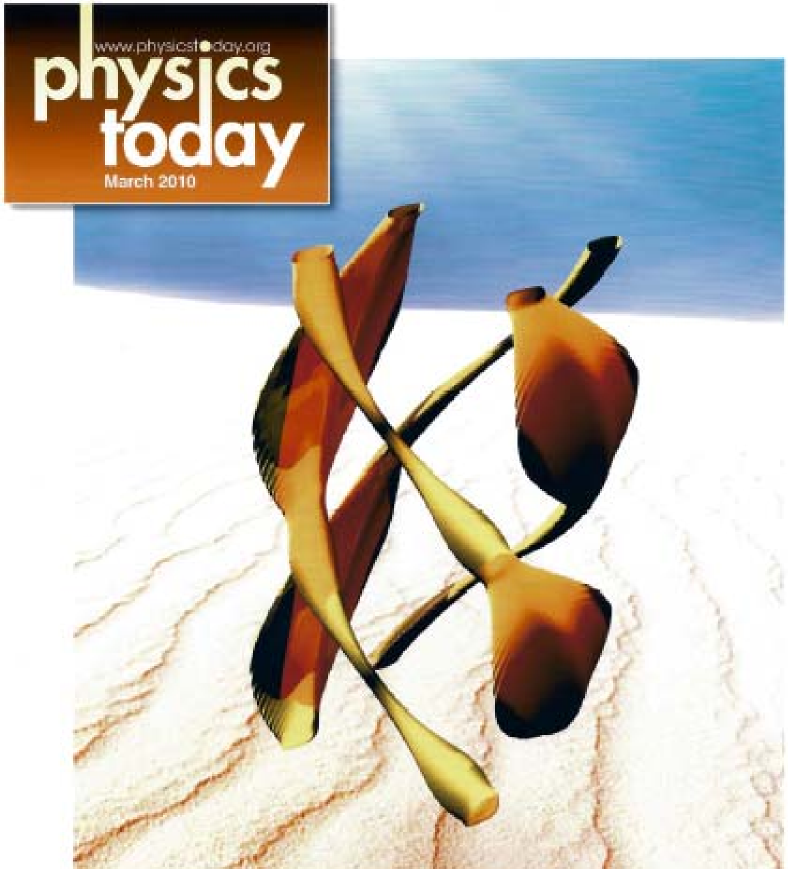
Four-body collisions, Universality, and the Efimov effect

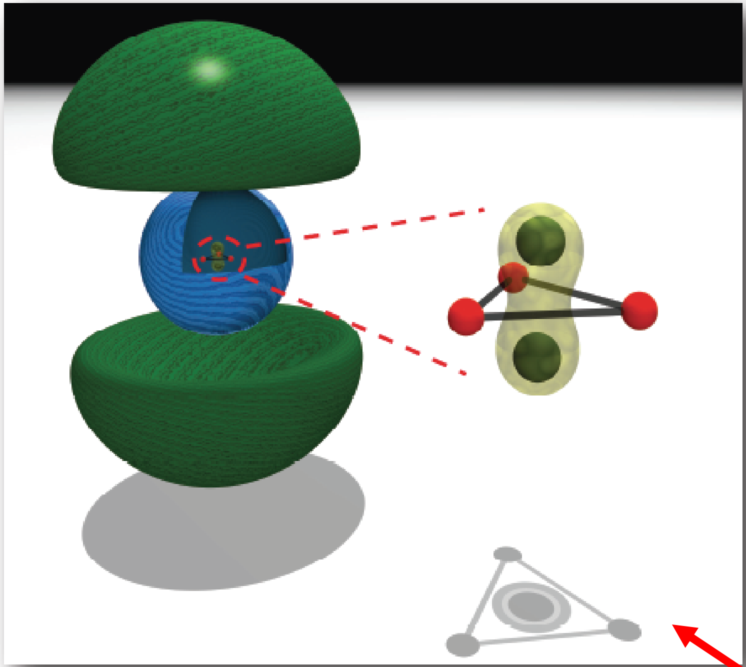
Chris H. Greene, JILA and the Physics Dept., University of Colorado at Boulder



cover: The four cobralike forms here represent one view of a wavefunction for a collision between two weakly bound dimers composed of fermionic atoms. The calculations underlying the figure are at the forefront of theoretical work that explores universal properties in few-body systems—that is, features that are independent of the details of particle interactions. Chris Greene's article, beginning on page 40, surveys universal physics in few-body systems, from a startling prediction offered in 1970 to recent theoretical and experimental advances. (Courtesy of Seth Rittenhouse.)

Universality in few-body land

A publication of the American Institute of Physics



Depicted:

A big universal molecule with 4 atoms, "attached" to a 3-body Efimov state

•Thanks, NSF!

JILA CENTER FOR ATOMIC, MOLECULAR, & OPTICAL PHYSICS
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Thanks at the outset to my collaborators on the ultra-cold four-boson project:

Jose D'Incao, Senior Research Associate at JILA

Javier von Stecher, co-recipient of the 2009 DAMOP Thesis Prize, now a JILA postdoc with Ana Maria Rey

Seth Rittenhouse, newly arrived postdoc at Harvard/ITAMP

Nirav Mehta, now a Research Ass't Prof. at Grinnell College, Iowa

Thanks to Jose and Javier for many of the slides in this talk.

+ long time collaborators, Brett Esry and Doerte Blume

Outline of this talk:

1. **Brief review of Efimov physics for 3 particles (atoms) with short-range interactions**
2. **The four-boson problem and the methods we developed to solve it**
3. **Results for 4-body bound and resonant states predicted in Nature Physics 5, 417 (2009), and experimentally confirmed by Ferlaino, Knoop, Berninger, Harm, D'Incao, Nägerl, and Grimm, Phys. Rev. Lett.102, 140401 (2009)**
4. **4-body recombination and its surprising importance**
5. **Theory (Mehta et al., PRL 103, 153201 (2009)) of 4-body and N-body recombination processes, e.g.**

$$A + A + A + A \rightarrow A_3 + A$$

6. ***Very recent headway for 3-bodies, a 4-body experiment at Rice Univ., and new theory for 5, 6, 7…8… bosonic atoms***

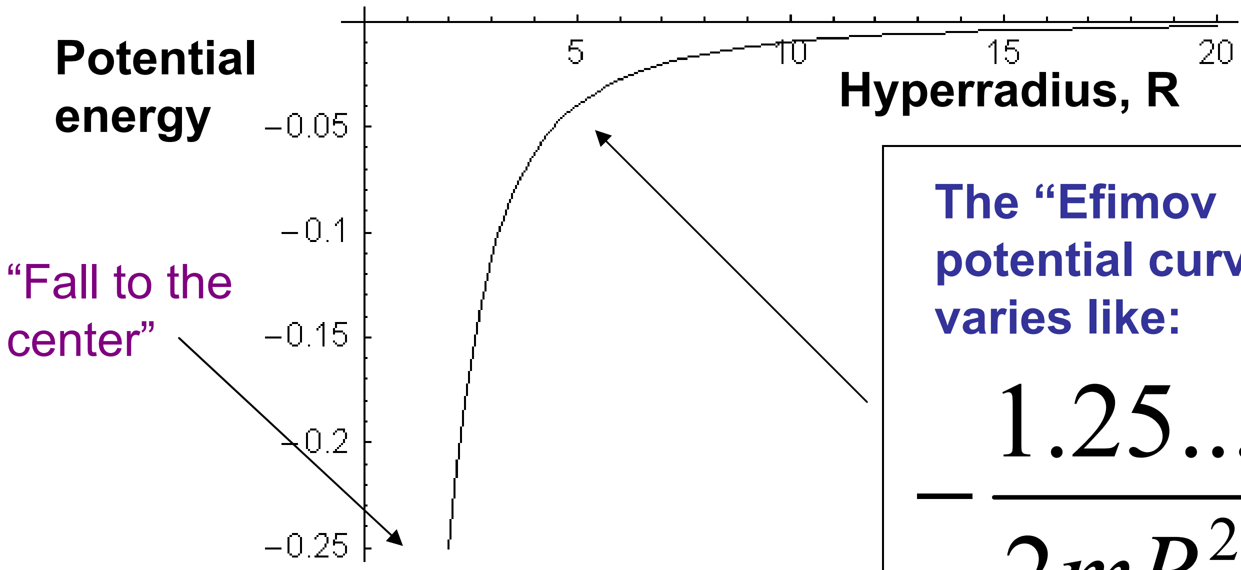
Universality in Few-Body Physics

When a few particles interact via short-range interactions, and their scattering length is much larger than the range r_0, their properties are observed to exhibit universality – i.e. binding energies scale in the same way from one level to the next, etc.

The behavior is UNIVERSAL in the sense that very different systems, from nucleons in a nucleus, to atoms in an ultracold cluster, to molecules, all exhibit these same universal properties.

In a sense, few-body physics is a great unifier, as it embraces theoretical (and experimental) studies across the sub-disciplines of physics, initially <u>nuclear physics</u> (Efimov) but more recently the torch has been taken up also by <u>atomic physics</u>, by <u>high-energy physics</u>, and even <u>condensed-matter</u> physics.

To understand the Efimov effect, look at the effective potential energy curve at unitarity, as a function of the hyperradius:



The “Efimov potential curve” varies like:

$$-\frac{1.25\ldots}{2mR^2}$$

Mathematical Detail. Once you have this “effective dipole-type attractive potential curve”, the rest is ‘TRIVIAL’!

Here, ‘trivial’ means that the solutions are simply Bessel functions (of imaginary *order*, and imaginary *argument*).

$$E_{n+1} = E_n e^{-2\pi/s_0}, \text{ where } s_0 = 1.00624\ldots \text{ is a universal constant.}$$

Aside: It can be helpful to systematize the discussion of long-range field effects, in order to include as much physics as possible analytically. Ideas like this have arisen in different subfields independently, often with different names:

Multichannel effective range theory (centrifugal only)

Bethe, Fermi, Breit...

Multichannel quantum defect theory (attractive Coulomb)

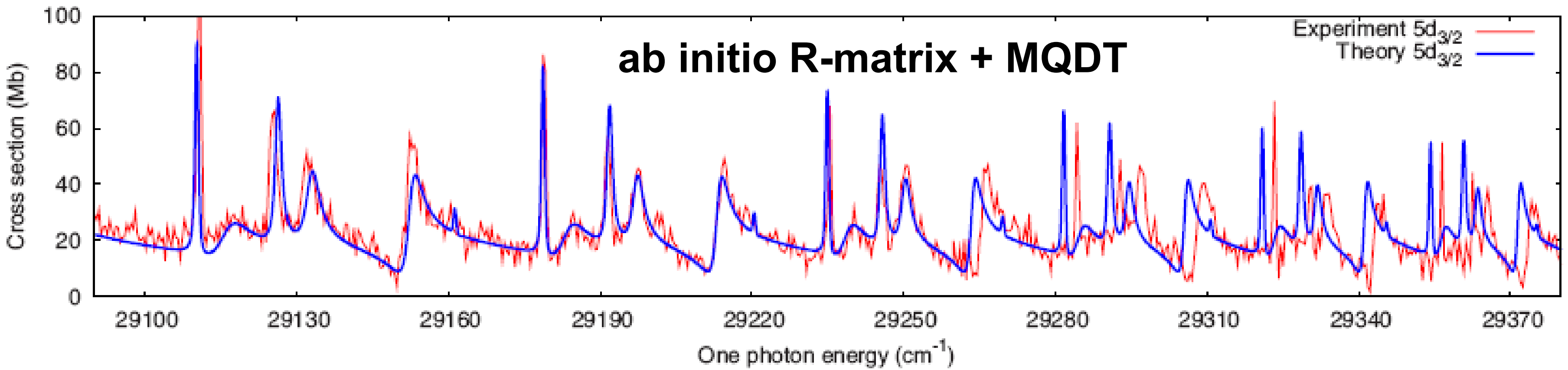
Seaton, Fano... (e.g. Seaton, Rep. Prog. Phys. 46, 1983)

Generalized quantum defect theory (arbitrary long range)

CHG, Fano, Rau, Mies, Gao, Bohn, ... e.g. PRL 81, 3355 (1998); PRA 26, 2441 (1982)

e.g., Ba excited state photoionization

PHYSICAL REVIEW A **80**, 033401 (2009)



Qualitative and quantitative understanding of Efimov's result

At a qualitative level, it can be understood in hindsight, because two particles that are already attracting each other and are infinitesimally close to binding, just need a bit of additional attraction from a third particle in order to push them over that threshold to become a bound three-body system.

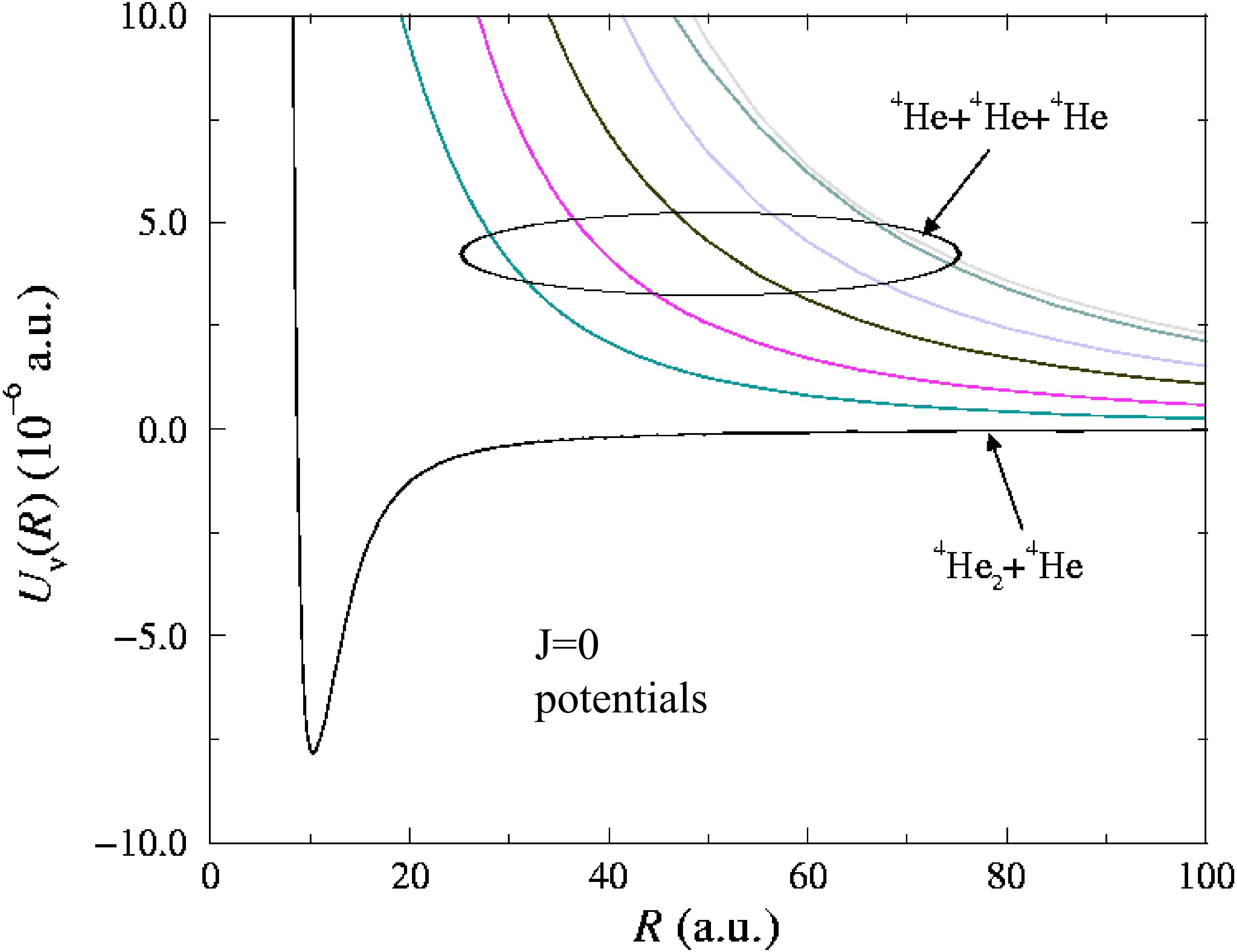
Quantitatively, Efimov (and later others) showed that a simple wavefunction can be written down at each hyperradius,

$$R^2 \propto r_{12}^2 + r_{23}^2 + r_{31}^2$$

i.e. R measures the SIZE of the 3-particle triangle, but not its shape. The shape is measured by 2 hyperangles, (θ, ϕ)

Aside: Efimov's method is similar to Joe Macek's 1968 "adiabatic hyperspherical potential curve" method, although these two theorists were not aware of each others' work until decades later. See also Werner and Castin's papers about the unitary gas limit.

Methodology: diagonalize the fixed-R Schrödinger equation at each R to obtain potential curves and couplings, thus mapping the many-dimensional Sch. Eqn. onto 1D coupled equations.



10.0
5.0
0.0
−5.0
−10.0
0
20
40
60
80
100
$U_v(R)$ (10^{-6} a.u.)
R (a.u.)
^{4}He+^{4}He+^{4}He
$^4He_2$+^{4}He
J=0
potentials

PHYSICAL REVIEW A 75, 062713 (2007)

3-body recombination in ^4He

J. R. Shepard

Fadeev treatment in momentum space, effective theory, comparison with hyperspherical calculation:

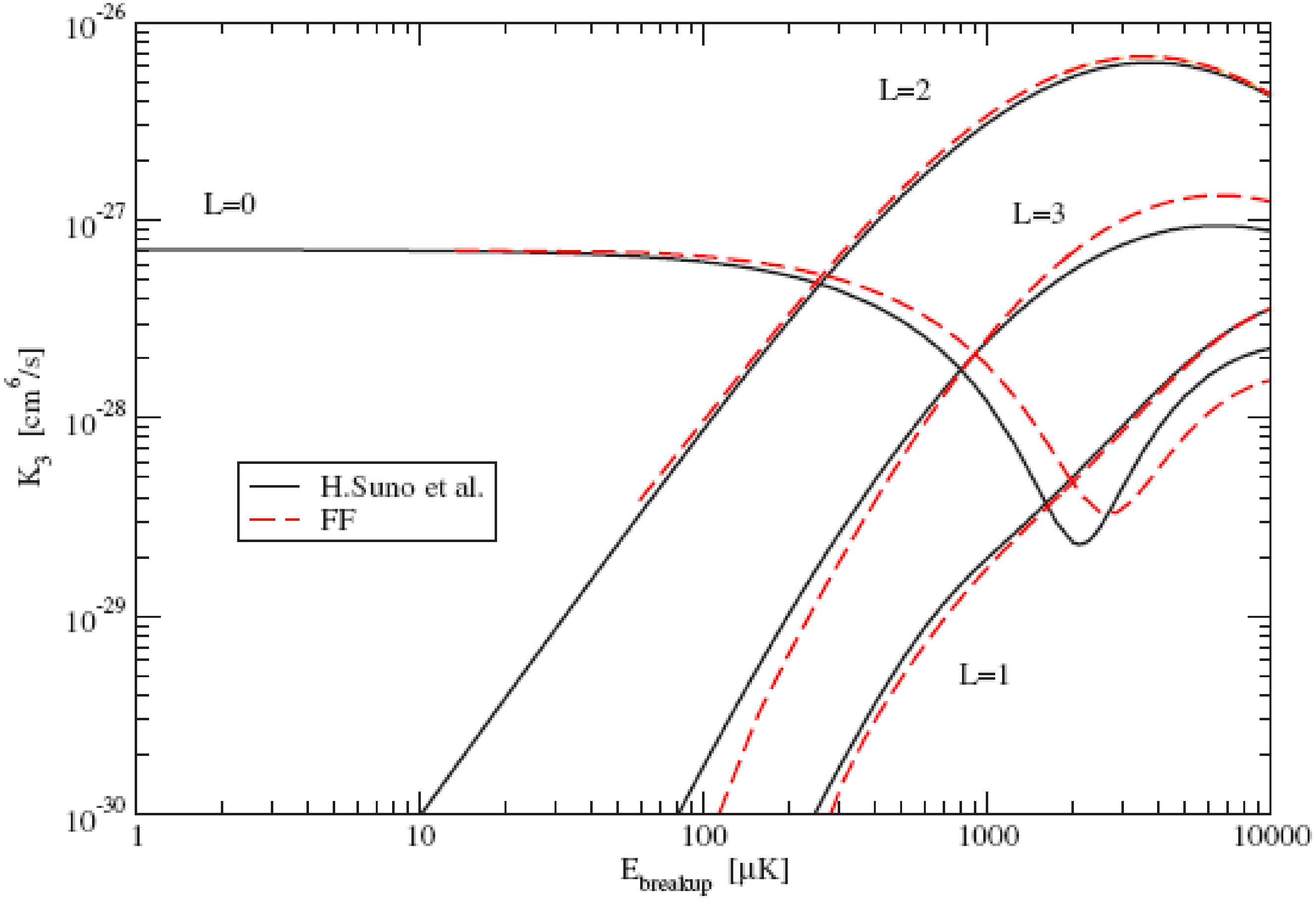


FIG. 2. (Color online) $K_3(L)$ computed using the form factor $g(q)$ (FF) and atom-atom potential HFD-B3-FCII. Present results (dashed) are compared with those of Suno *et al.* [16] (solid).

$$\left(\frac{\Lambda^2}{2\mu R^2} + \underline{E_{th}} + \underline{V}(R,\Omega)\right)\vec{\Phi}(R;\Omega) = U(R)\vec{\Phi}(R;\Omega)$$

$$\Lambda^2 = \frac{-1}{\sin\alpha\cos\alpha}\frac{\partial^2}{\partial\alpha^2}(\sin\alpha\cos\alpha) + \frac{\vec{l}_1^{\,2}}{\sin^2\alpha} + \frac{\vec{l}_2^{\,2}}{\cos^2\alpha} - 4$$

Green's function method to find the hyperspherical potentials

$$\left(\Lambda^2 - \nu_\Sigma(\nu_\Sigma + 4)\right) G_{\Sigma\Sigma}(\Omega,\Omega') = \delta(\Omega,\Omega')$$

Defining $g^{\nu_\Sigma}_{l_1 l_2}(\alpha,\alpha') = N_{\nu_\Sigma l_1 l_2} f^{(-)}_{\nu_\Sigma l_1 l_2}(\alpha_<) f^{(+)}_{\nu_\Sigma l_1 l_2}(\alpha_>)$, The most useful solution is written [34],

$$G_{\Sigma\Sigma}(\Omega,\Omega') = \sum_{l_1,m_1,l_2,m_2} g^{\nu_\Sigma}_{l_1 l_2}(\alpha,\alpha') Y_{l_1 m_1}(\omega_1) Y^*_{l_1 m_1}(\omega_1) Y_{l_2 m_2}(\omega_2) Y^*_{l_2 m_2}(\omega_2') \tag{7}$$

$$N_{\nu_\Sigma l_1 l_2} = \frac{1}{2}\frac{\Gamma(\frac{l_1+l_2-\nu_\Sigma}{2})\Gamma(\frac{l_1+l_2+\nu_\Sigma+4}{2})}{\Gamma(l_1+\frac{3}{2})\Gamma(l_2+\frac{3}{2})} \tag{8}$$

$$f^{(-,+)}_{\nu_\Sigma l_1 l_2}(\alpha) = (\sin\alpha)^{l_1}(\cos\alpha)^{l_2} {}_2F_1\left(\frac{l_1+l_2-\nu_\Sigma}{2}, \frac{\nu_\Sigma+l_1+l_2+4}{2}, l_{1,2}+\frac{3}{2}, \frac{1\mp\cos(2\alpha)}{2}\right) \tag{9}$$

where $l_{1,2} = l_1$ for f^+ and $l_{1,2} = l_2$ for f^-. The hyperangular Lippmann-Schwinger (L-S) equation,

$$\Phi_\Sigma(\Omega) = -2\mu R^2 \sum_{\Sigma',k} \int d\Omega'\, G_{\Sigma\Sigma}(\Omega,\Omega') V^{(k)}_{\Sigma\Sigma'}(R,\Omega')\Phi_{\Sigma'}(\Omega') \tag{10}$$

is then solved by evaluating each integral in a *different* coordinate system. The integral over

