z^4} + \frac{-2dtd\xi + dx^i dx^i + dz^2}{z^2} \quad i = 1, \ldots, D$$

- Isometries of Sch_{D+3} obey the Schrödinger albegra $Sch(D)$
- ∂_ξ corresponds to the particle number generator N and coordinate ξ is possibly compact

3-point function G_3 in cold atoms

- We compute $G_3 = \langle \psi_1(\bar{x}_1)\psi_2(\bar{x}_2)\phi^*(\bar{x}_3)\rangle$ at unitarity regime in position space by performing integration over $\bar{x}$



- Two important points
  1. no condensate in non-relativistic vacuum $\langle \psi_i \rangle = 0$, $\langle \phi \rangle = 0$
  2. Yukawa vertex is not renormalized in vacuum
- G_3 is completely determined by $\Delta_\psi = \frac{D}{2}$ and $\Delta_\phi = 2$

3-point function G_3

- Agreement with Schrödinger Ward identities — Henkel 94
- We determined the non-universal scaling function for $D > 2$

$$\Psi(y) \sim y^{-\frac{D}{2}+1}\gamma(\frac{D}{2} - 1, y),$$

where $\gamma(n, y) = \int_0^y t^{n-1} e^{-t} dt$

- AdS/CFT gives non-universal scaling function

$$\Psi(y) \sim \int_{\mathbf{R}+i\epsilon} dv \int_{\mathbf{R}+i\epsilon'} dv'\, e^{-iM_1 v - iM_2 v'} \times$$
$$(v - v' + iy)^{-\Delta_{12,3}/2} (v')^{-\Delta_{23,1}/2} v^{-\Delta_{13,2}/2},$$

where $\Delta_{ij,k} = \Delta_i + \Delta_j - \Delta_k$

3-point function G_3 from AdS/CFT

- We take unitarity scaling $\Delta_1 = \Delta_2 = D/2$, $\Delta_3 = 2$ and perform double contour integration

$$\Psi(y) \sim y^{-\frac{D}{2}+1}\gamma(\frac{D}{2} - 1, y)$$

- Non-universal scaling function $\Psi(y)$ agrees with unitarity cold atoms result! Fuertes& SM 09
- We take free scaling dimension $\Delta_1 = \Delta_2 = D/2$ and $\Delta_3 = D$ and perform double contour integration

$$\Psi(y) = \text{const}$$

- Agreement with free QFT
- Possibly AdS/CFT describes both free and unitarity regime Son 08

Bosons at unitarity

- The action is similar

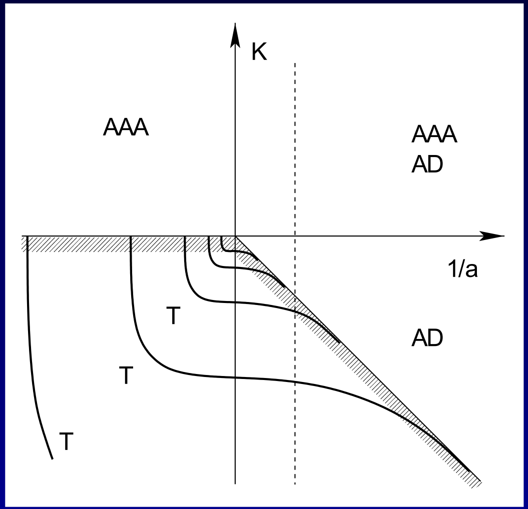
$$S[\psi,\phi] = \int dt d^D x \ \ [\psi^*(i\partial_t + \frac{\Delta}{2m})\psi - \frac{1}{c_0}\phi^*\phi \\ + (\phi^*\psi\psi + \phi\psi^*\psi^*)]$$

- Unitarity regime $a^{-1} \to 0 \Rightarrow$ no intrinsic length scale in theory
- Can be prepared in cold atoms experiments, e.g. ^{7}Li, ^{133}Cs...
- Two-body problem is similar to fermions
- It is not NRCFT due to the Efimov effect

Three-body problem and the Efimov effect

Energy spectrum near the unitarity regime

Efimov 70



- At unitarity $a = \pm\infty$ spectrum becomes geometric

$$\frac{E_T^{(n+1)}}{E_T^{(n)}} \to e^{-2\pi/s_0} \quad \text{as n} \to \infty \quad s_0 \approx 1.0062$$

- The spectrum is manifestation of scale quantum anomaly
- In RG language $\to$ limit cycle solution

Breitenlohner-Freedman bound in AdS_{d+1}

- Free complex scalar

$$S[\phi,\phi^*] = -\int dz d^d x \sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + m^2\phi^*\phi\right)$$

in AdS_{d+1} spacetime

$$ds^2 = \frac{dz^2 + \eta^{\mu\nu}dx_\mu dx_\nu}{z^2}$$

- Fourier transform $x^\mu \longrightarrow q^\mu$ on the boundary and change variables $\psi = z^{(1-d)/2}\phi$

$$-\partial_z^2\psi + \frac{m^2 + \frac{d^2-1}{4}}{z^2}\psi = -q^2\psi, \qquad q^2 \equiv -(q^0)^2 + \vec{q}^2$$

- Map onto 1D QM problem with inverse square potential!

