Lattice approach for studies of interacting non-relativistic fermions

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May 20, 2010

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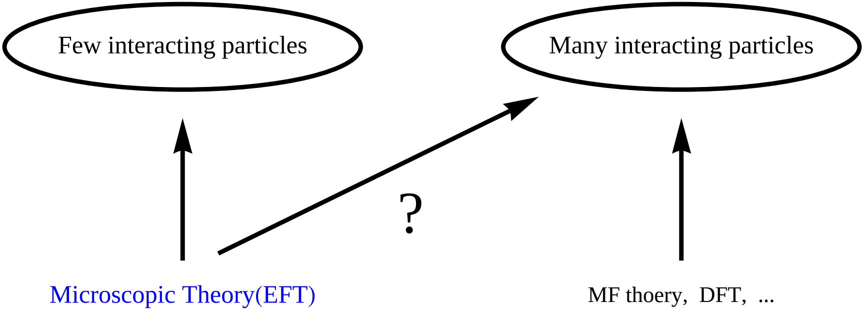
Outline

1. Introduction
2. Lattice theory for interacting non-relativistic fermions
3. Some results for unitary fermions in a finite box

Motivation



Can we study the interacting many fermions from EFT?

Advertisement : Lattice effective theory for interacting non-relativistic fermions

Advantages :

- 1. Fast - No $det(\phi)$, Pairing wave function
- 2. Accurate - Improve coupling tuning
- 3. Not severe signal-to-noise(S/N) problem

Goal : Calculate the ground state energy of interacting non-relativistic fermions by measuring the many-particle correlation functions :

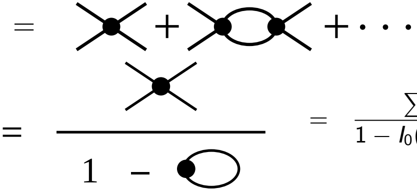
$$G_N(\tau) \sim Z_0 e^{-m_0 \tau} \quad \text{in the limit of large } \tau$$

Pionless effective theory - Continuum

- Continuum theory for non-relativistic fermions with contact interactions

$$\mathcal{L} = \bar{\psi}\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2M}\right)\psi + C_0(\bar{\psi}\psi)^2 + \cdots$$

$\mathcal{A}$ =



$$= \frac{\sum C_{2n}(\mu)p^{2n}}{1 - I_0(\mu)\sum C_{2n}(\mu)p^{2n}}$$

- Effective range expansion

$$\mathcal{A} = \frac{4\pi}{M}\frac{1}{p\cot\delta - ip}, \qquad p\cot\delta = -\frac{1}{a} + \frac{1}{2}r_0 p^2 \sum_{i=0}^{\infty}(r_i^2 p^2)^i$$

- Example : fermions at unitarity

$$p\cot\delta = 0, \quad \text{strongly interacting fermions} \quad \Rightarrow \quad \text{nonperturbative}$$

Lattice construction for interacting non-relativistic fermions

- Euclidean lattice action with auxiliary field $\phi \in Z_2$(Kaplan & Chen, 04')

$$S = \sum_{\mathbf{n}} \bar{\psi}_{\mathbf{n}} \left[\psi_{\mathbf{n}} - \psi_{\mathbf{n}-\hat{\mathbf{e}}_0} - \sum_{i=1}^{3} \frac{(\psi_{\mathbf{n}+\hat{\mathbf{e}}_\mathbf{i}} - 2\psi_{\mathbf{n}} + \psi_{\mathbf{n}-\hat{\mathbf{e}}_\mathbf{i}})}{2M} - \sqrt{C_0}\phi_{\mathbf{n}}\psi_{\mathbf{n}-\hat{\mathbf{e}}_0} + \cdots \right]$$

Lattice construction for interacting non-relativistic fermions

- Euclidean lattice action with auxiliary field $\phi \in Z_2$(Kaplan & Chen, 04')

$$S = \sum_{\mathbf{n}} \bar{\psi}_{\mathbf{n}} \left[\psi_{\mathbf{n}} - (1 + \sqrt{C_0}\phi_{\mathbf{n}} + \cdots)\psi_{\mathbf{n}-\hat{\mathbf{e}}_0} - \sum_{i=1}^{3} \frac{(\psi_{\mathbf{n}+\hat{\mathbf{e}}_\mathbf{i}} - 2\psi_{\mathbf{n}} + \psi_{\mathbf{n}-\hat{\mathbf{e}}_\mathbf{i}})}{2M} \right]$$