The hyperspherical potential curves are obtained as roots of a transcendental determinantal self-consistency condition:

$$\left(\frac{3^{1/4}}{2^{1/2}R}\left(\underline{M}^{(1)}+\underline{M}^{(2)}\underline{P_-}+\underline{M}^{(3)}\underline{P_+}\right)-\underline{1}\right)\vec{C}^{(1)}=0 \tag{6}$$

where, for bosons,

$$M^{(i)}_{\Sigma\Sigma'}=\begin{cases}A^{(i)}_{\Sigma\Sigma'}\lambda_\Sigma\cot\left(\lambda_\Sigma\pi/2\right) & i=1\\ A^{(i)}_{\Sigma\Sigma'}\frac{-4\sin\left(\lambda_\Sigma\pi/6\right)}{\sqrt{3}\sin\left(\lambda_\Sigma\pi/2\right)} & i=2,3\end{cases} \tag{7}$$

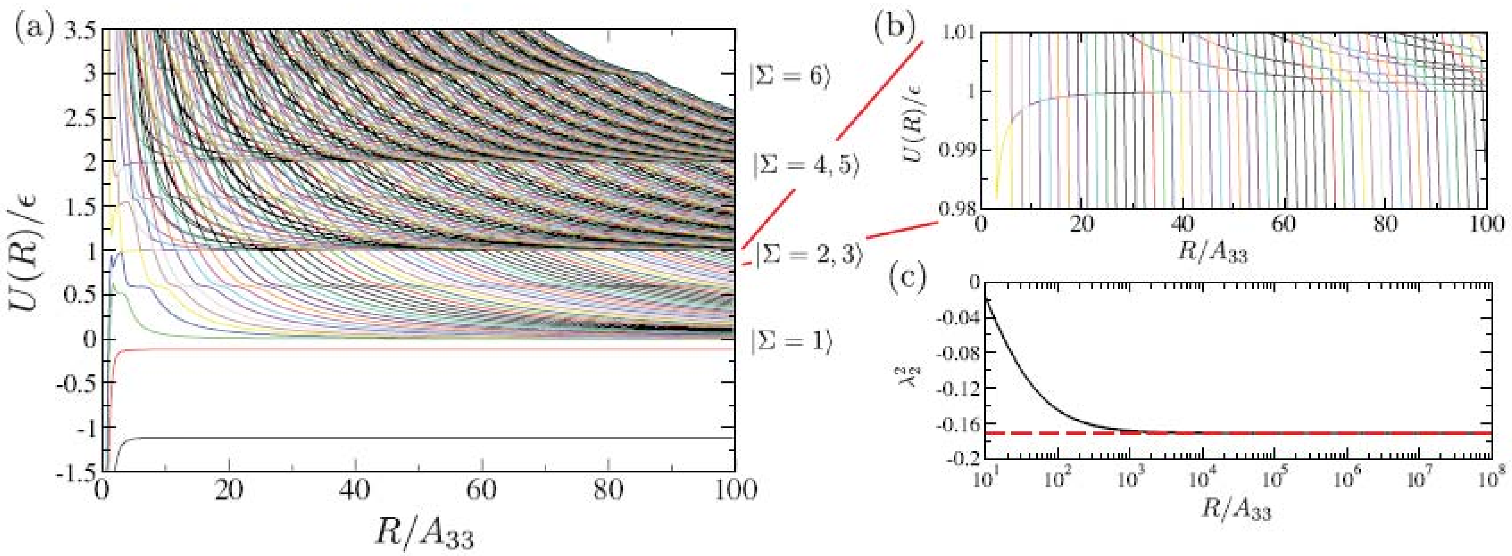


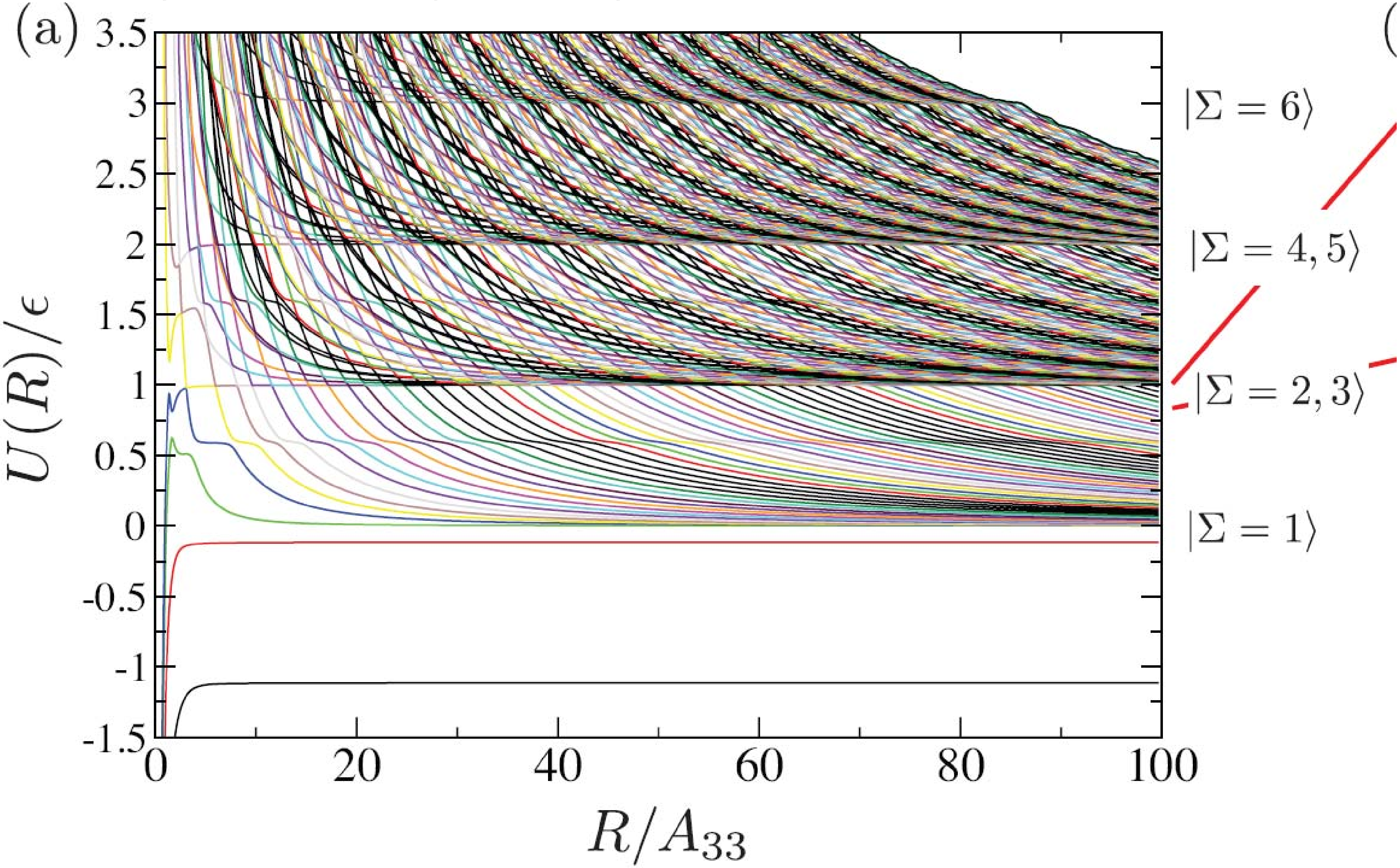
FIG. 3: In (a), we show approximately the lowest 300 potential curves up to 3.5ϵ, while (b) shows an enlarged view of the region near the $E=\epsilon$ threshold, and the attractive R^{-2} diabatic Efimov potential. Note also the series of avoided crossings in (a) near $E=0.6\epsilon$ indicating the presence of a two-body quasi-bound state. In (c) we show the eigenvalue near $E=\epsilon$ converging to the universal value for two identical bosons and one distinguishable particle $\lambda^2\rightarrow-0.171145$ indicated by the dashed red line.

Zero-range multichannel 3-boson potential curves versus hyperradius:
Efimov effect at excited 3-body dissociation thresholds.

PHYSICAL REVIEW A 78, 020701(R) (2008)

Efimov states embedded in the three-body continuum

N. P. Mehta,[*] Seth T. Rittenhouse,[†] J. P. D'Incao,[‡] and Chris H. Greene[§]



March 2010 issue of Physics Today:

Universal insights from few-body land

The ability to tune atomic interactions has inspired theorists and experimentalists to investigate properties of few-particle systems that hold universally, regardless of the specific nature of the inter-particle force.

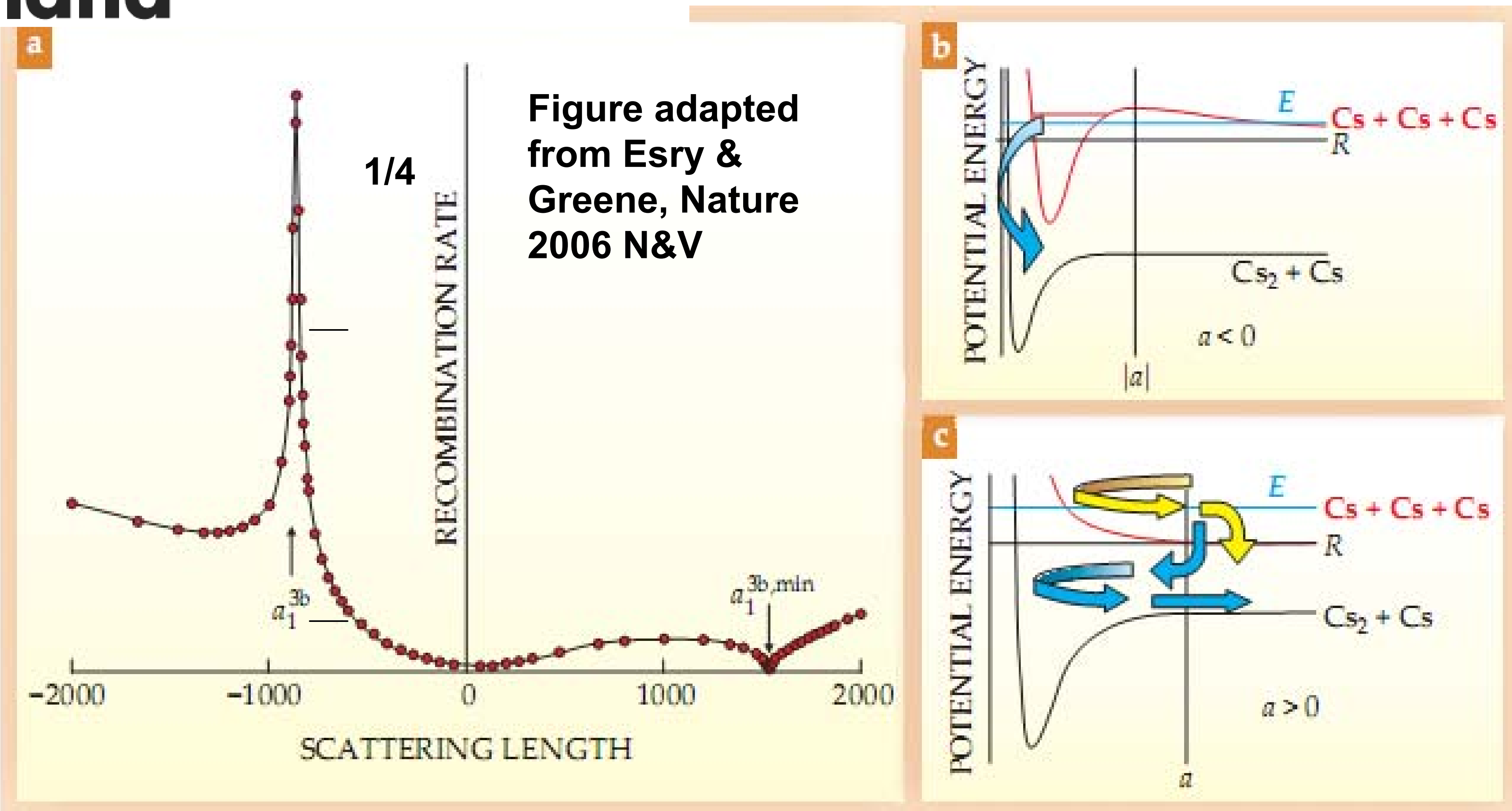
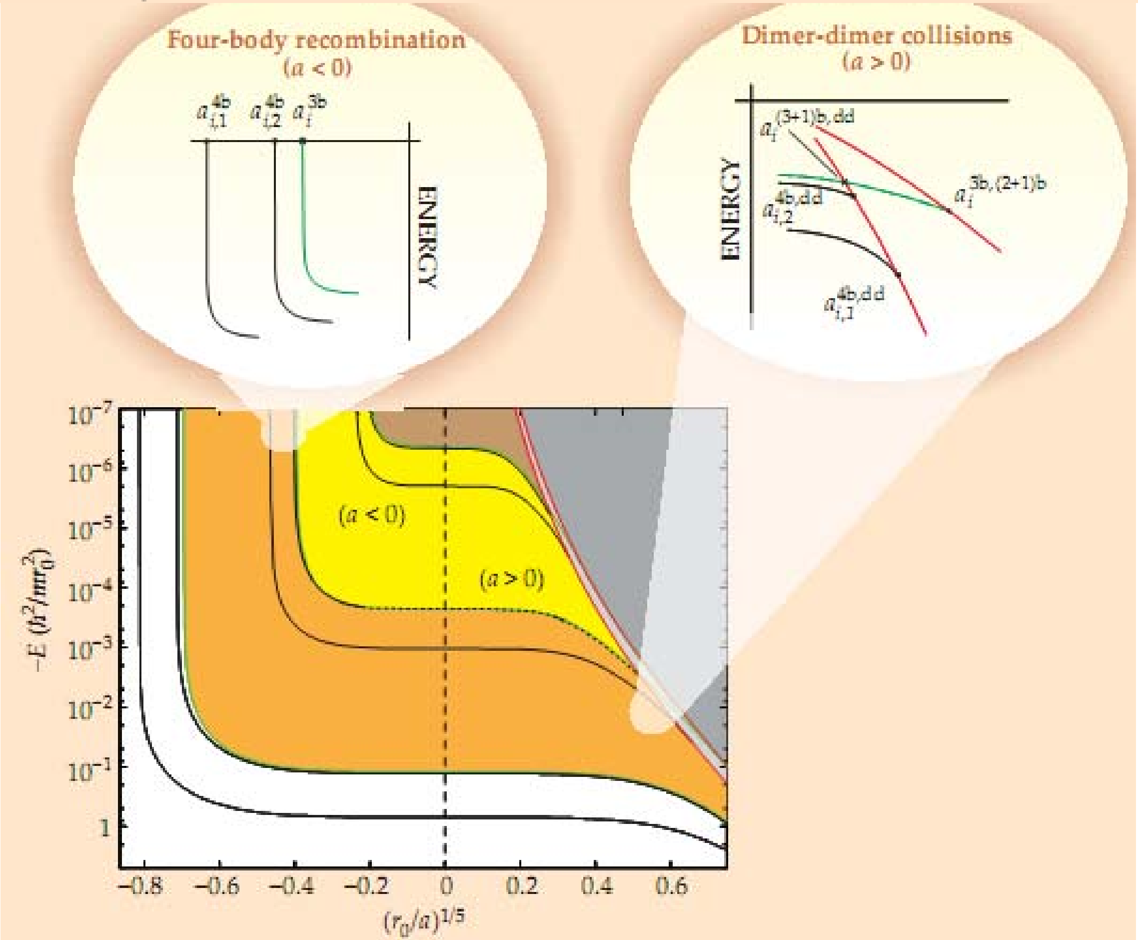


Table 1. Special values of the 2-body scattering length:

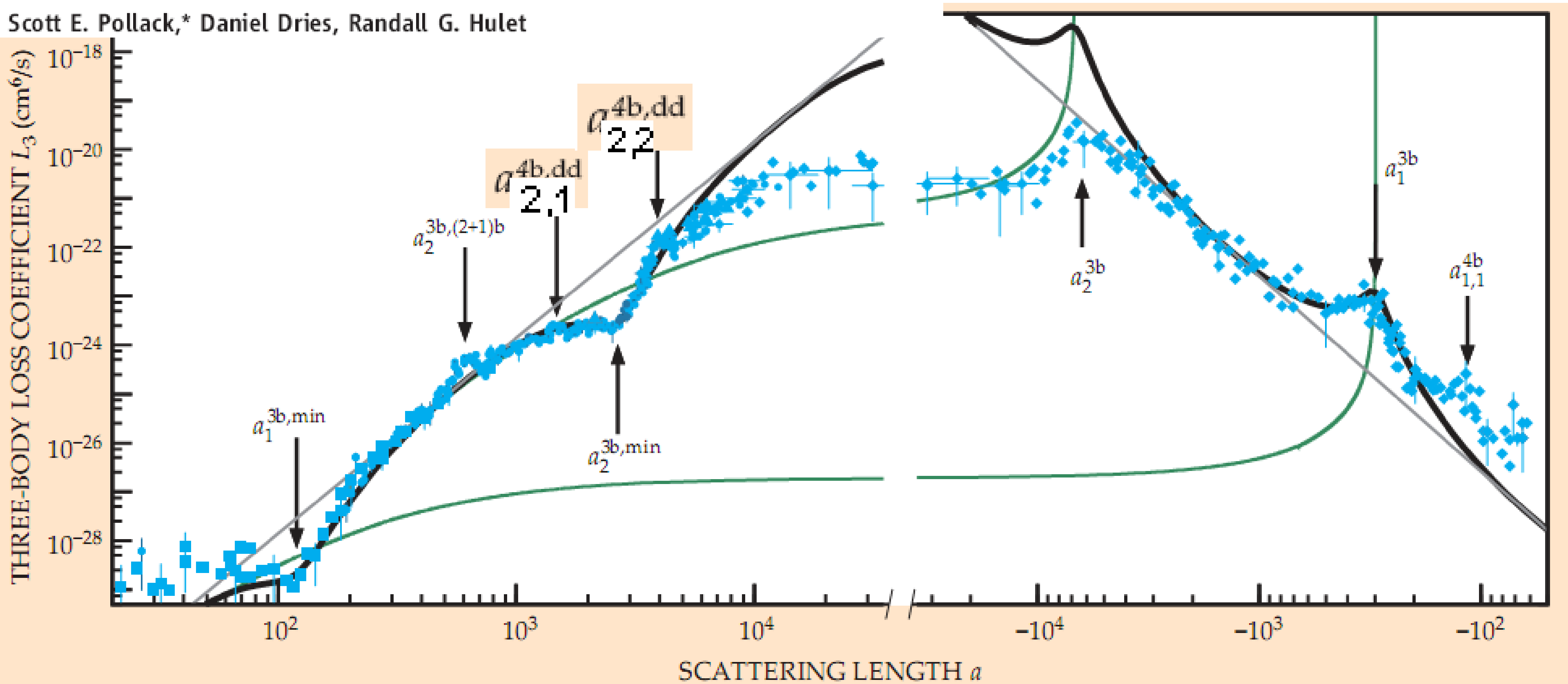
a_i^{3b}	<0, where the ith Efimov trimer state hits $E = 0$; $i = 1,2, \ldots \infty$
$a_{i,k}^{4b}$	<0, where the kth four-body state attached to the ith three-body Efimov state lies at $E = 0$; $k = 1,2$
$a_i^{3b,\min}$	>0, the ith minimum expected in the three-body recombination rate
$a_i^{3b,(2+1)b}$	>0, where the ith Efimov trimer crosses the energy of the dimer + atom threshold, $E = -\hbar^2/ma^2$
$a_{i,k}^{4b,dd}$	>0, where the kth four-body state ($k = 1,2$) attached to the ith Efimov trimer crosses the dimer + dimer threshold, $E = -2\hbar^2/ma^2$
$a_i^{(3+1)b,dd}$	>0, where the ith Efimov trimer + atom threshold crosses the dimer + dimer threshold, $E = -2\hbar^2/ma^2$
$a_{i,1}^{Nb}$	<0, where the lowest N-body state built on the ith Efimov trimer crosses $E = 0$



Universality in Three- and Four-Body Bound States of Ultracold Atoms

Science, Dec.2009

Scott E. Pollack,* Daniel Dries, Randall G. Hulet



a_i^{3b}	< 0, where the ith Efimov trimer state hits $E = 0$; $i = 1,2, \ldots \infty$
$a_{i,k}^{4b}$	< 0, where the kth four-body state attached to the ith three-body Efimov state lies at $E = 0$; $k = 1,2$
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$a_{i,k}^{4b,dd}$	> 0, where the kth four-body state ($k = 1,2$) attached to the ith Efimov trimer crosses the dimer + dimer
$a_i^{(3+1)b,dd}$	> 0, where the ith Efimov trimer + atom threshold crosses the dimer + dimer threshold, $E = -2\hbar^2/ma^2$

Alternative theoretical method – low energy effective field theory

Physics Reports 428 (2006) 259–390

Eric Braaten[a], H.-W. Hammer

Universality in few-body systems with large scattering length

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\vec{\nabla}^2}{2m} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 - \frac{D_0}{6} (\psi^\dagger \psi)^3 + \cdots , \qquad (1)$$

where the dots represent higher-order derivative terms which are suppressed at low energies. The strength of the two-body interaction C_0 is determined by the scattering length a, while D_0 depends on a three-body parameter to be introduced below. For momenta k of the order of the inverse scattering length $1/|a|$, the problem is nonperturbative in ka. The exact two-body

A formula for 3-body recombination into a shallow weakly bound dimer has been derived by Petrov, and in effective field theory by Braaten and Hammer, as:

$$\alpha_{\text{shallow}} = \frac{128\pi^2(4\pi - 3\sqrt{3})}{\sinh^2(\pi s_0) + \cosh^2(\pi s_0)\tan^2[s_0 \ln(a\kappa_*) + \gamma]} \frac{\hbar a^4}{m}$$

This improves somewhat over a simplified version containing much of the same physical content, derived by Esry, Burke, and CHG, 1999 PRL

Two independent theoretical studies in 1999 considered the problem of 3-body recombination (i.e. $Rb+Rb+Rb \rightarrow Rb_2+Rb$) from an adiabatic hyperspherical point of view.