Inverse square potential in QM

$$-\partial_z^2\psi - \frac{\kappa}{z^2}\psi = E\psi$$

- The potential is singular and must be regularized
- Two branches of solution
  - $\kappa < \kappa_{cr} = \frac{1}{4} \longrightarrow$ no bound states, continuous spectrum
  - $\kappa > \kappa_{cr} \longrightarrow$ infinite geometric bound state spectrum
- In our mapping

$$E < 0 \Rightarrow (q^0)^2 < 0$$

$$\kappa > \kappa_{cr} \Rightarrow m^2 < m_{BF}^2 = -\frac{d^2}{4}$$

- The bound was first derived from positivity of conserved energy functional of scalar fluctuations Breitenlohner&Freedman 82

No BF bound in Sch_{D+3}

- Free complex scalar in Sch_{D+3}

$$S[\phi,\phi^*] = -\int dz dt d\xi d^D x \sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + m_0^2\phi^*\phi\right)$$

- Mapping onto Schrödinger equation

$$-\partial_z^2\psi + \frac{m^2 + \frac{(D+2)^2-1}{4}}{z^2}\psi = -\tilde{q}^2\psi, \qquad \tilde{q}^2 \equiv -2M\omega + \vec{q}^2$$

- Due to nonrelativistic dispersion

$$E < 0 \Rightarrow \omega < 0$$

- Nothing special happens at $m^2 = m_{BF}^2 = -\frac{(D+2)^2}{4}$
- No stability bound in nonrelativistic AdS/CFT!

Son

Two-point correlator $\langle OO^\dagger \rangle$ for $m^2 < m_{BF}^2$

- Using standard AdS/CFT machinery we can calculate SM 09

$$\langle OO^\dagger \rangle \sim \tan \{ |\nu| \ln \widetilde{q} + \gamma \} ,$$

where $\nu = \sqrt{\frac{(D+2)^2}{4} + m^2}$ and $\widetilde{q}^2 \equiv -2M\omega + \vec{q}^2$

- Properties
  - $\langle OO^\dagger \rangle$ is log-periodic in $\widetilde{q}$
  - Operator O describes infinitely many particles

$$\frac{\omega_{n+1}}{\omega_n} = \exp\left(-\frac{2\pi}{|\nu|} \right)$$

  - Continuous scale symmetry is broken $\longrightarrow$ limit cycle solution
  - γ determines initial UV position on RG limit cycle

Limit cycles in QM and complex Δ

- Efimov effect
  - Trimer operator $O = \psi\phi$ has

$$\Delta_{\pm} = \frac{5}{2} \pm i s_0$$

- QM with $1/r^2$ potential in D dimensions
  - For $\kappa > \kappa_{cr} = \frac{(D-2)^2}{4}$, composite $O = \psi\psi$ acquires complex scaling dimension

$$\Delta_{\pm} = \frac{D+2}{2} \pm \sqrt{\frac{(D-2)^2}{4} - \kappa}$$

- If described by AdS/CFT $\rightarrow m^2 < m_{BF}^2$

Conclusion and outlook

- AdS/CFT was extended to nonrelativistic physics
- Schrödinger symmetry is powerful
- Agreement of specific 3-point function, but better understanding?
- Limit cycles can be realized in nonrelativistic AdS/CFT
- Calculate limit cycle two-point function in QM and compare with holographic prediction

Extra slides

Applications to condensed matter physics

- Holographic systems with Schrödinger symmetry
- Holographic superfluids
- Holographic non-Fermi liquids
- Holographic systems with Lifshitz symmetry

Schrödinger algebra

- Centrally extended Galilei algebra

$$[M_{ij}, M_{kl}] = i(\delta_{ik}M_{jl} - \delta_{jk}M_{il} + \delta_{il}M_{kj} - \delta_{jl}M_{ki}) \quad ,$$

$$[M_{ij}, K_k] = i(\delta_{ik}K_j - \delta_{jk}K_i), \quad [M_{ij}, P_k] = i(\delta_{ik}P_j - \delta_{jk}P_i),$$

$$[P_i, K_j] = -i\delta_{ij}N \quad , \qquad [H, K_j] = -iP_j \quad .$$

- Additionally

$$[P_i, D] = -iP_i \quad , \quad [P_i, C] = -iK_i \quad , \quad [K_i, D] = iK_i \quad ,$$

$$[D, C] = -2iC \quad , \quad [D, H] = 2iH \quad , \quad [C, H] = iD \quad .$$

- The generators H, D and C close a subalgebra $sl(2, R)$