- Single fermion propagation over background ϕ-field

$$G_1(T) = D^{-1} X(T) D^{-1} \cdots D^{-1} X(0) D^{-1}$$

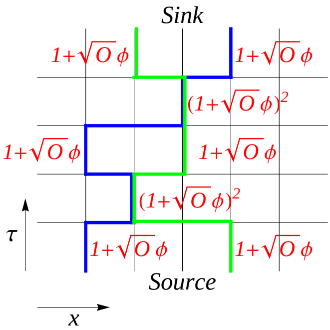
where

$$D = 1 + \frac{\Delta(\mathbf{p})}{M}, \qquad X(\tau) = 1 + \sqrt{O(\mathbf{p})}\phi(\tau)$$

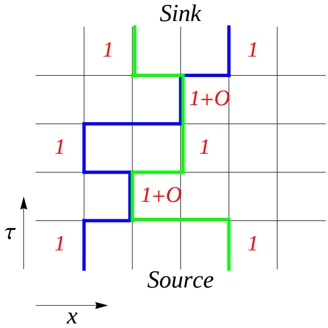
- Perfect dispersion relation

$$\Delta(\mathbf{p}) = \begin{cases} M(e^{p^2/2M} - 1) & \text{for} \quad |\mathbf{p}| \leq \pi \\ 0 & \text{for} \quad |\mathbf{p}| \geq \pi \end{cases}$$

Transfer Matrix



Transfer Matrix



- 2-point correlation function

$$\begin{aligned} G_2(T) &= \langle G_1(T) G_1(T) \rangle \\ &= \mathcal{D}^{-1}(1-\mathcal{V})\mathcal{D}^{-1} \cdots \mathcal{D}^{-1}(1-\mathcal{V})\mathcal{D}^{-1} \\ &= \mathcal{D}^{-1/2}\mathcal{T}^{T}\mathcal{D}^{-1/2} \end{aligned}$$

- Transfer Matrix

$$\mathcal{T} = \mathcal{D}^{-1/2}(1-\mathcal{V})\mathcal{D}^{-1/2}$$

$$\langle mn|\mathcal{D}^{-1}|m'n'\rangle = \mathcal{D}^{-1}_{m,m'}\mathcal{D}^{-1}_{n,n'}, \qquad \langle mn|\mathcal{V}|m'n'\rangle = -O\delta_{mn}\delta_{mm'}\delta_{n,n'},$$

Tuning coupling constants

$$\mathcal{T}\psi = \lambda\psi$$

- Eigenvalue equation

$$1 = \frac{1}{V}\sum_{\mathbf{q}} \frac{O(\mathbf{q})}{\lambda D^2(\mathbf{q}) - 1}$$

where

$$O(\mathbf{q}) = \sum_n C_{2n}\mathcal{O}_{2n}(\mathbf{q}), \qquad \mathcal{O}_{2n} = \frac{4\pi}{M} M^{2n}(1 - 1/D^2(\mathbf{q}))^{2n}$$

- Solve the equation for C_{2n} to reproduce Luscher's energy eigenvalues up to $(\mathbf{p}^2)^k$ in a finite box

N-particle correlation function

- Slater-determinent for 2-component fermions($N_1 = N_2 = N' = N/2$)
  - without pairing wave function - very small overlap to true ground state
  - with pairing wave function ψ

$$G_{2N'}(\tau) = \begin{vmatrix} \langle\psi|\mathcal{P}(\tau)\mathcal{P}(\tau)|11\rangle & \langle\psi|\mathcal{P}(\tau)\mathcal{P}(\tau)|12\rangle & \dots & \langle\psi|\mathcal{P}(\tau)\mathcal{P}(\tau)|1N'\rangle \\ \langle\psi|\mathcal{P}(\tau)\mathcal{P}(\tau)|21\rangle & \langle\psi|\mathcal{P}(\tau)\mathcal{P}(\tau)|22\rangle & \dots & \langle\psi|\mathcal{P}(\tau)\mathcal{P}(\tau)|2N'\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi|\mathcal{P}(\tau)\mathcal{P}(\tau)|N'1\rangle & \langle\psi|\mathcal{P}(\tau)\mathcal{P}(\tau)|N'2\rangle & \dots & \langle\psi|\mathcal{P}(\tau)\mathcal{P}(\tau)|N'N'\rangle \end{vmatrix}$$

- For $N_1 = N_2 - 1$, replace j'th row by

$$\langle j|\mathcal{P}(\tau)|1\rangle \quad \langle j|\mathcal{P}(\tau)|2\rangle \quad \dots \quad \langle j|\mathcal{P}(\tau)|N_1\rangle$$