Nielsen and Macek, 1999 PRL

Esry, Greene, and Burke, 1999 PRL

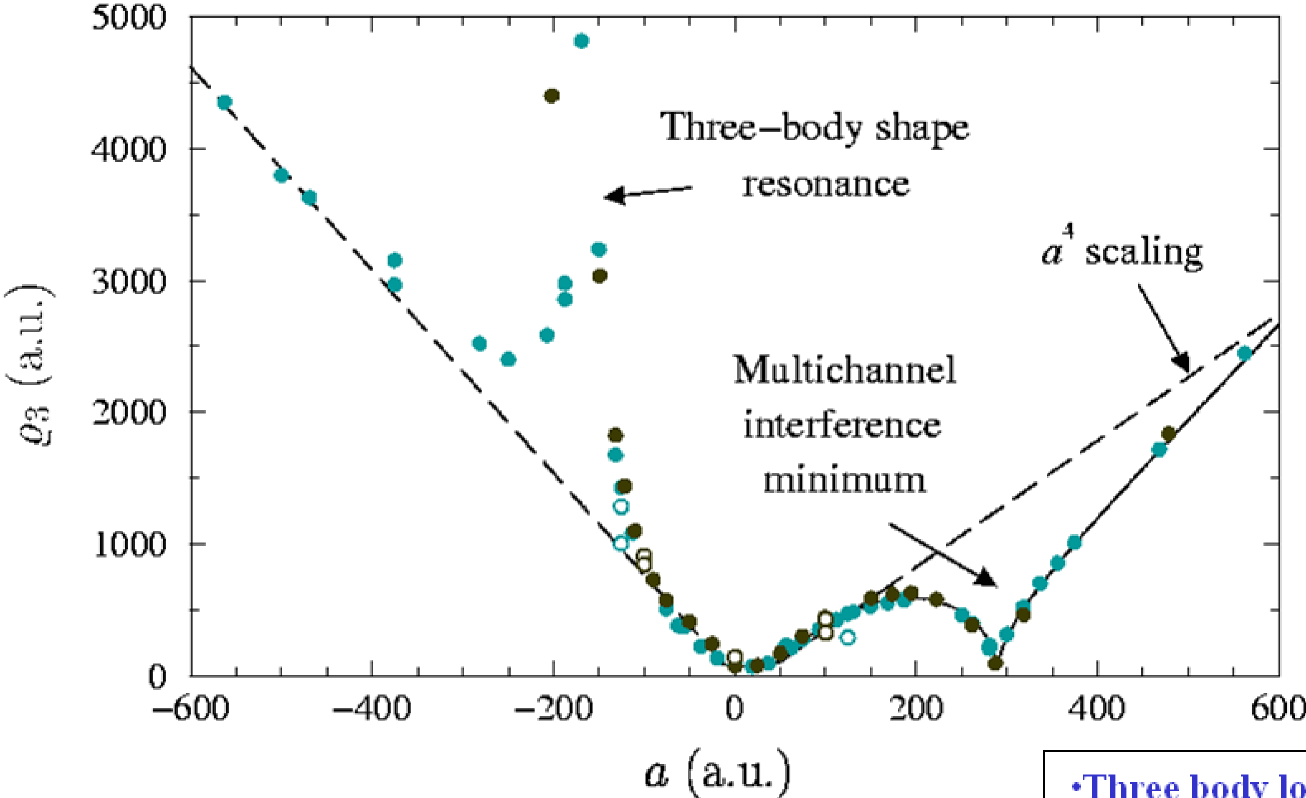
Conclusions:

(1) Both groups found that there should be a series of minima in the 3-body recombination rate coefficient $K_3(a)$ at zero energy, for positive scattering lengths

(2) Esry et al. further predicted an infinite series of strong resonances in $K_3(a)$ at negative scattering lengths, having Efimov character.

In subsequent years, these conclusions were confirmed independently using effective field theory methods, by Braaten, Bedaque, and Hammer (see, e.g., Physics Reports 2006), who derived a convenient parameterization at large $|A|$.

General features of three-body recombination



Predicted dependence of the 3-body recombination rate on the two-body scattering length,

Esry, Greene, Burke, 1999 PRL:

""...the zero-energy rate should be enhanced at an infinite number of Efimov-like shape resonances as A approaches –infinity."

"Recombination length" defined as

$$\varrho_3 = \left(\frac{\mu}{\hbar} K_3\right)^{\frac{1}{4}} \propto a$$

•Three body losses are important for Bose-Einstein condensates, since the loss of atoms goes as:

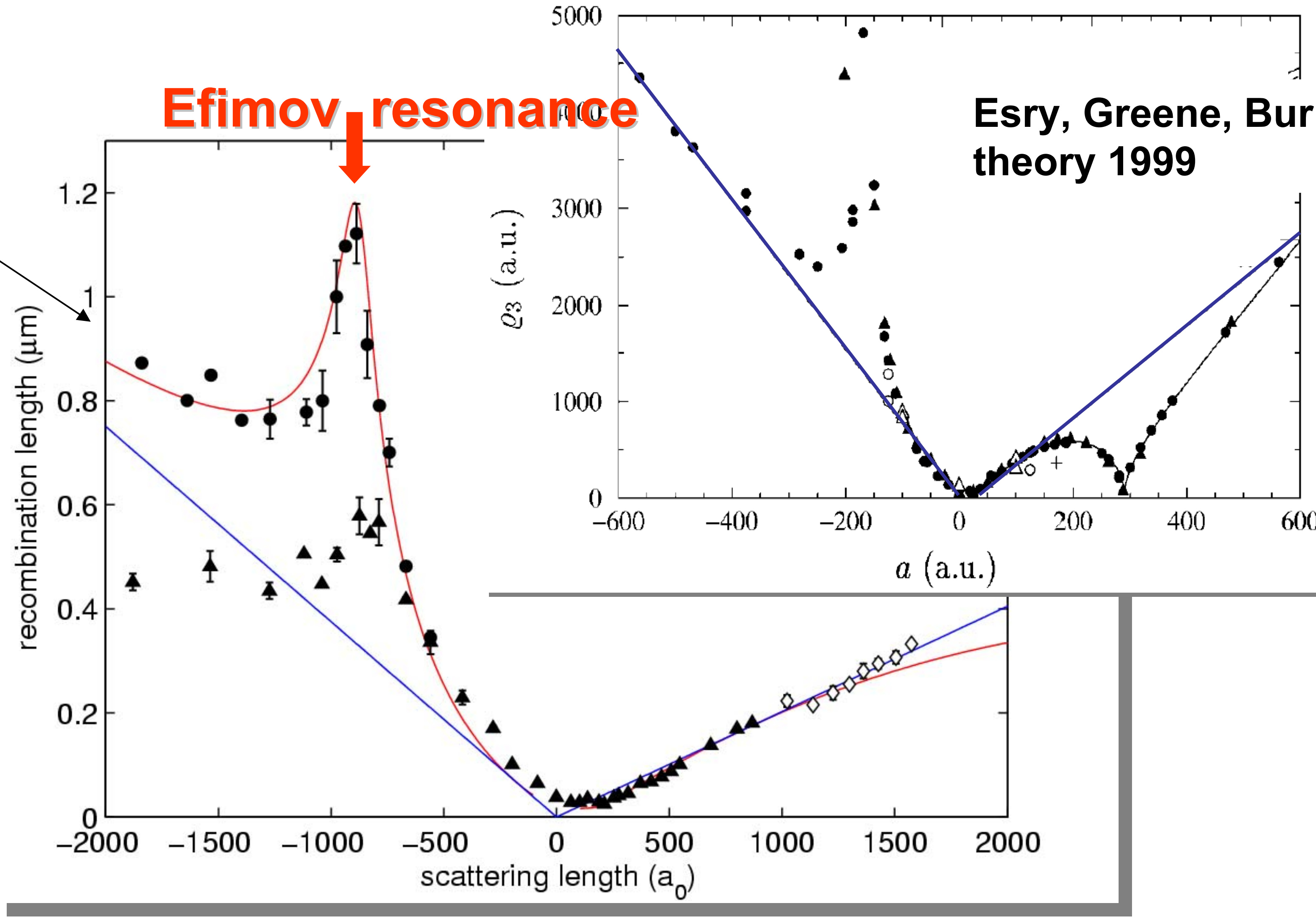
$$\frac{dn}{dt} = -L_3 n^3 - L_2 n^2$$

The 3-body term is important at high density n, or whenever L_3 gets large, $L_3 \propto a^4$

2006 exp. results theory

Grimm group, Nature 2006

Efimov resonance



Esry, Greene, Burke theory 1999

Other groups have subsequently rederived the Efimov physics in the universality regime of large two-body scattering lengths, especially relevant for 3-body recombination, using other methods:

Braaten and Hammer, 2000-2006 – Effective field theory approach

Shepard, 2007 – Fadeev treatment in momentum space, effective theory

Lee, Köhler, Julienne, 2007 – 3-body Green's function approach based on a transition matrix, basic formulation was developed in nuclear physics by Sandhas, Alt, and Grassman.

Gogolin, Mora, Egger, 2008 – Analytic solution of a model

Wetterich and coworkers, 2009 – functional renormalization approach

Observation of an Efimov spectrum in an atomic system

NATURE PHYSICS | VOL 5 | AUGUST 2009 586

M. Zaccanti[1]*, B. Deissler[1], C. D'Errico[1], M. Fattori[1,2], M. Jona-Lasinio[1], S. Müller[3], G. Roati[1], M. Inguscio[1] and G. Modugno[1]

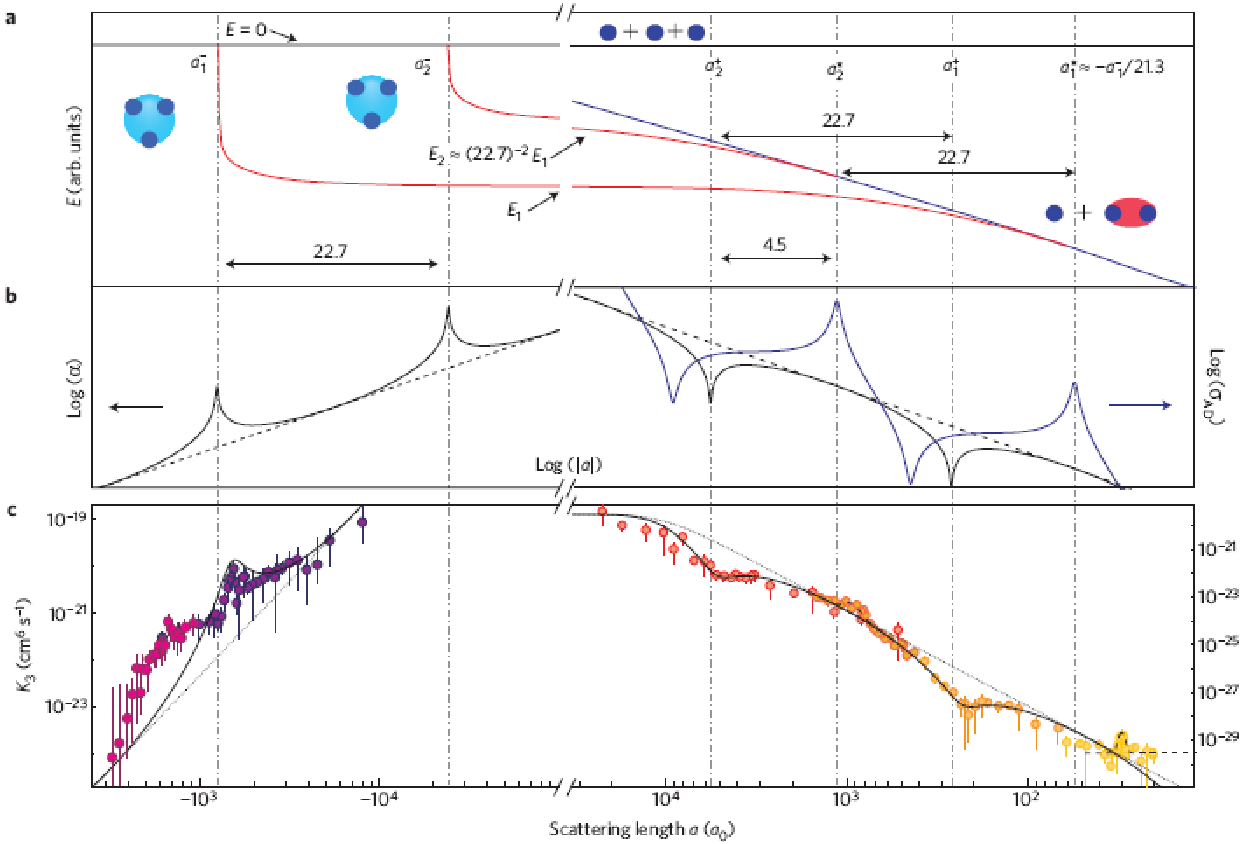


Figure 1 | Efimov spectrum. a Theoretical binding energy of two consecutive Efimov states (red) and of the dimer state (blue) in the universal regime versus the scattering length a. **b**, Theoretical three-body recombination rate α (black) and atom–dimer elastic cross-section σ_{AD} (blue). The vertical dash-dotted lines indicate the position of the detectable maxima and minima in the three-body observables, for which the relevant scaling rules are summarized in **a**. The dashed lines indicate the a^4 behaviour of the three-body recombination rate expected in the absence of Efimov states. **c**, Measured recombination coefficient K_3 in an ultracold potassium gas (circles), featuring deviations from the bare a^4 trend (dotted line), and fitted behaviour assuming a local universal trend for K_3 in the vicinity of the two recombination minima at $a > 0$ and of the Efimov resonance at $a < 0$ (solid line), see text. The other two features due to the atom–dimer resonances a_1^* and a_2^*, not expected by theory, are locally fitted with a Gaussian profile superimposed to a constant background and to the universal behaviour, respectively (dashed lines). The various colours correspond to different data sets. For all data points, the error bars are the root sum squared of the standard error of the mean value resulting from the fit and of the uncertainty on the trap frequencies (see the Methods section).

Observation of universality in ultracold ^{7}Li three-body recombination

arXiv:0906.4731 and **PRL 103, 163202 (2009)**

Noam Gross[1], Zav Shotan[1], Servaas Kokkelmans[2] and Lev Khaykovich[1]

We report on experimental evidence of universality in ultracold ^{7}Li atoms' three-body recombination loss in the vicinity of a Feshbach resonance. We observe a recombination minimum and an Efimov resonance in regions of positive and negative scattering lengths, respectively, which are connected through the pole of the Feshbach resonance. Both observed features lie deeply within the range of validity of the universal theory and we find that the relations between their properties, i.e. widths and locations, are in an excellent agreement with the theoretical predictions.

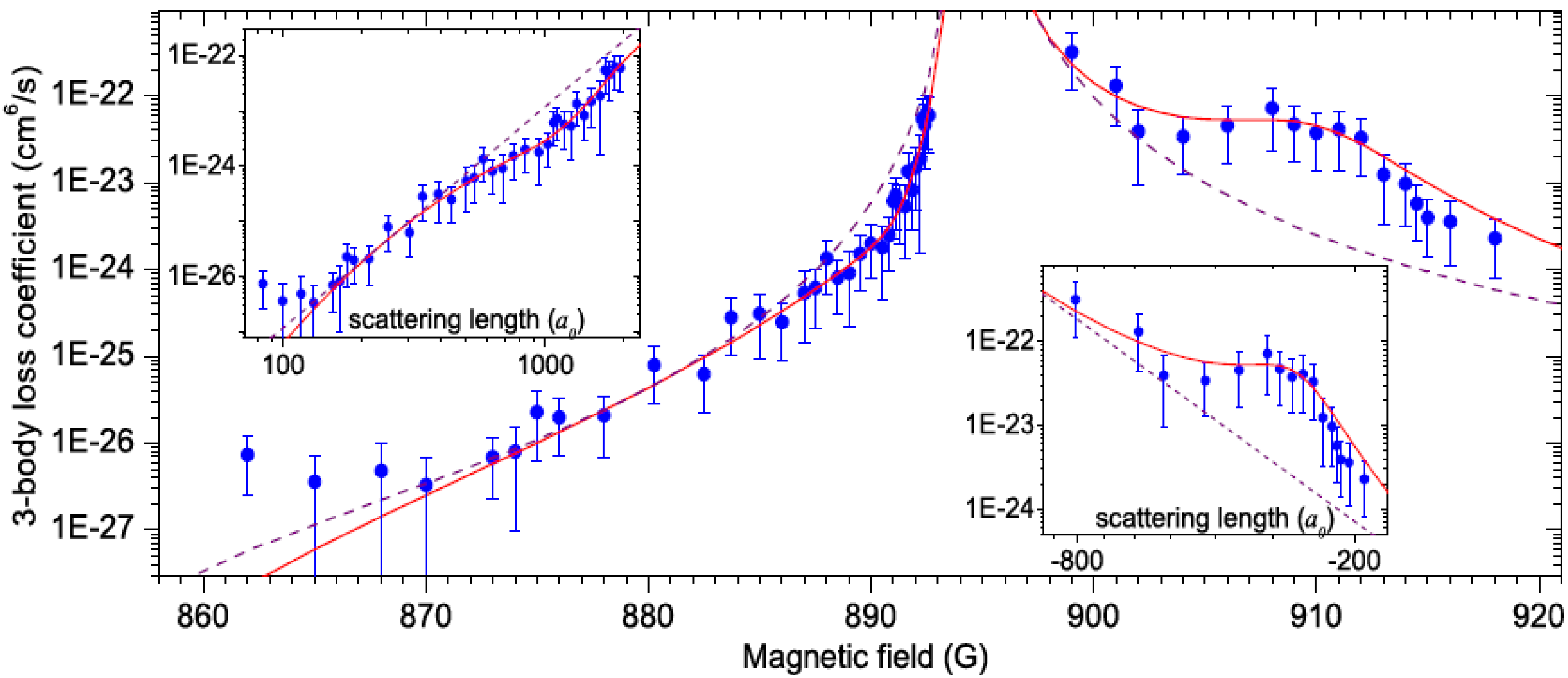


Figure 2: Three-body loss coefficient K_3 is shown as a function of magnetic field and scattering length (insets). The solid lines represent fittings to the analytical expressions of universal theory. The dashed lines represent the upper (lower) limit of K_3 for $a > 0$ ($a < 0$).

T. B. Ottenstein,* T. Lompe, M. Kohnen, A. N. Wenz, and S. Jochim†

Universal 3-body physics for 3-component Fermionic ^{6}Li

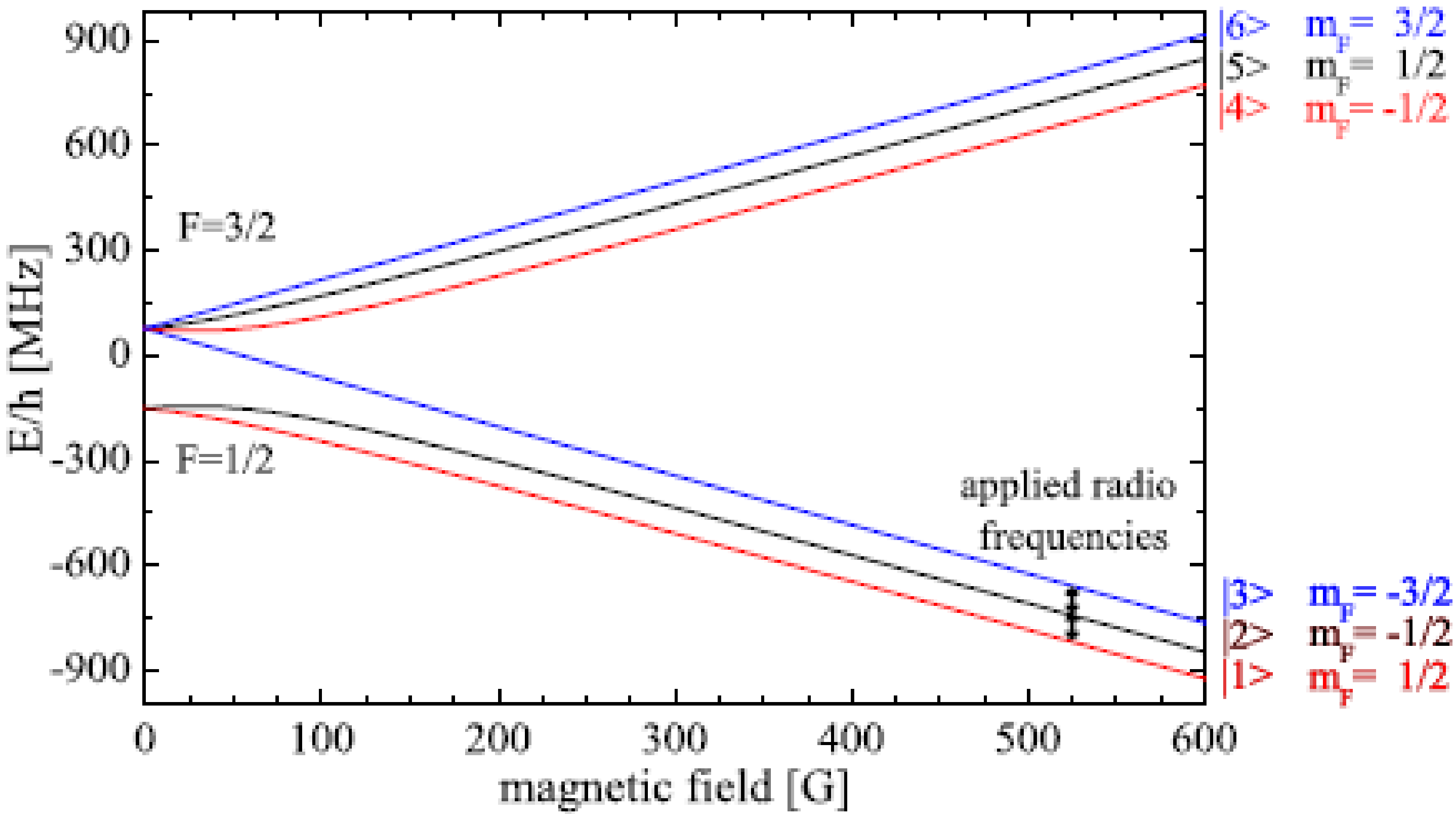


FIG. 1 (color online). Zeeman hyperfine levels of ^{6}Li in the electronic ground state. Transitions between adjacent hyperfine states can be driven with external rf fields.

PRL 101, 203202 (2008)

Collisional Stability of a Three-Component Degenerate Fermi Gas

T. B. Ottenstein,* T. Lompe, M. Kohnen, A. N. Wenz, and S. Jochim†

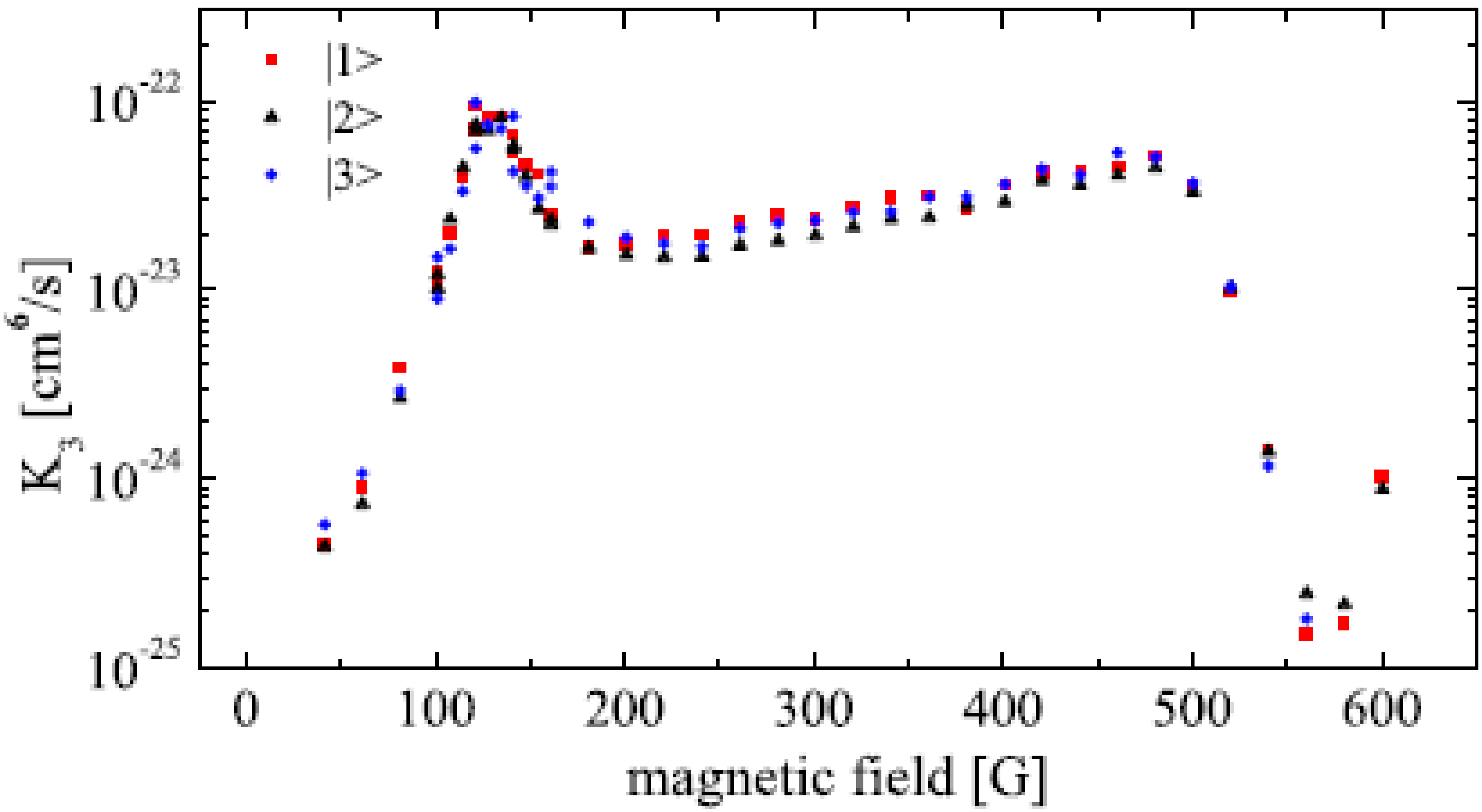


FIG. 4 (color online). Three-body loss coefficient K_3 vs magnetic field.

PRL 102, 165302 (2009)

Three-Body Recombination in a Three-State Fermi Gas with Widely Tunable Interactions

J. H. Huckans, J. R. Williams, E. L. Hazlett, R. W. Stites, and K. M. O'Hara

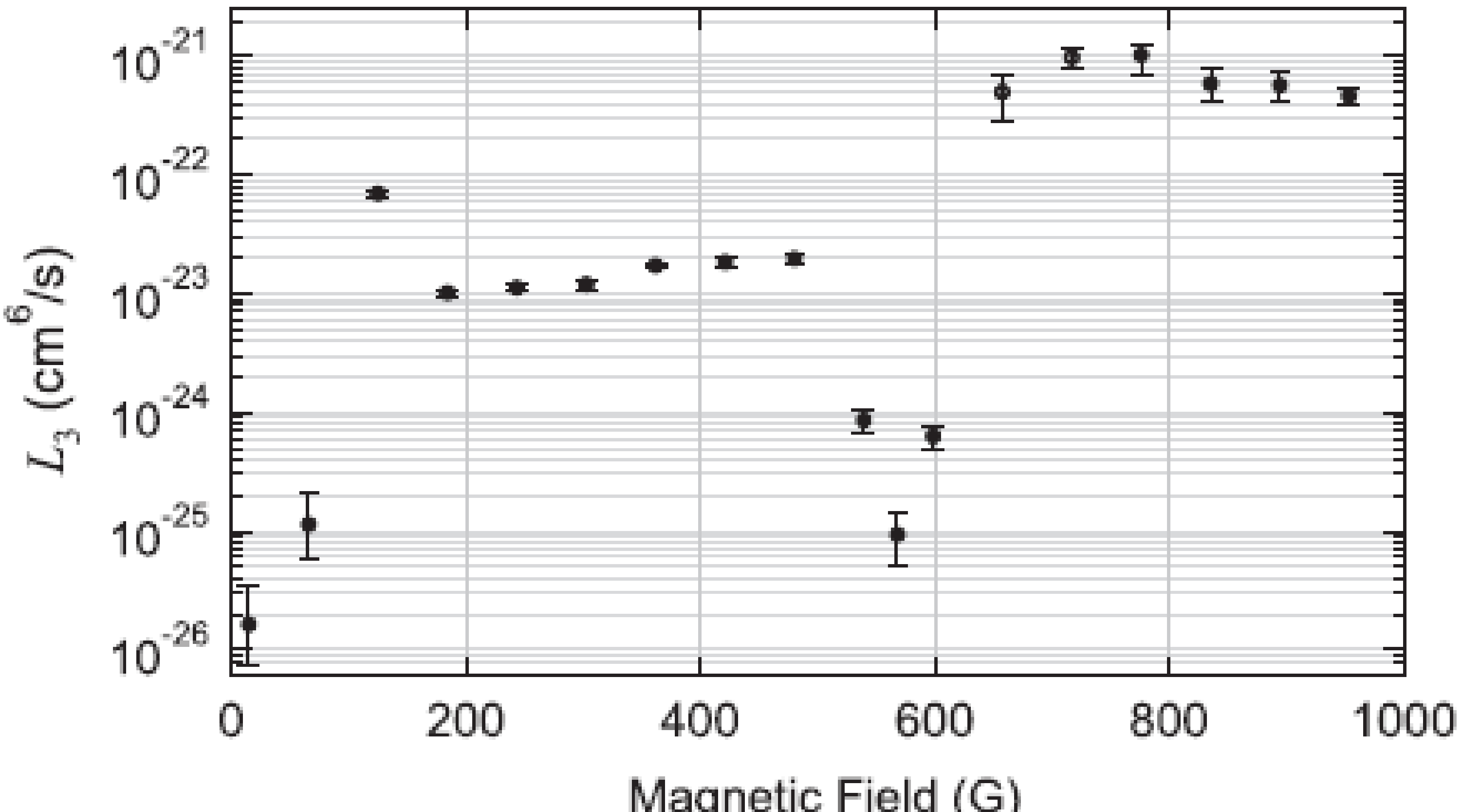


FIG. 4. Three-body loss rate coefficient, L_3, at various fields.

Typical result of universal theory with the three-body parameter adjusted to put the Efimov resonance(s) in the correct place.

FLOERCHINGER, SCHMIDT, AND WETTERICH

PHYSICAL REVIEW A **79**, 053633 (2009)

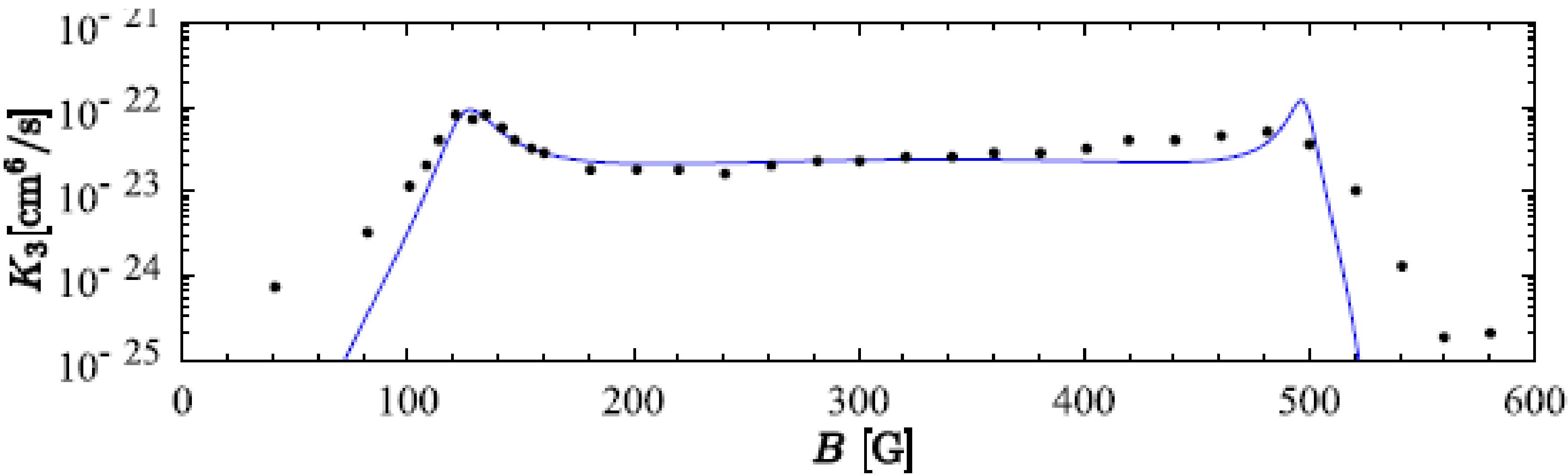


FIG. 4. (Color online) Loss coefficient K_3 in dependence on the magnetic field B as measured in [8] (dots). The solid line is the fit of our model to the experimental curve. We use here a decay width Γ_χ that is independent of the magnetic field B.

Three-Body Recombination of ^{6}Li Atoms with Large Negative Scattering Lengths

Eric Braaten,[1] H.-W. Hammer,[2] Daekyoung Kang,[1] and Lucas Platter[1]

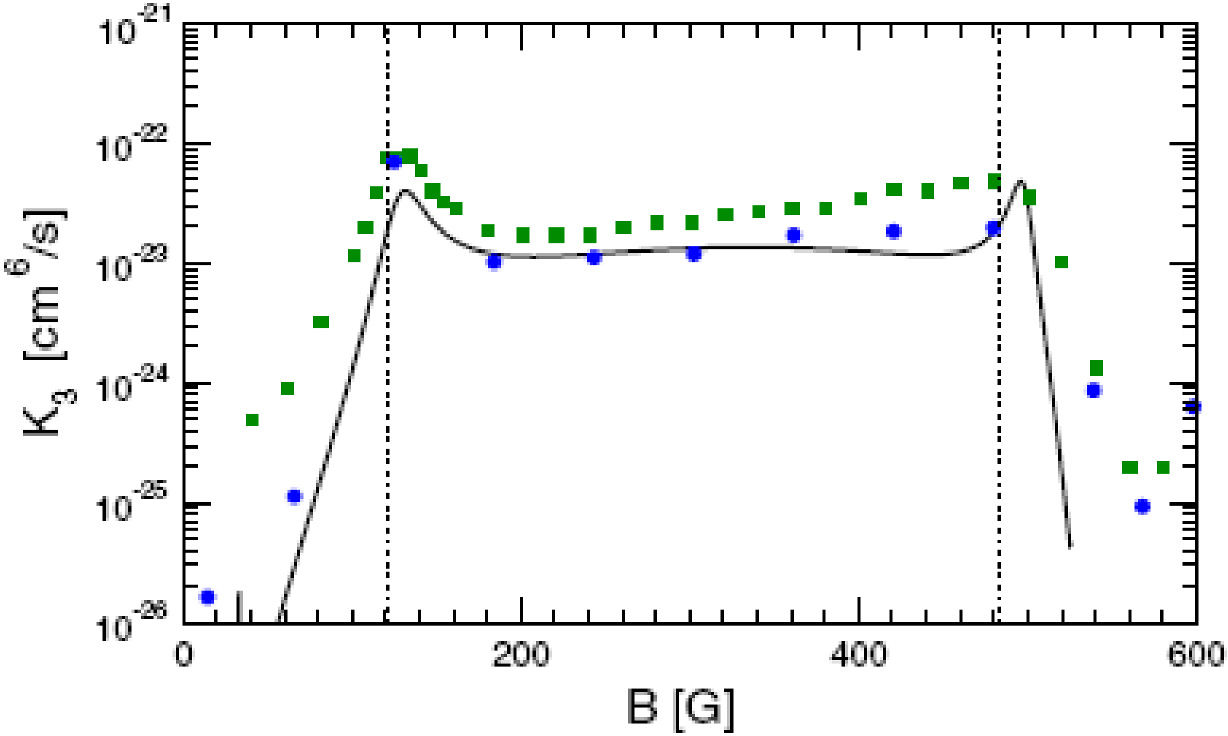
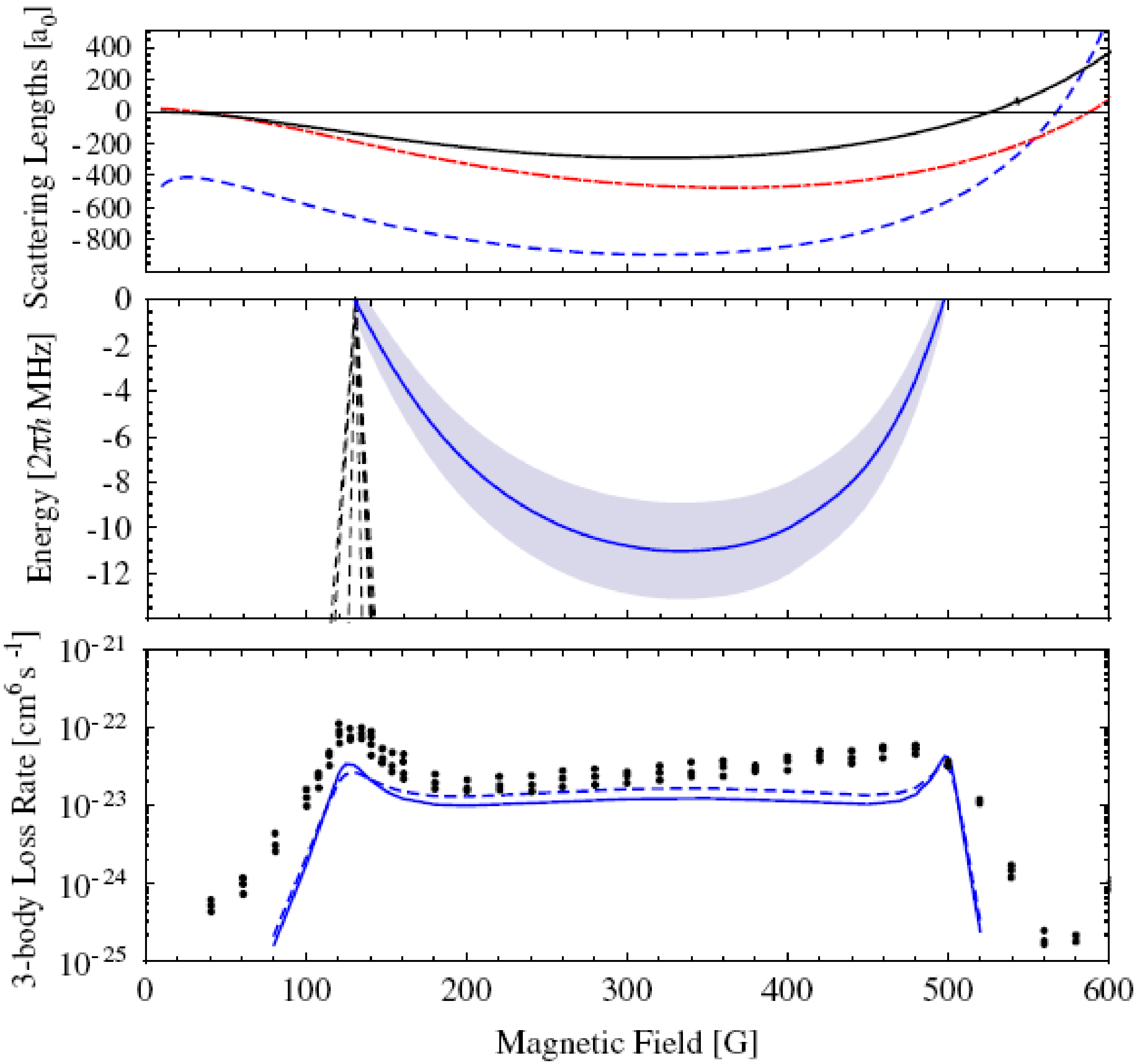


FIG. 3 (color online). The three-body recombination rate constant K_3 as a function of the magnetic field B. The two vertical dotted lines mark the boundaries of the region in which $|a_{12}| > 2\ell_{\mathrm{vdW}}$. The solid squares and dots are data points from Refs. [6,7], respectively. The curve is a 2-parameter fit to the shape of the data from Ref. [6].

PRL 103, 073203 (2009)

Possible Efimov Trimer State in a Three-Hyperfine-Component Lithium-6 Mixture

Pascal Naidon* and Masahito Ueda



And a third theoretical study looks similar, with reasonable overall qualitative agreement but again the higher-field resonance appears to be sharper in theory than in experiment

PHYSICAL REVIEW A 80, 040702(R) (2009)

Universal trimer in a three-component Fermi gas

A. N. Wenz,* T. Lompe, T. B. Ottenstein, F. Serwane, G. Zürn, and S. Jochim†

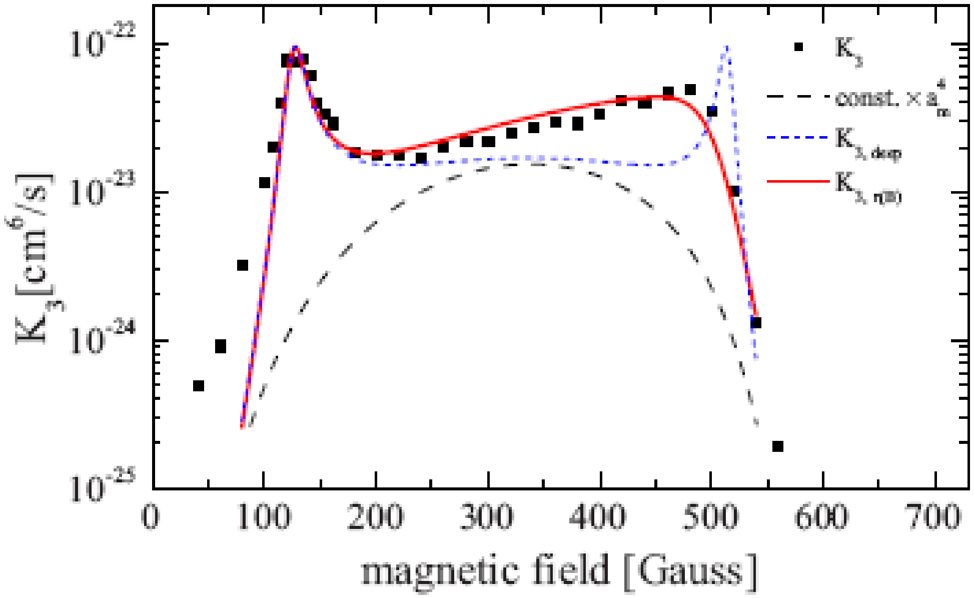


FIG. 3. (Color online) Measured three-body loss coefficient (black squares) and different models as a function of the magnetic field. The dashed line shows the prediction for three-body loss without resonant enhancement due to trimer states, the short dashed line gives the theoretical result for $K_{3,\mathrm{deep}}(a_m)$, and the solid line is the model for K_3 including a magnetic field dependence of η_*.

A very recent preprint → arXiv:1002.4891

Magnetic Field Dependence and Efimov Resonance Broadening in Ultracold Three-Body Recombination

Seth T. Rittenhouse
ITAMP, Harvard-Smithsonian Center for Astrophysics, Cambridge, MA 02138
(Dated: February 25, 2010)

We derive an analytic formula which describes the final bound state dependence in ultracold three-body recombination. Using an energy-dependent loss parameter, the recently observed broad resonance in an ultracold gas of ^{6}Li atoms [1, 2] is described quantitatively. We also provide an analytic and approximation for the three-body recombination rate which encapsulates the underlying physics of the universal three-body recombination process.

$$K_3 = \frac{\hbar 4590}{m} \frac{a^4 \sinh 2\eta}{\cos^2 \phi_{WKB} + \sinh^2 \eta}$$

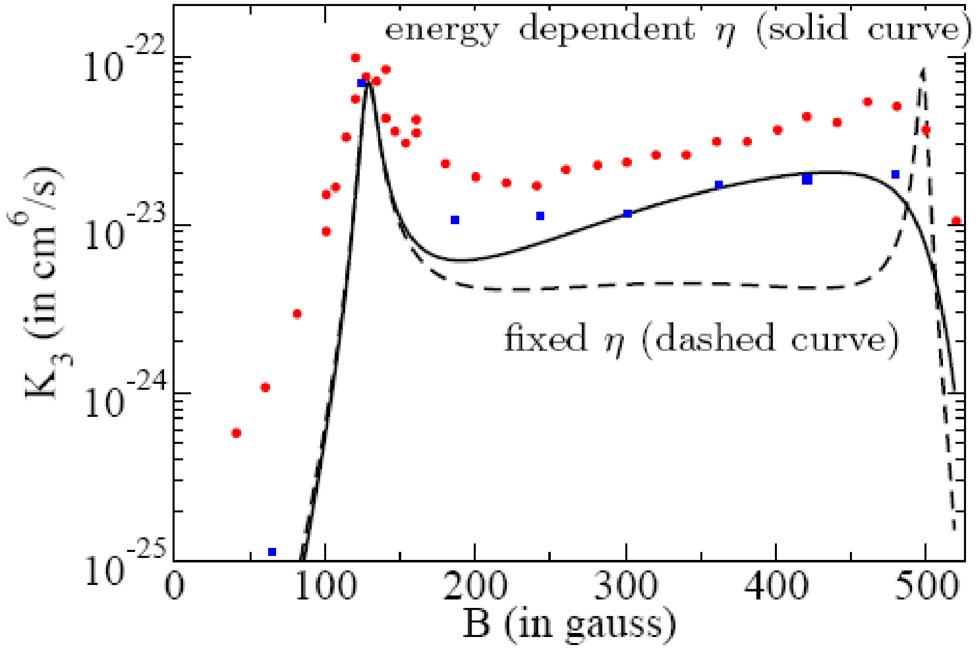
$$\eta = \frac{1}{2} \ln \left[\frac{1}{1 - \exp(-\beta\Delta)} \right]$$

$$\beta = \frac{\pi}{\hbar v} \frac{1}{4P_{\max}}.$$

$$P_{na} = e^{-2\pi\gamma},$$

$$\gamma = \frac{1}{\hbar v} \frac{\Delta}{8P_{\max}},$$

$$\frac{|\Psi_{out}|^2}{|\Psi_{in}|^2} = e^{-4\eta}$$



Since 2006 – We initiated a concerted effort on the 4-body problem using hyperspherical coordinates

Recent papers either published or about to be:

Correlated Gaussian Hyperspherical Method for Few-Body Systems

Javier von Stecher and Chris H. Greene

arXiv:0904.1405 (PRA **80**, 022504, 2009)

Universal Four-Boson States in Ultracold Molecular Gases: Resonant Effects in Dimer-Dimer Collisions