Pairng wave functions

- Example of pairing wave functions
  - Exact 2-particle wave function

$$\psi_{PAIR1}(p) = \frac{1}{\lambda - 1/D^2(p)}$$

  - Wave function interpolating between free and unitary fermions

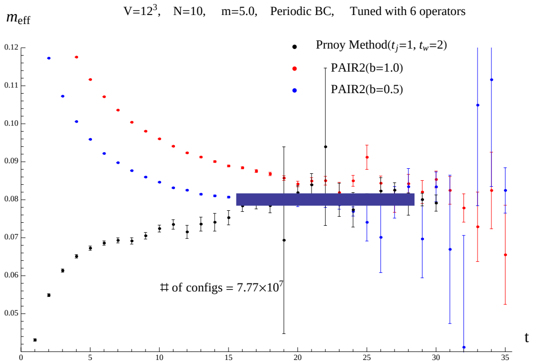
$$\psi_{PAIR2}(x) = \frac{1}{\sqrt{x^2 + b^2}}, \qquad \psi_{PAIR2}(p) = \frac{e^{-bp}}{p^2}$$

Unitary fermions in a finite box(I)

- Measure the effective mass up to $N = 32$ in a finite box(Periodic BC)

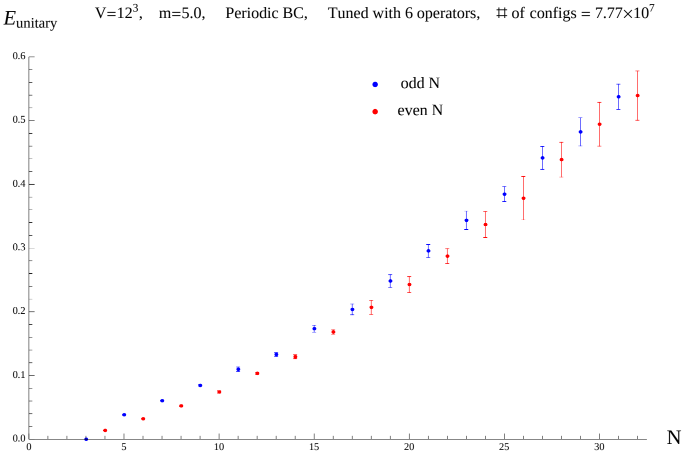
$$m_{eff} \equiv \mathrm{Log}[G_N(\tau+1)/G_N(\tau)] \underset{\text{large } \tau}{\sim} m_0$$

- Analysis
  - Jackknife error estimate
  - Correlated χ-square fit
  - Prony Method
- Example of m_{eff} plot



Unitary fermions in a finite box(II)

- Ground state energy of unitary fermions(Preliminary)

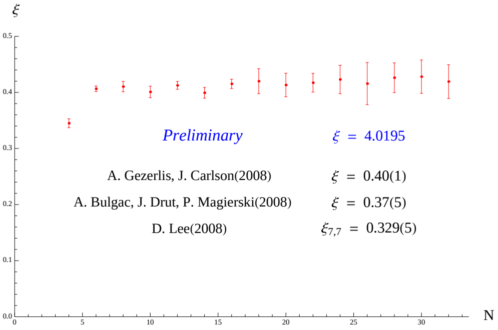


Bertsch parameters

- Universal feature of fermions at unitarity

$$E_{unitary}(\rho) = \xi E_{free}(\rho)$$

- Plot of Bertsch parameters

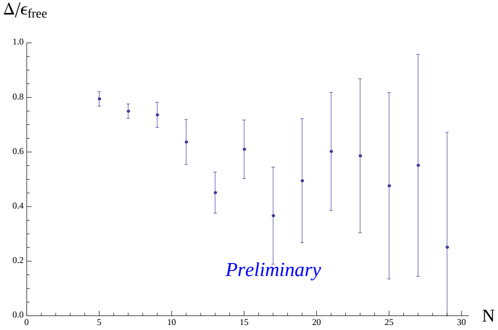


Pairng Gap

- Pairng gap : energy cost to break a pair(even N)

$$\Delta = E(N+1) - \frac{E(N) + E(N+2)}{2}$$

- Plot of pairng gaps



Summary and Discussion

- Lattice effective theory - large number of interacting non-relativistic fermions
  - No severe signal-to-noise problem
  - Couplings can be tuned to reproduce Lucsher's energy values in a finite box up to p^{2k}
  - Pairing wave functions works well
- Fermions at unitarity(up to $N = 32$)
  - Bertsch parameter : $\xi = 0.41095$

Effective mass plot : unpaired vs paired