J. P. D'Incao,[1,2] J. von Stecher,[1] and Chris H. Greene[1]

arXiv:0903.3348 (PRL **103**, 033004, 2009)

A general theoretical description of N-body recombination

N. P. Mehta,[1,2] Seth T. Rittenhouse,[1] J. P. D'Incao,[1,3] J. von Stecher,[1] and Chris H. Greene[1]

arXiv:0903.4145 PRL **103**, 153201 (2009)

Signatures of universal four-body phenomena and their relation to the Efimov effect

J. von Stecher, J. P. D'Incao and Chris H. Greene*

Nature Phys. June 2009

Dimer-dimer collisions at finite energies in two-component Fermi gases

J. P. D'Incao, Seth T. Rittenhouse, N. P. Mehta,* and Chris H. Greene

Phys. Rev. A **79**, 030501(R), (2009)

2-component fermions: key observable that has to come out correctly is that dimer-dimer scattering length A_{dd}

Weakly Bound Dimers of Fermionic Atoms

D. S. Petrov,[1,2,*] C. Salomon,[3] and G. V. Shlyapnikov[1,2,3,†]

Phys. Rev. Lett 93, 090404 (2004)

Predicted A_{dd}*=0.6 a +/- 0.01a*

Previous (ancient) theory had been based on the perturbative result A_{dd}***=2 a***

Others confirming this 0.6a result include, e.g. Radzihovsky and Gurarie

Subsequent work: von Stecher and Greene, 2007 PRL:

Our new results, from subtracting the noninteracting energy of two bosonic molecules from the ground state energy on the BEC side of the crossover, have now pinned down the second digit:

A_{dd}=0.608 a +/- 0.003a (distributed Gaussian diagonalization)

Add=0.636 a +/- 0.01a (fixed-node Diffusion Monte Carlo – D. Blume)

Add=0.604 a (4-body hyperspherical calculation)

PRL Referee: *"…the accuracy … was not established in a convincing manner …**As a matter of fact I know for sure that the present approach is definitely not accurate enough**… I do not believe that the authors can prove it any way, as they seem to allude, that they can extract the dimer-dimer scattering length using this set of wave functions. A scattering state can not be expanded in gaussians… "*

Dimer-dimer scattering length and effective range, versus mass ratio

The dimer-dimer effective range is quite large, and it can be important for the behavior of the molecular BEC formed. We obtain: r_{dd}=0.134 a_s in CG (or 0.116 a_s in MC)

J. von Stecher,[1] Chris H. Greene,[1] and D. Blume[1,2]

PHYSICAL REVIEW A **76**, 053613 (2007)

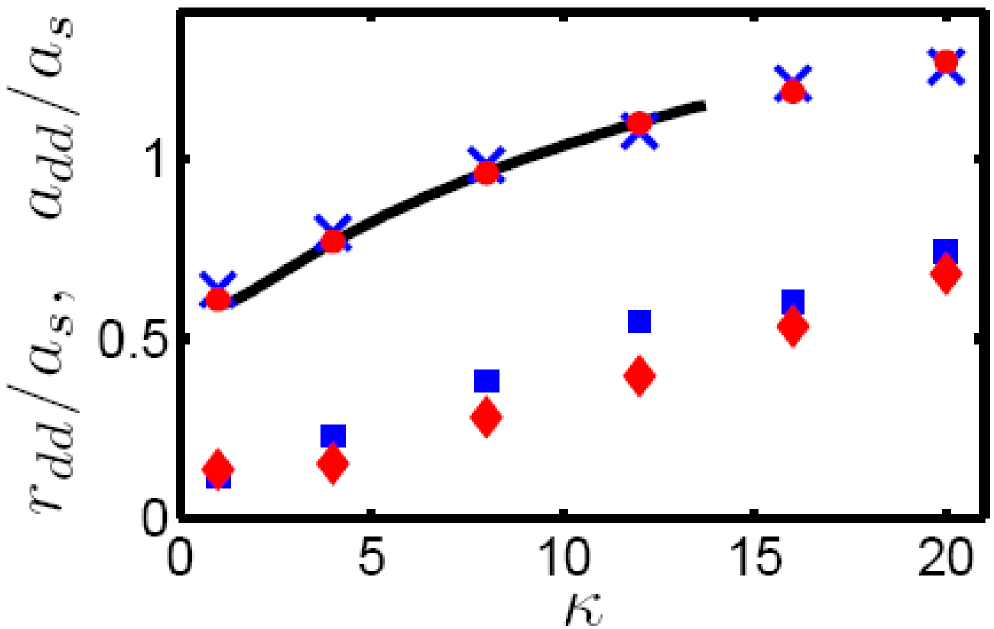


FIG. 3: (Color online) Circles and crosses show a_{dd}/a_s as a function of κ extracted from the four-fermion CG and FN-DMC energies, respectively. For comparison, a solid line shows the results from Fig. 3 of Ref. [8]. Diamonds and squares show r_{dd}/a_s extracted from the four-fermion CG and FN-DMC energies, respectively.

How does Efimov physics extend to four bosons?

A few details for the specialists, our correlated Gaussian hyperspherical method (PRA 2009):

Hamiltonian studied:

$$H = \sum_i \left(\frac{-\hbar^2}{2m_i} \nabla_i^2 + V_{ext}(\mathbf{r}_i) \right) + \sum_{i<j} V_0(r_{ij}).$$

Basis set expansion:

$$\Psi(\mathbf{x}_1, \cdots, \mathbf{x}_N) = \sum_A C_A \Phi_A(\mathbf{x}_1, \cdots, \mathbf{x}_N)$$

Ritz variational optimization:

$$\Phi_A(\mathbf{x}_1, \cdots, \mathbf{x}_N) = \psi_0(\mathbf{R}_{CM}) \mathcal{S} \left\{ \exp\left(-\sum_{j>i=1}^{N} \alpha_{ij} r_{ij}^2 / 2 \right) \right\}$$

$$\mathcal{H}\vec{C}_i = E_i \mathcal{O} \vec{C}_i$$

Def. of hyperradius R:

$$\mu R^2 = \sum_{i=1}^{N} m_i \mathbf{r}_i^2$$

Expansion in hyperangular basis gives coupled 1D ordinary diff equations:

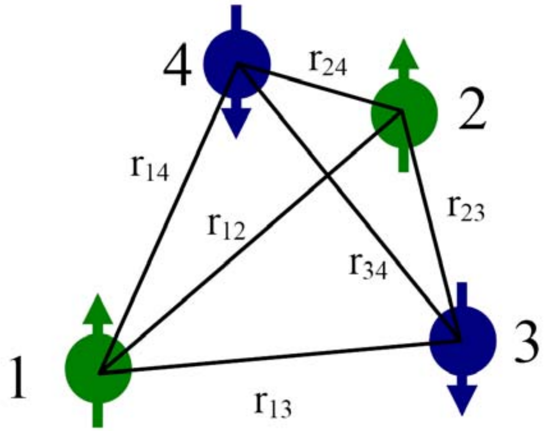
$$\Psi_E(R,\Omega) = \sum_{\nu} F_{\nu E}(R) \Phi_\nu(R;\Omega).$$

$$\left(\frac{\hbar^2 \Lambda^2}{2\mu R^2} + \frac{(d-1)(d-3)\hbar^2}{8\mu R^2} + V(R,\Omega) \right) \Phi_\nu(R;\Omega) = U_\nu(R) \Phi_\nu(R;\Omega).$$

$$\left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dR^2} + U_\nu(R) \right] F_{\nu E}(R) - \frac{\hbar^2}{2\mu} \sum_{\nu'} \left[2P_{\nu\nu'}(R) \frac{d}{dR} + Q_{\nu\nu'}(R) \right] F_{\nu' E}(R) = E F_{\nu E}(R)$$

Four-body Formulation – How to diagonalize the fixed-hyperradius Hamiltonian for 4 (or more) particles?

Correlated Gaussian basis set:



$$\Psi_T(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_3) = \sum_{\{d_{ij}\}} C_{\{d_{ij}\}} \Psi_{\{d_{ij}\}}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$$

$$\Psi_{\{d_{ij}\}}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) = \psi_0(\mathbf{R}_{CM})\mathcal{S}\left\{e^{-r_{12}^2/2d_{12}^2}e^{-r_{13}^2/2d_{13}^2}e^{-r_{14}^2/2d_{14}^2}e^{-r_{23}^2/2d_{23}^2}e^{-r_{24}^2/2d_{24}^2}e^{-r_{34}^2/2d_{34}^2}\right\}$$

$\mathcal{S} = (1-\mathcal{P}_{1,2})(1-\mathcal{P}_{3,4}) \longleftarrow$ **e.g., symmetry operator relevant for 4 fermionic atoms in two-components**

- **The matrix elements can be calculated analytically.**
- **The same basis set can be used through the complete crossover (for fermions).**
- **Linear dependence issues.**
- **Can describe bound states, as usually implemented.**

Revisiting the 2006 Grimm group experiment that was the first to see 3-body Efimov states

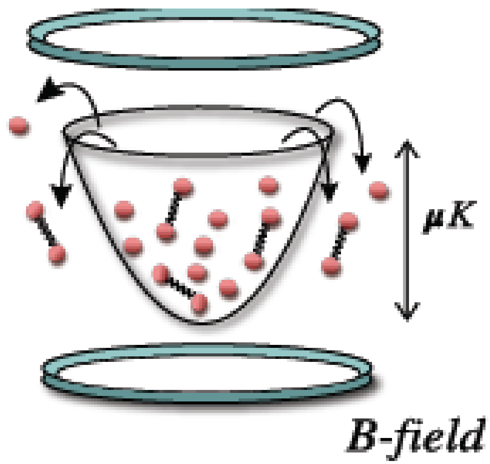
Signatures of universal four-body phenomena and their relation to the Efimov effect

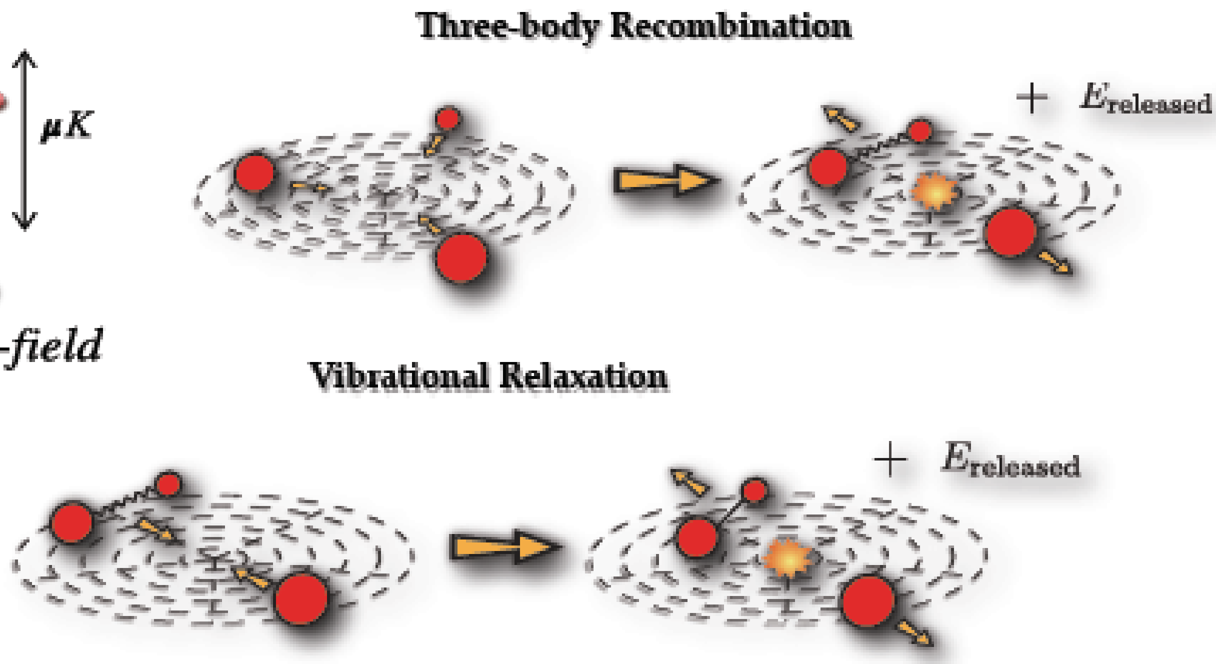
J. von Stecher, J. P. D'Incao and Chris H. Greene*

NATURE PHYSICS | VOL 5 | JUNE 2009 | p.417

Why do we care ?

- Losses / Lifetime / Stability





Previous important studies of the 4-boson system in the universality regime in 3D:

Platter, L., Hammer, H. & Meißner, U. Four-boson system with short-range interactions. *Phys. Rev. A* 70, 52101 (2004).

Hammer, H. W. & Platter, L. Universal properties of the four-body system with large scattering length. *Eur. Phys. J. A* 32, 113–120 (2007).

"We have conjectured, that there are always two four-body resonances between any two three-body states." (i.e. below each Efimov state) + Also, no 4-body param.

Hanna, G. J. & Blume, D. Energetics and structural properties of three-dimensional bosonic clusters near threshold. *Phys. Rev. A* 74, 063604 (2006).

…also found general correlations between N-body bound levels and (N-1)-body bound levels

---

Yamashita, M. T., Tomio, L., Delfino, A. & Frederico, T. Four-boson scale near a Feshbach resonance. *Europhys. Lett.* 75, 555–561 (2006).

…conclude that a "4-body parameter" is in fact needed, but they only studied low (non-universal states), which is presumably why they reach a different conclusion from that of Platter and Hammer and also different from ours.

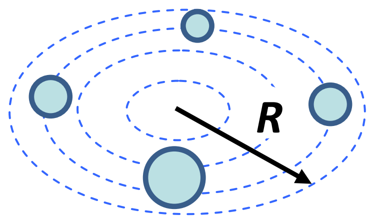
J. von Stecher and C. H. Greene, arXiv:0904.1405

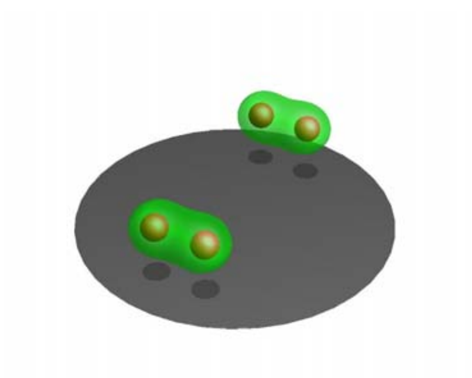
PRA **80**, 022504, (2009)

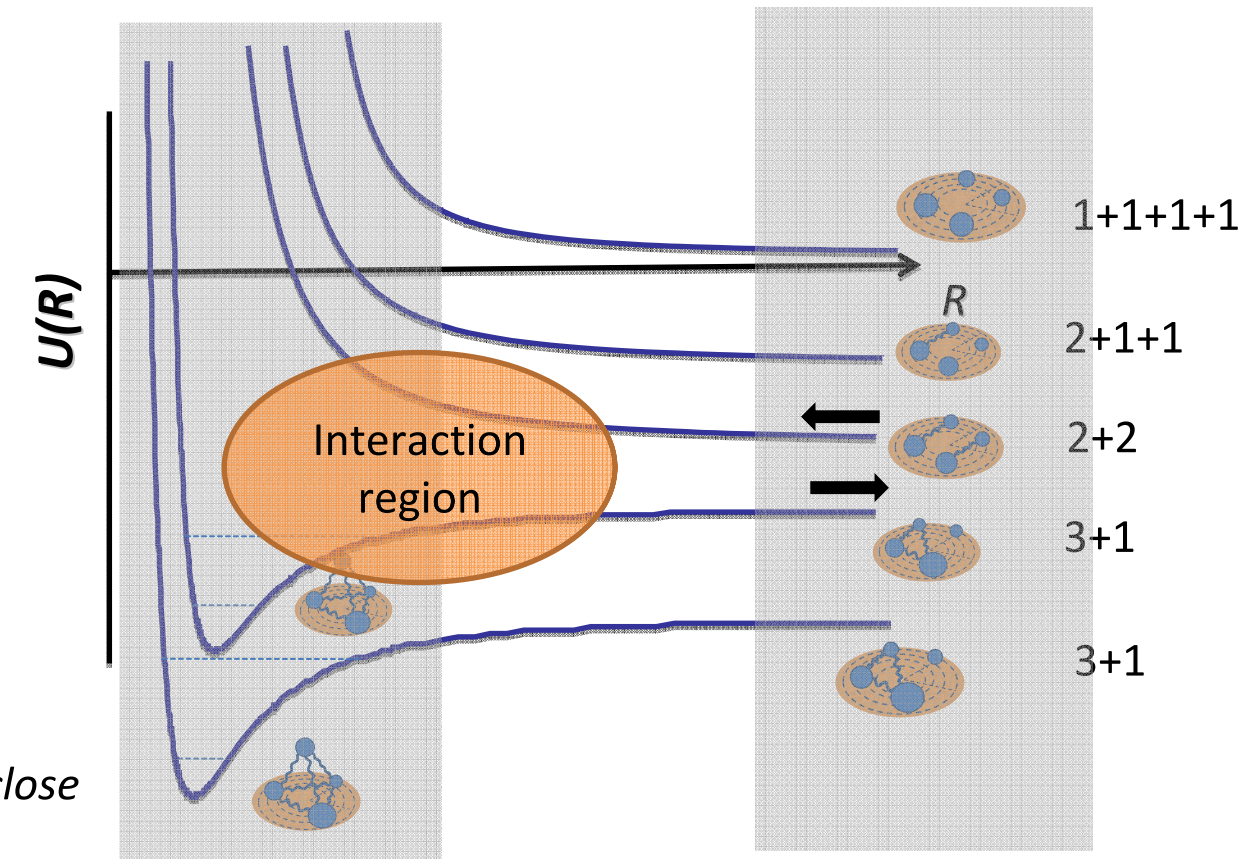
Hyperspherical Picture

… think Born Oppenheimer

Fragmentation thresholds







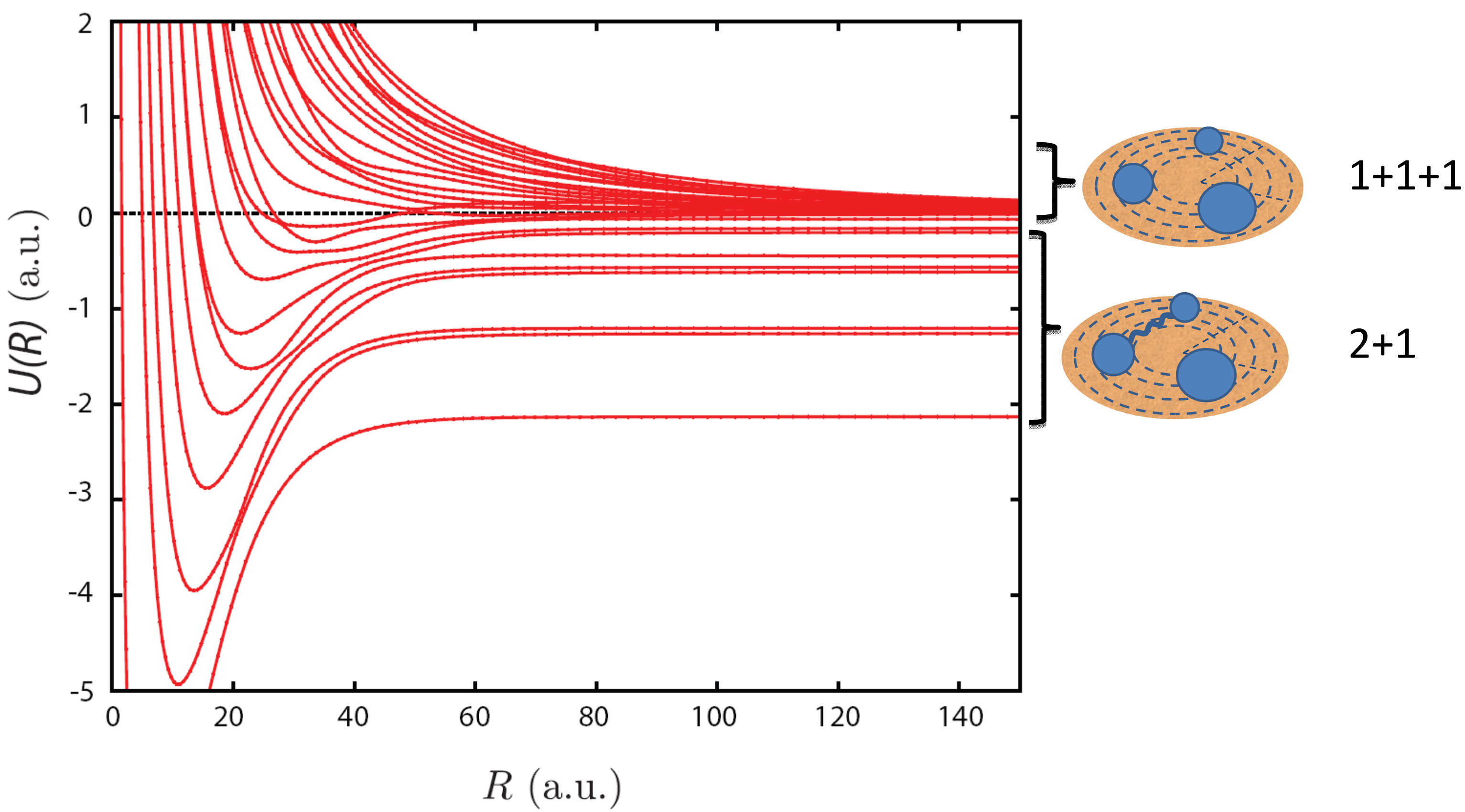
All particles close together:
Bound and quasi-bound states

J. von Stecher and C. H. Greene, arXiv:0904.1405

PRA **80**, 022504, (2009)

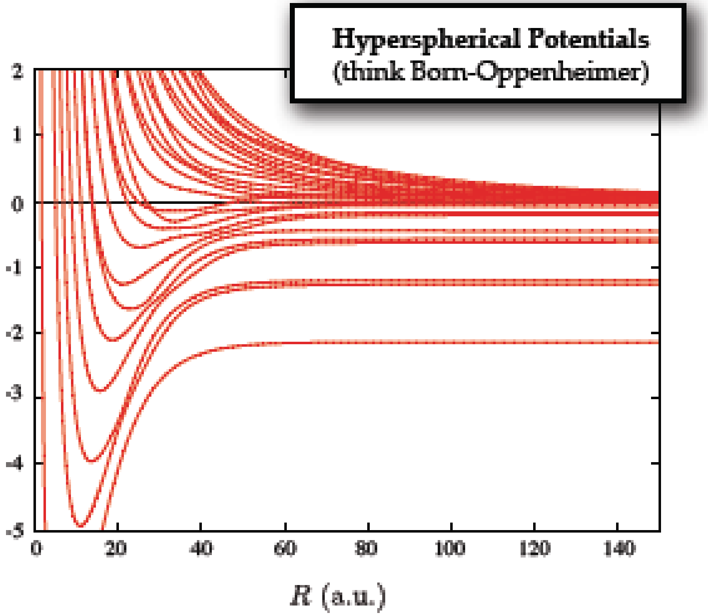
Hyperspherical Picture

Example: Even a three-body calculation already gives quite a bit of complexity, so can we really expect that there could be some simple physics lurking in the 4-body problem?

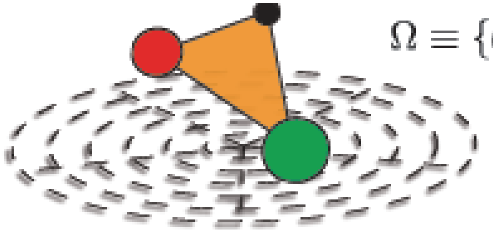


... the hyperspherical way !!!

$$\hat{H} = -\frac{1}{2\mu}\frac{d^2}{d^2R} + \frac{\Lambda^2(\Omega)}{2\mu R^2} + V(R,\Omega)$$

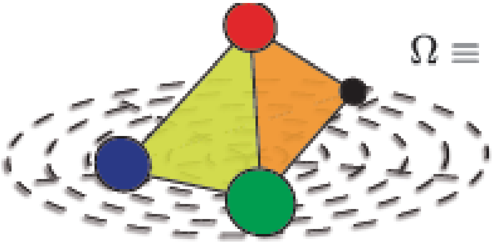


Three-body Problem



$\Omega \equiv \{\theta, \varphi\}$: 2D PDE

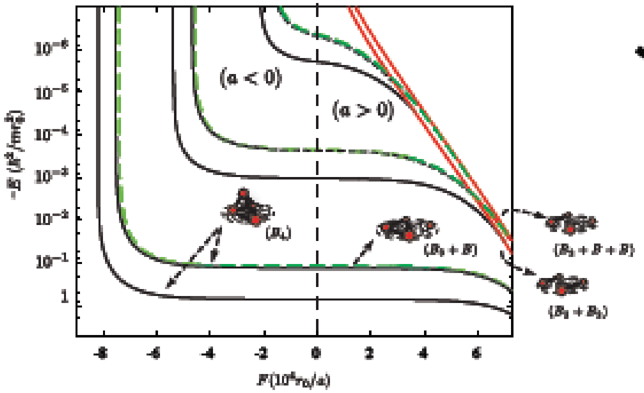
Four-body Problem



$\Omega \equiv \{\theta_1, \theta_2, \phi_1, \phi_2, \phi_3\}$: 5D PDE

... crank up the computers !!!

Four-boson Spectrum



$(a = \infty)$

Two four-body states per Efimov trimer !!!

$$E_{4b}^{(n,m)} = c_m E_{3b}^{(n)} \qquad m = 1, 2$$
$$n = 1, 2, ..., \infty$$

$$(c_1 \approx 4.58, \; c_2 \approx 1.01)$$

(no four-body parameter)

Our findings

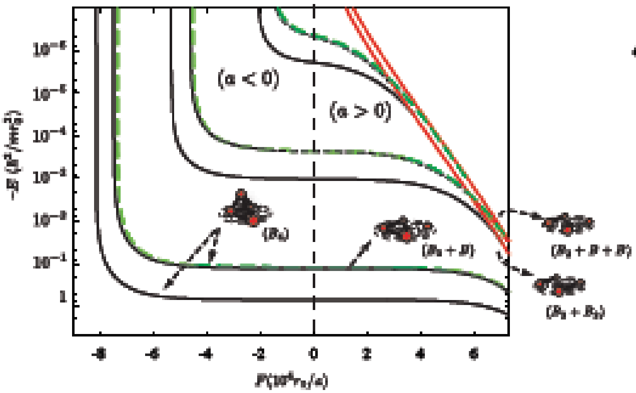
Controversy

Hammer, Platter (2007) (no four-body parameter) **VS** **Yamasita et. al (2006)** (four-body parameter)

$$E_{4b}^{(0,2)} \approx 1.01 E_{3b}^{(0)}$$
$$E_{4b}^{(0,1)} \approx 5.0 E_{3b}^{(0)}$$

Our results are consistent with Hammer & Platter's insightful conjecture

Four-boson Spectrum



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Two four-body states per Efimov trimer !!!

$$E_{4b}^{(n,m)} = c_m E_{3b}^{(n)} \qquad m = 1, 2$$
$$n = 1, 2, ..., \infty$$

$(c_1 \approx 4.58,\ c_2 \approx 1.01)$

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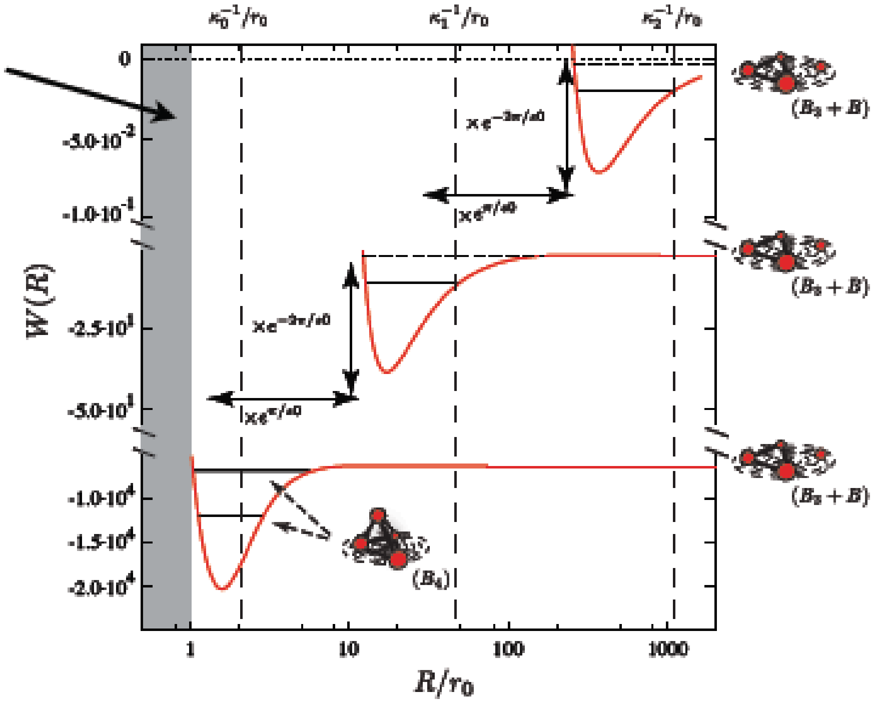
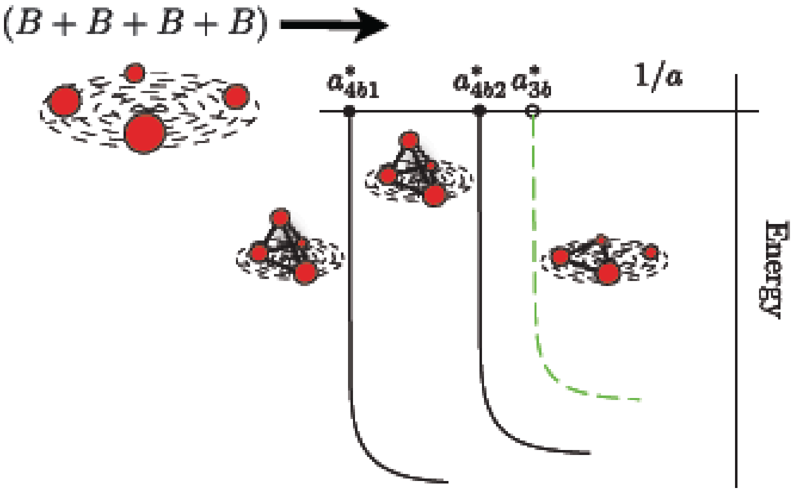
details !



Four-body physics is truly Universal !!!
(geometric scaling: Efimov physics)

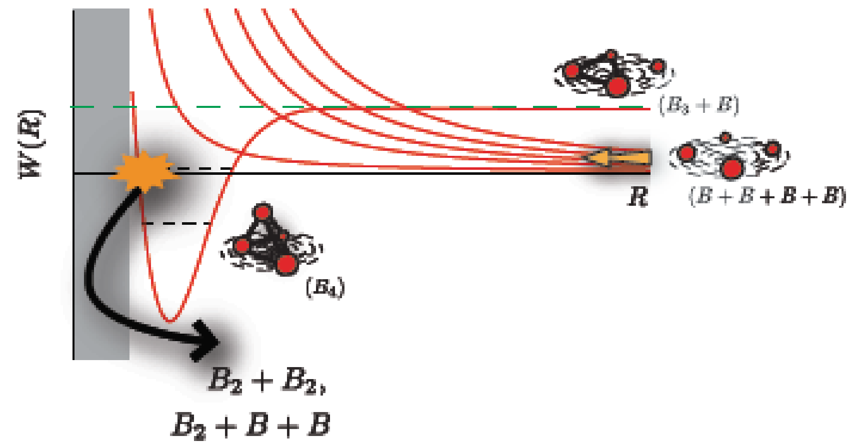
Four-body recombination $(a < 0)$



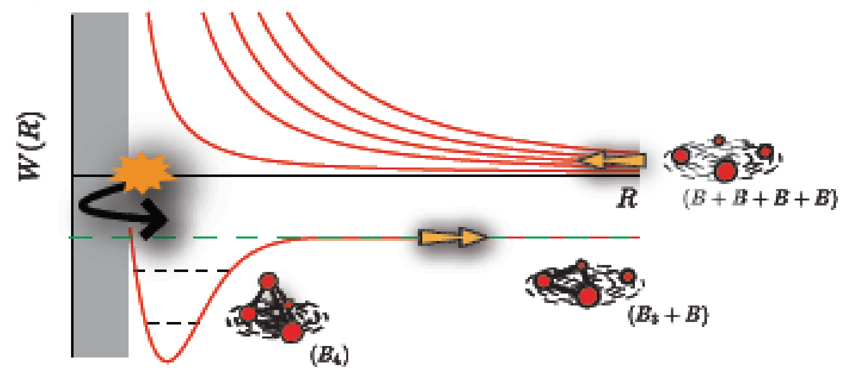
Before the trimer becomes bound,
we should have formed two four-body states !!!

$a^*_{4b,1} = 0.43\ a^*_{3b}$ $\quad$ $a^*_{4b,2} = 0.90\ a^*_{3b}$

- $|a| < |a^*_{3b}|$



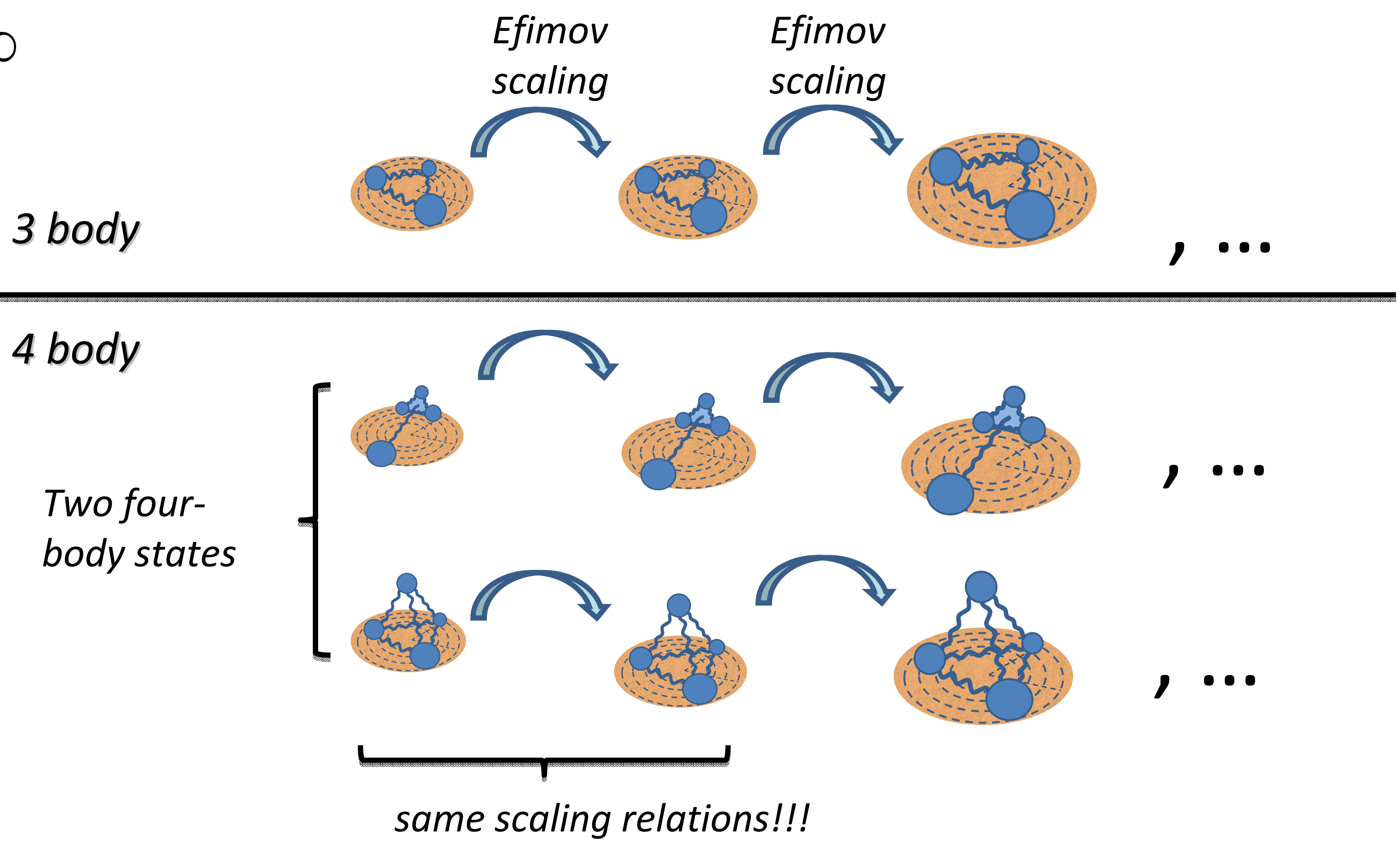
- $|a| > |a^*_{3b}|$



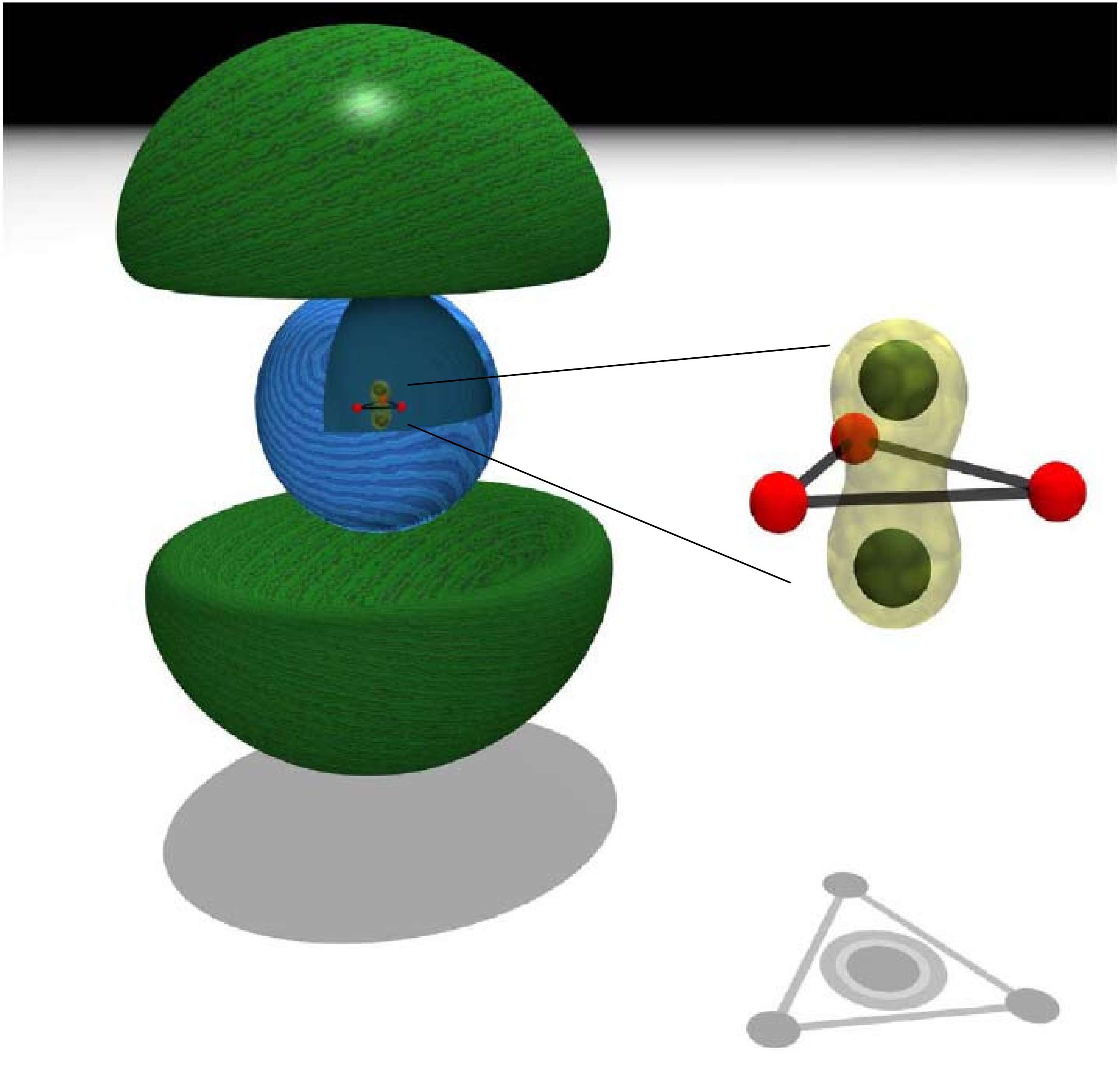
J. von Stecher, J. P. D'Incao and C. H. Greene, Nat. Phys. (2009)

Universality study in the four-boson system

$a=\infty$

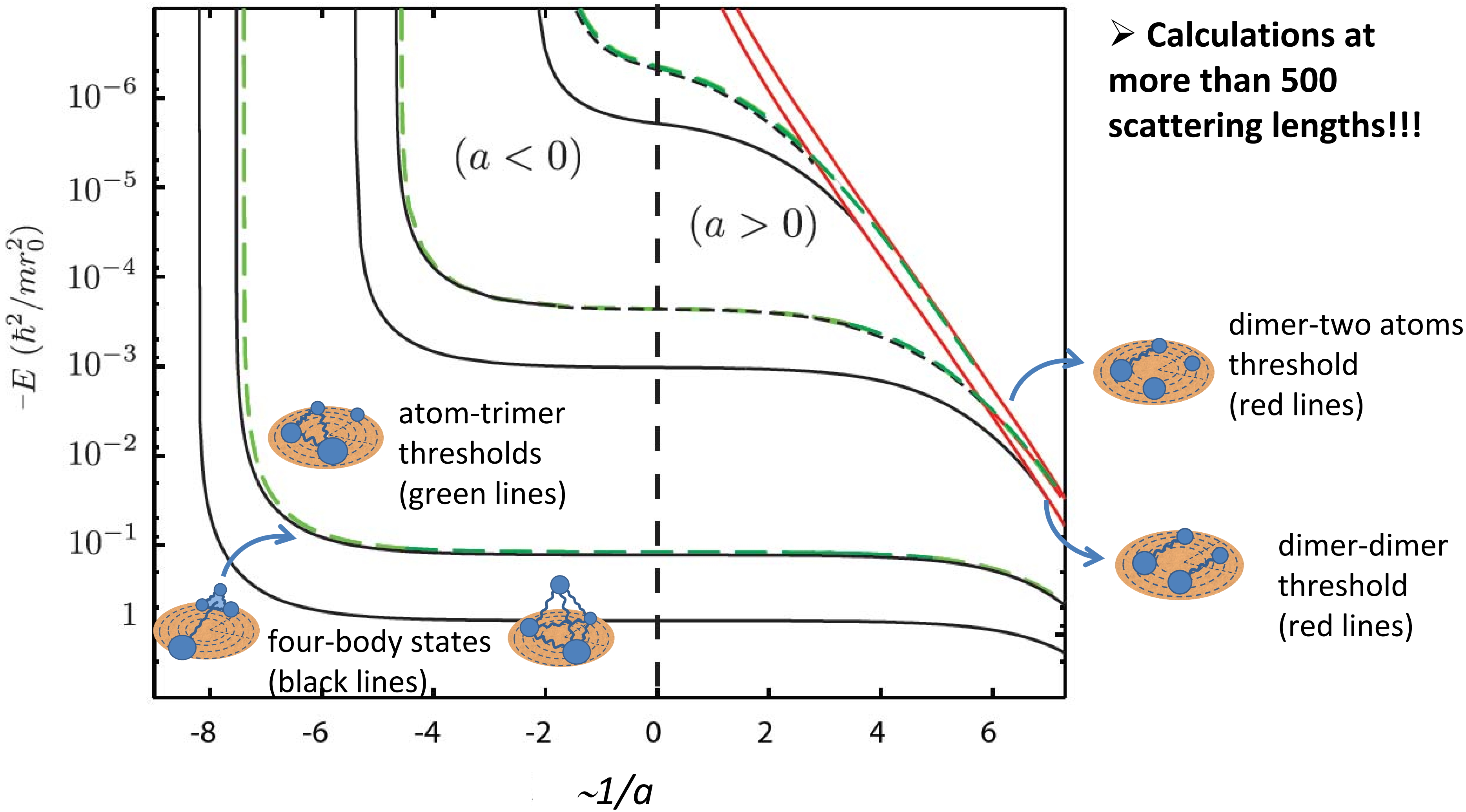


Bird's-eye view of the higher-energy tetramer, very weakly bound



Four-Bosons J. von Stecher, J. P. D'Incao and C. H. Greene, Nat. Phys. (2009)

Spectrum: Extended Efimov plot



- **Calculations at more than 500 scattering lengths!!!**

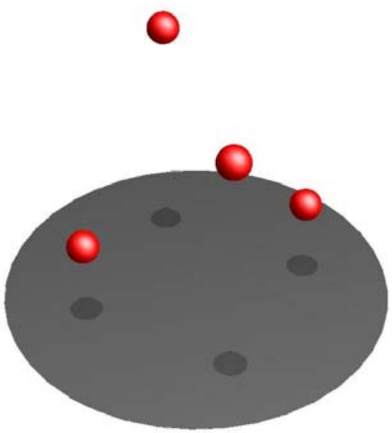
What is the effect of the four-body states on recombination processes?

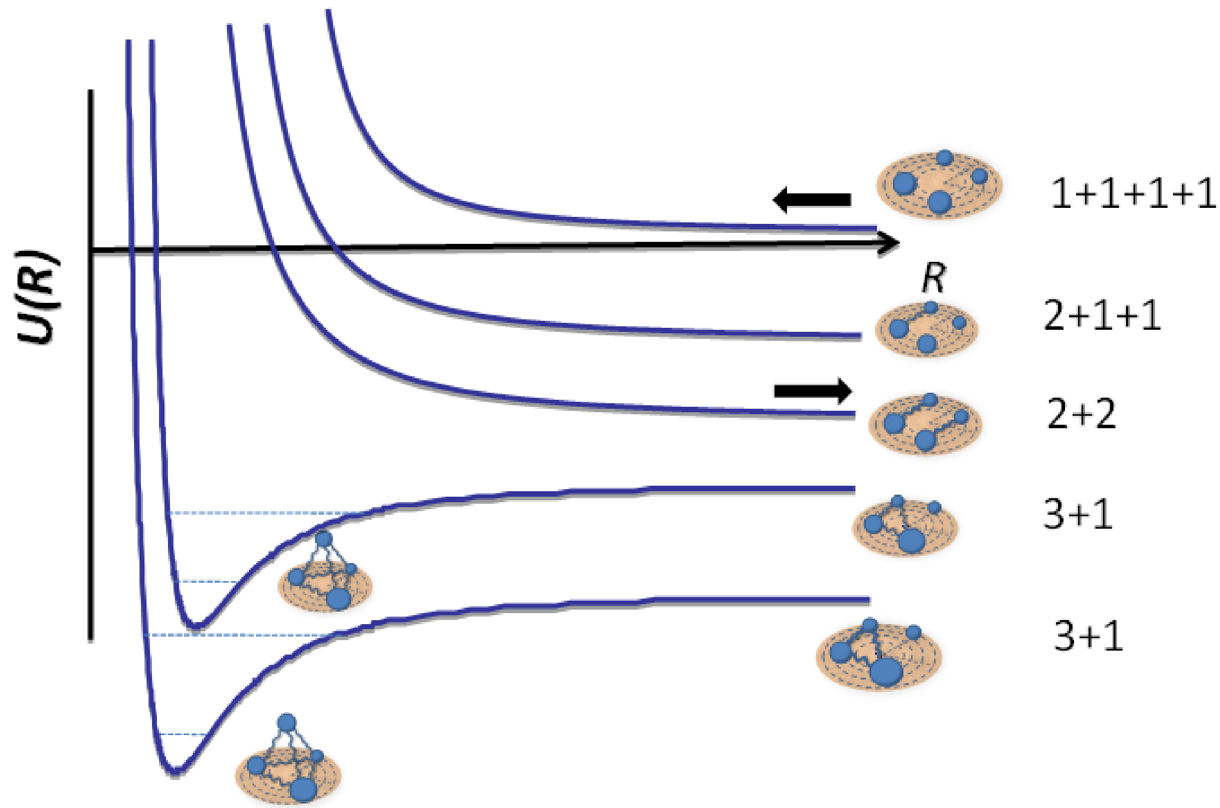
J. von Stecher and C. H. Greene, arXiv:0904.1405

Hypersphericaal Picture of 4-body recombination

... think Born Oppenheimer

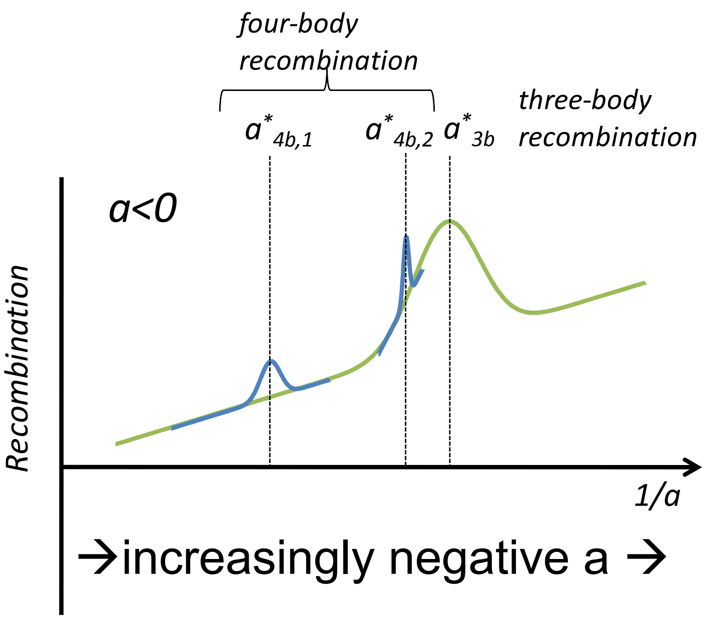
Fragmentation thresholds



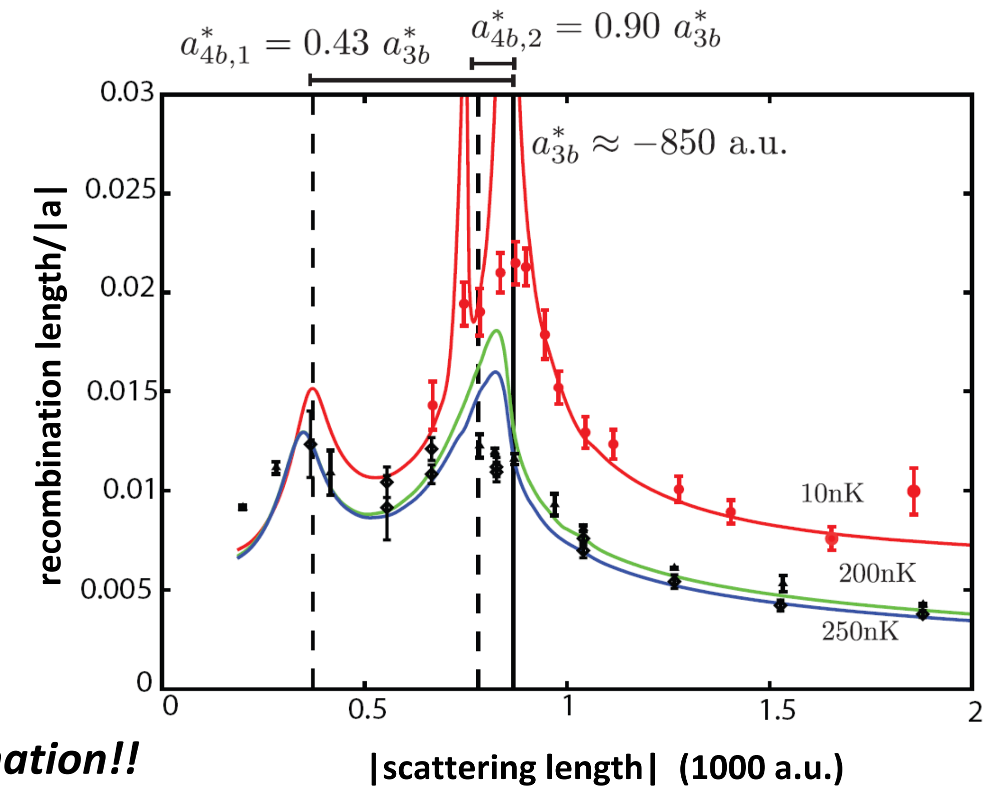


J. von Stecher, J. P. D'Incao and C. H. Greene, Nat. Phys. (2009)

Four-body recombination



Experimental evidence



➢ ***Include four-body effects in the recombination!!***

See also →

Viewpoint *Physics* **2**, 26 (2009)

Ultracold experiments strike universal physics—again

B. D. Esry

Department of Physics, Kansas State University, Manhattan, KS 66506

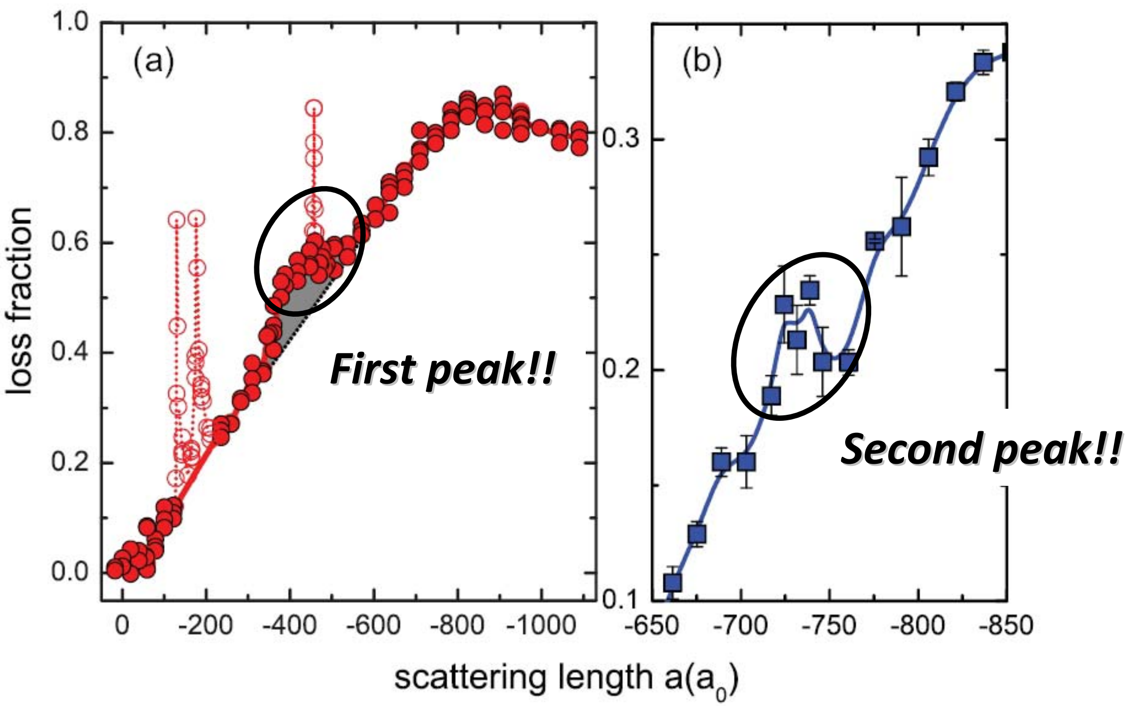
Published April 6, 2009

Just a bit later, some experimental evidence!

Selected for a Viewpoint in *Physics*
PRL **102,** 140401 (2009) PHYSICAL REVIEW LETTERS week ending 10 APRIL 2009

Evidence for Universal Four-Body States Tied to an Efimov Trimer

F. Ferlaino,[1] S. Knoop,[1] M. Berninger,[1] W. Harm,[1] J. P. D'Incao,[2,3] H.-C. Nägerl,[1] and R. Grimm[1,2]



More experimental evidence:
M. Zaccanti et al., arXiv: 0904.4453

Observation of both four-body resonances!!!

But before we can actually calculate the rate of 4-body recombination in an ultracold gas, we have to develop some scattering theory:

PRL 103, 153201 (2009)

A general theoretical description of N-body recombination

N. P. Mehta,[1,2] Seth T. Rittenhouse,[1] J. P. D'Incao,[1] J. von Stecher,[1] and Chris H. Greene[1]

[1] *Department of Physics and JILA, University of Colorado, Boulder, CO 80309*

[2] *Grinnell College, Department of Physics, Grinnell, IA 50112**

(Dated: March 24, 2009)

We present a formula for the cross section and event rate constant describing recombination of N particles in terms of general S-matrix elements. Our result immediately yields the generalized Wigner threshold scaling for the recombination of N bosons. We find that four-boson recombination is resonantly enhanced by the presence of metastable states in the entrance channel. Hence, recombination into a trimer-atom channel could be an effective mechanism for the formation of Efimov trimers.

And here it is, THE FORMULA for N-body recombination, i.e. for the process: A+A+A+….+..A → A_{N-1}+A or A_{N-2}+A+A +…etc.

$$K_N^{0^+} = \frac{2\pi\hbar}{\mu_N} N! \left(\frac{2\pi}{k}\right)^{(3N-5)} \frac{\Gamma\left((3N-3)/2\right)}{2\pi^{(3N-3)/2}} \left|S_{f0}^{0^+}\right|^2$$

And using WKB ideas, we can derive a semi-analytic expression giving the structure of the N-body recombination rate at zero energy:

$$K_N^{0^+} = \frac{\pi \hbar N!}{\mu_N \Omega(3N-3)} \left(\frac{4\pi\alpha|a|}{3N-5}\right)^{3N-5} \frac{e^{-2\gamma_I}\sinh(2\eta)}{\cos^2\phi + \sinh^2\eta}$$

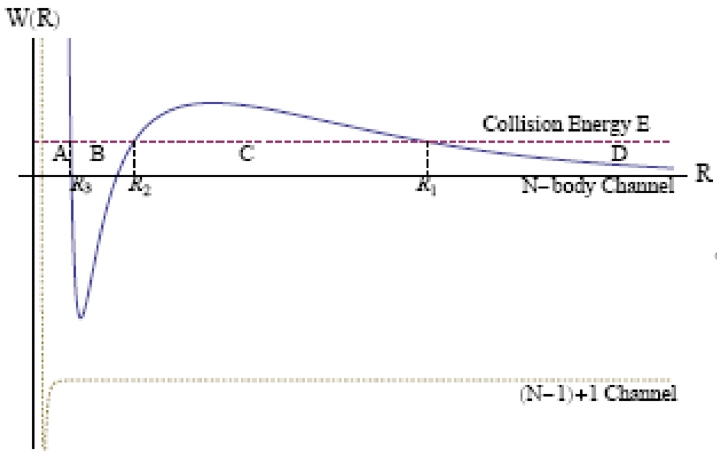


FIG. 1: A schematic representation of the N-boson hyp
dial potential curves is shown. When a metastable N-l
state crosses the collision energy threshold at $E = 0$, N-
recombination into a lower channel with $N-1$ atoms b
plus one free atom is resonantly enhanced.

4-body recomb. rate versus a

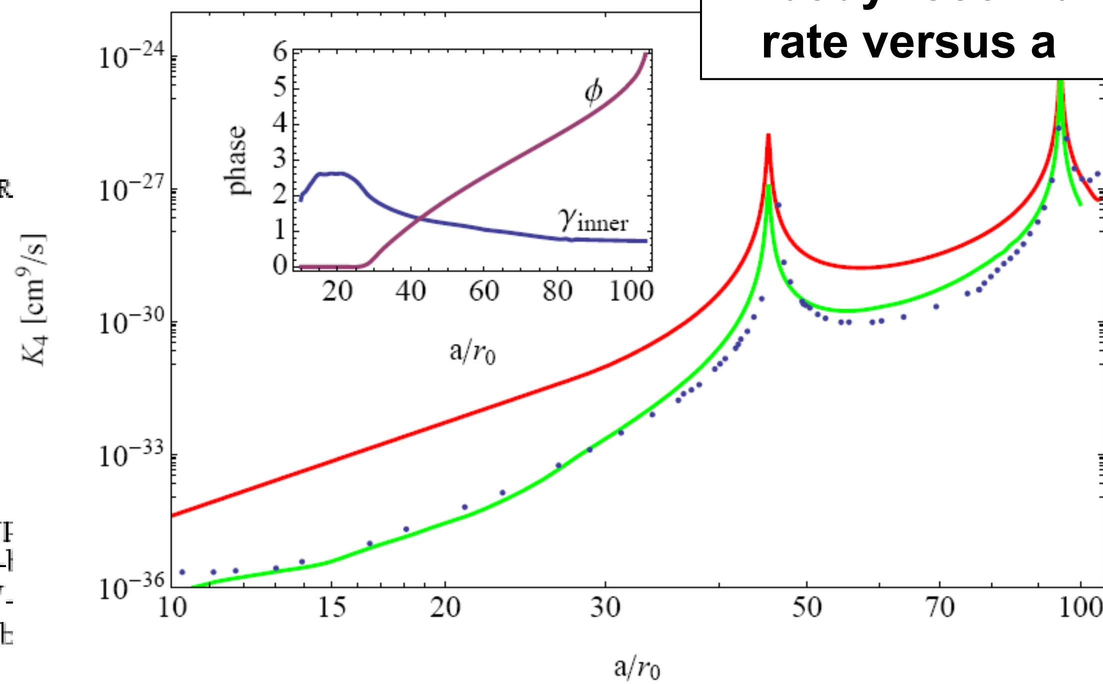
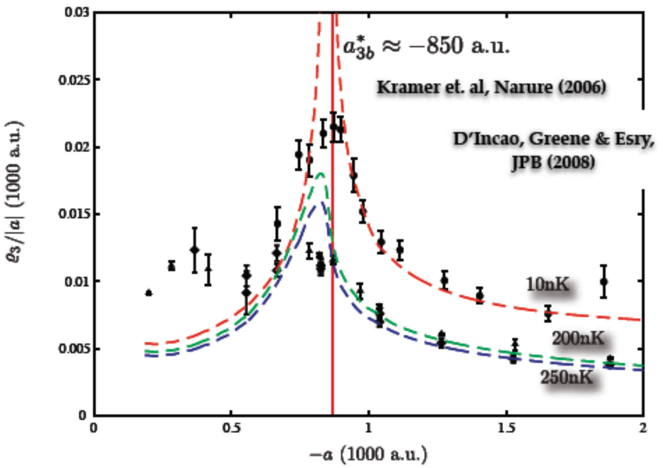


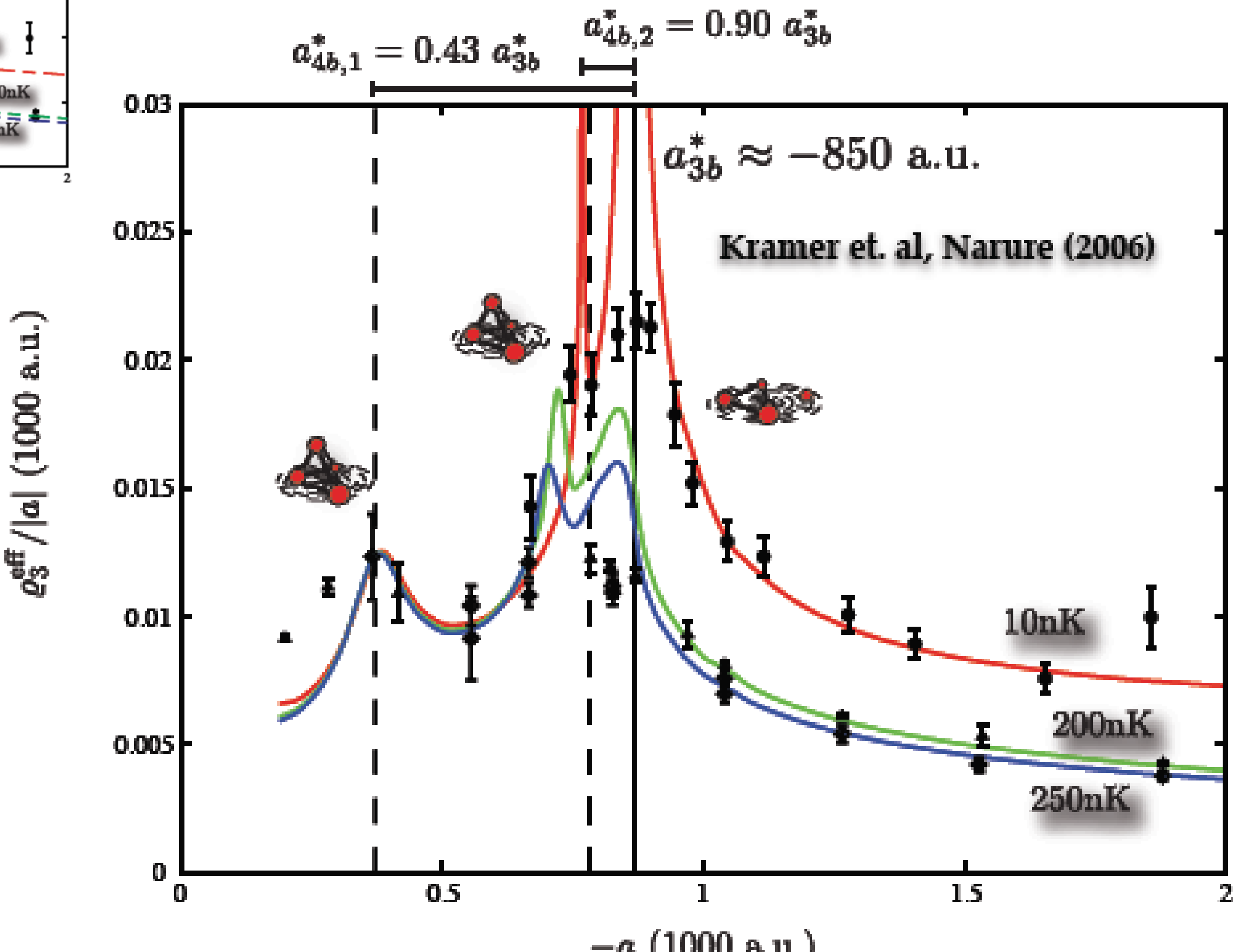
FIG. 2: The four-boson recombination rate constant is shown in the region of interest, $a^*_{3B1} \lesssim a \lesssim a^*_{3B2}$. Four-boson recombination into the existing 3+1 channel is resonantly enhanced at the universal values given in Eq. (16). The dots are numer-

Considering only three-body recombination ...



Considering three- and four-body recombination ...

$$K_3^{\mathrm{eff}}(a,t) = K_3(a) + n(t)K_4(a)$$



Measurement of four-body state 1

Measurement of four-body state 2

Estimated size of this 4-body molecule is around 200-400 nm

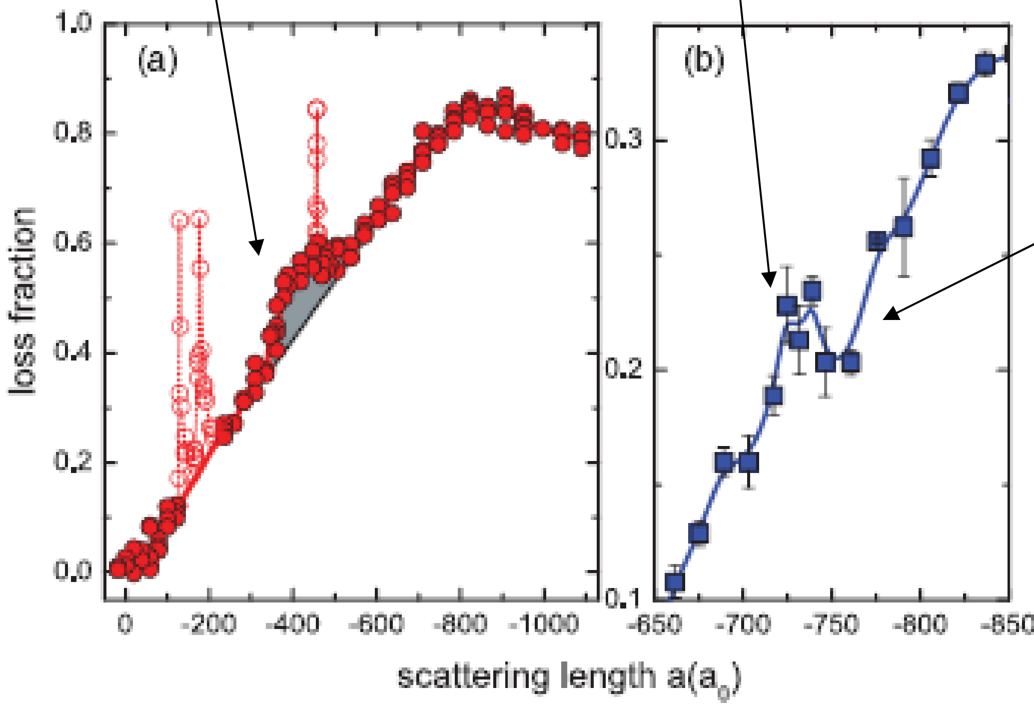
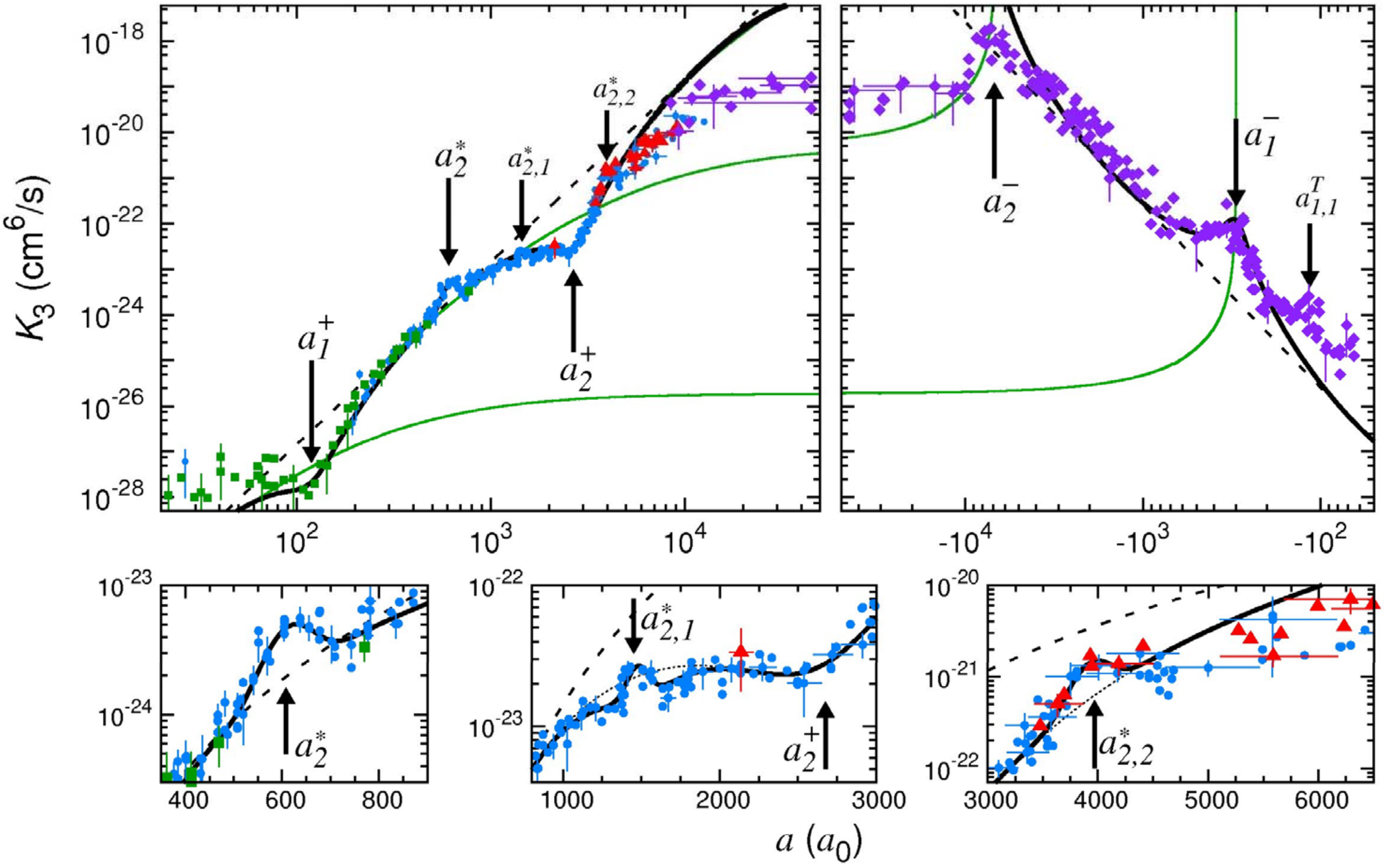
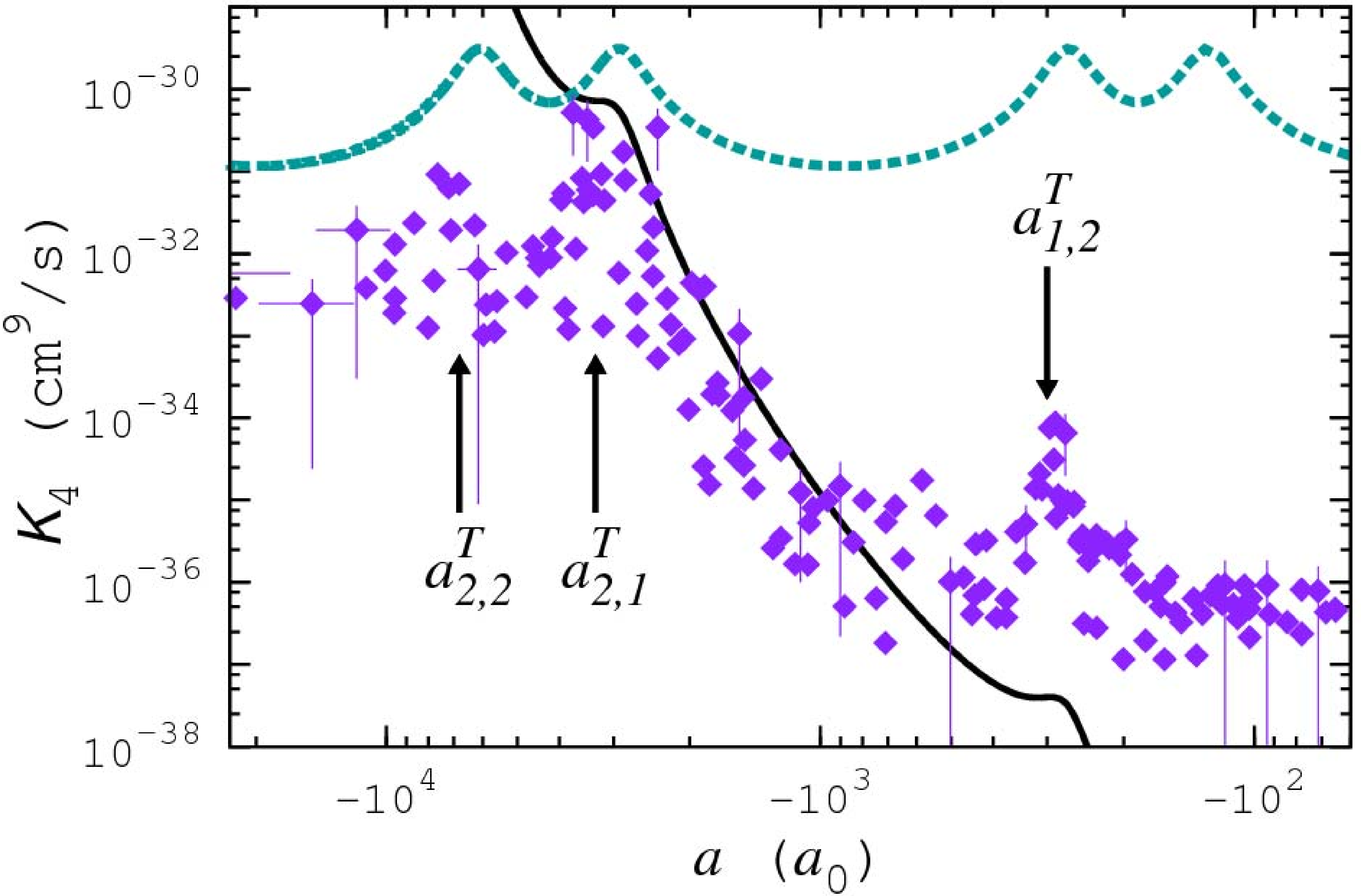


FIG. 2 (color online). Recombination losses in an ultracold sample of Cs atoms. (a) Loss fraction for a 50-nK sample after a storage time of 250 ms. Here we present all individual measurements to give an impression of the scatter of our data. The broad maximum at about $-870a_0$ is caused by a triatomic Efimov resonance [7] and the shaded area highlights the resonant loss enhancement that we attribute to the four-body state Tetra1. The

Scott Pollack, Dan Dries, and Randy Hulet, Science, Dec.2009

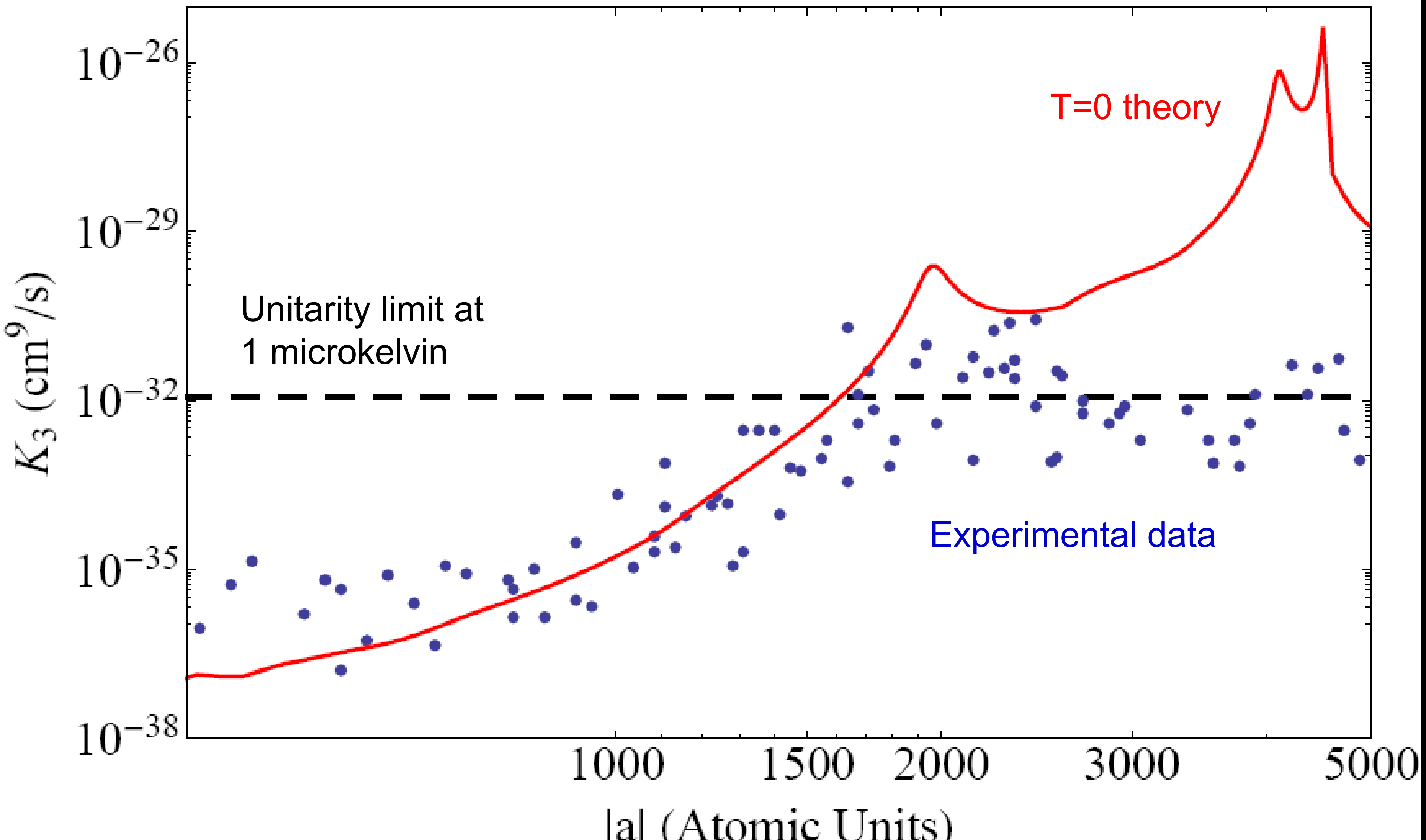


Scott Pollack, Dan Dries, and Randy Hulet, Science, Dec.2009



Calculated 4-body recombination rate, using the theoretical treatment of N-body recombination from Mehta et al., PRL 103, 153201 (2009), compared with the experimental measurement published by the Rice group in Science 2009.

Figure prepared by Seth Rittenhouse, 2010



Scott Pollack, Dan Dries, and Randy Hulet, Science, Dec.2009 – measured universality ratios

	Ratio	Data	Theory	%Δ
$a > 0$	a_2^+/a_1^+	22.5(2.2)	22.7*	−1(9)
	a_2^+/a_2^*	4.40(14)	4.46*	−1(3)
	$a_{2,1}^*/a_2^*$	2.4	2.37†	+1
	$a_{2,2}^*/a_2^*$	6.5	6.6†	−2
$a < 0$	a_2^-/a_1^-	23.1(9)	22.7*	+2(4)
	$a_{1,1}^T/a_1^-$	0.39	0.43†	−9
	$a_{1,2}^T/a_1^-$	1.01	0.90†	+12
	$a_{2,1}^T/a_2^-$	0.48	0.43†	+12
	$a_{2,2}^T/a_2^-$	0.99	0.90†	+10
$a \to \infty$	$\lvert a_1^-\rvert/a_1^+$	2.5(2)	4.9*	−49(4)
	$\lvert a_2^-\rvert/a_2^+$	2.6(3)	4.9*	−47(6)
	$\lvert a_1^-\rvert/a_2^*$	0.49(2)	0.97*	−49(2)
	$\lvert a_2^-\rvert/a_2^*$	11.3(3)	22.0*	−49(1)

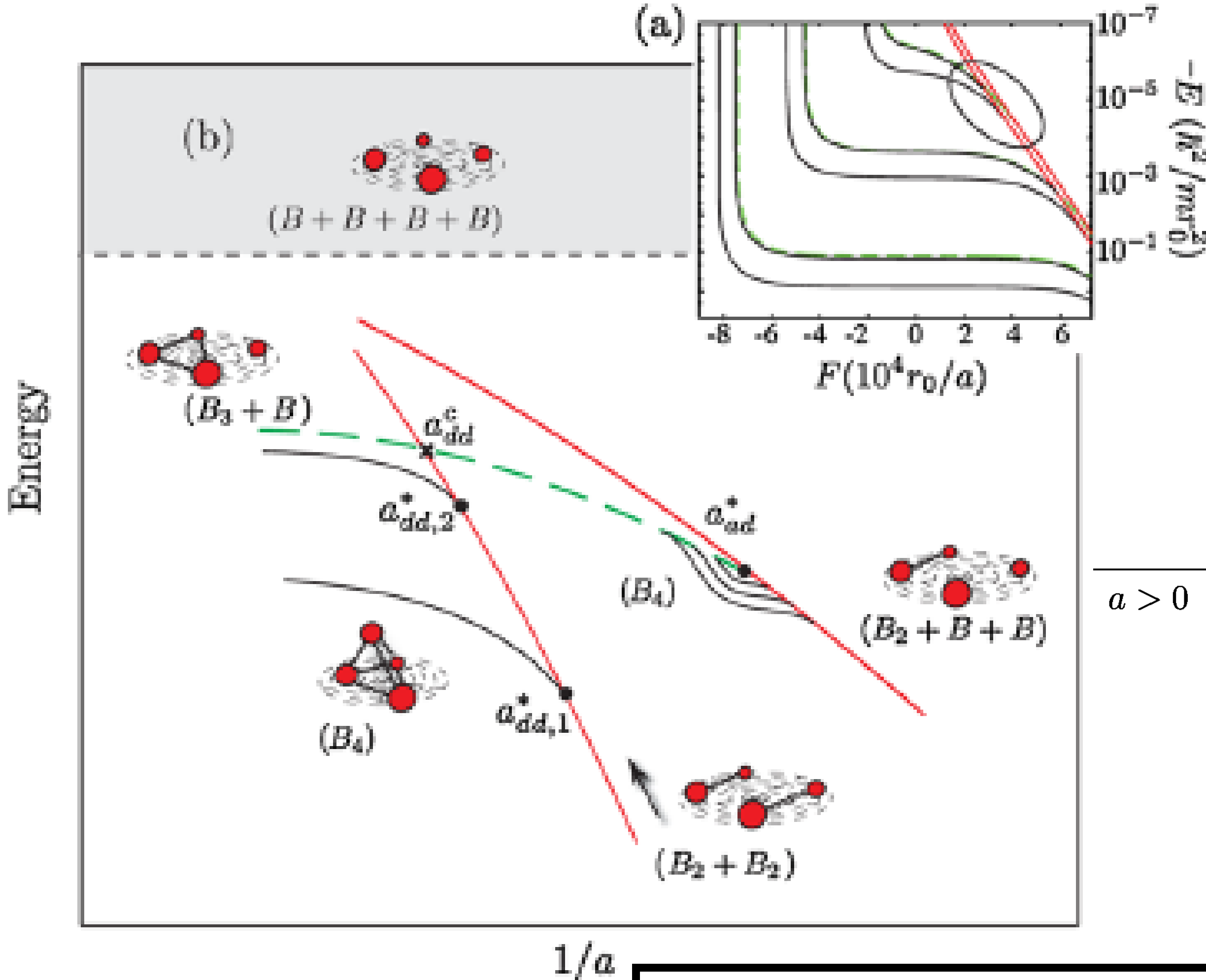
	Loss Feature
$a > 0$	$a_1^+ = 119(11)\, a_0$
	$a_2^+ = 2676(67)\, a_0$
	$a_2^* = 608(11)\, a_0$
	$a_{2,1}^* \approx 1450\, a_0$
	$a_{2,2}^* \approx 3970\, a_0$
$a < 0$	$a_1^- = -298(10)\, a_0$
	$a_2^- = -6869(146)\, a_0$
	$a_{1,1}^T \sim -115\, a_0$
	$a_{1,2}^T \approx -300\, a_0$
	$a_{2,1}^T \approx -3300\, a_0$
	$a_{2,2}^T \approx -6800\, a_0$

* c.f. E. Braaten and H.-W. Hammer, Phys Rep. **428**, 259 (2006).

† von Stecher, *et. al.*, Nature Phys. (2009); D'Incao, *et. al.*, PRL (2009).

PRL 103, 033004 (2009)

D'Incao, von Stecher, and CHG



Hulet group

	Ratio	Data	Theory	%Δ
$a > 0$	a_2^+/a_1^+	22.5(2.2)	22.7*	−1(9)
	a_2^+/a_2^*	4.40(14)	4.46*	−1(3)
	$a_{2,1}^*/a_2^*$	2.4	2.37†	+1
	$a_{2,2}^*/a_2^*$	6.5	6.6†	−2

Predictions of universal theory: →

$$\frac{a^*_{dd,1}}{a^*_{ad}} \approx 2.37, \qquad \frac{a^*_{dd,2}}{a^*_{ad}} \sim 6.6, \quad \text{and} \quad \frac{a^c_{dd}}{a^*_{ad}} \approx 6.73.$$

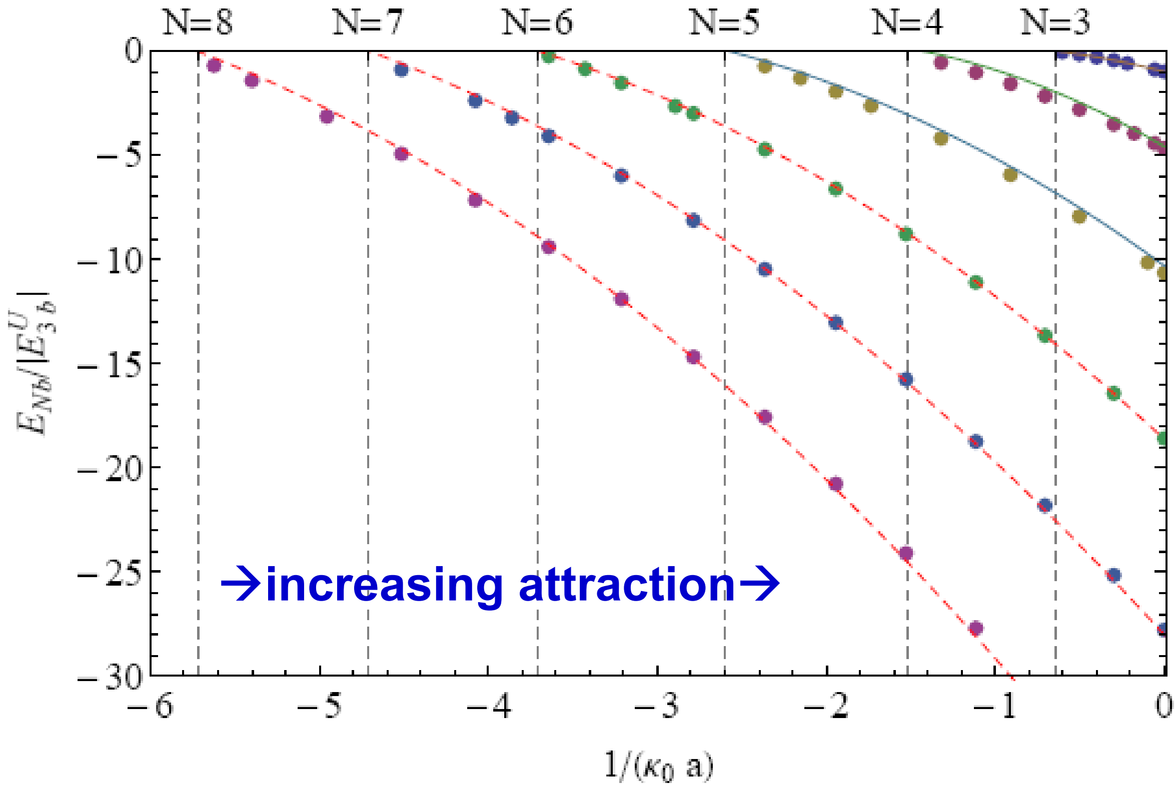
Preprint posted fall 2009:

Weakly Bound Cluster States of Efimov Character

Javier von Stecher
JILA, University of Colorado and National Institute of Standard and Technology, Boulder, CO 80309-0440

We study the behavior of weakly bound clusters and their relation to the well-known three-body Efimov states. We propose a model Hamiltonian that allows the independent control of two- and three-body physics. To test the validity of the model potential, we reproduce predictions of three- and four-body universal states. Then we extend our study to larger cluster systems by combining numerical techniques such as correlated-Gaussian and diffusion Monte Carlo methods. We identify a series of universal cluster states that can be qualitatively interpreted as adding one particle at a time to an Efimov trimer. Experimental signatures of these cluster states are discussed.

arXiv:0909.4056



von Stecher - arXiv:0909.4056

TABLE I: Energies at unitarity and scattering-length ratios that characterize weakly bound cluster states. The scattering length ratios can be transformed to an absolute scale using $1/(\kappa_0 a_{3b}) \approx 0.64$.

N	E_N^U/E_3^U	$a^*_{Nb}/a^*_{(N-1)b}$	N	E_N^U/E_3^U
4	4.66(4)	0.42(1)	9	49.9(6)
5	10.64(4)	0.60(1)	10	60.2(6)
6	18.59(5)	0.71(1)	11	70.1(7)
7	27.9(2)	0.78(1)	12	79.9(3)
8	38.9(3)	0.82(1)	13	88.0(7)

Conclusions:

In the "universal regime" where the largest length scale in the system is the atom-atom scattering length, there should always be 2 four-body states "attached to" and lying just below each Efimov 3-body state – this gives strong evidence to support the conjecture of Platter, Hammer, and Meißner about such states

The Grimm group experiment that saw a 3-body Efimov resonance now has *additional confirmation* that their main resonance is TRULY an Efimov state

A reanalysis of the Grimm group experiment (and their new Physical Review Letter!) suggests strongly that it is also seeing universal 4-body physics for the first time, both 4-body recombination and 4-body bound states "attached" to a 3-body Efimov state. Their new experiment (PRL 2009) strengthens this interpretation.

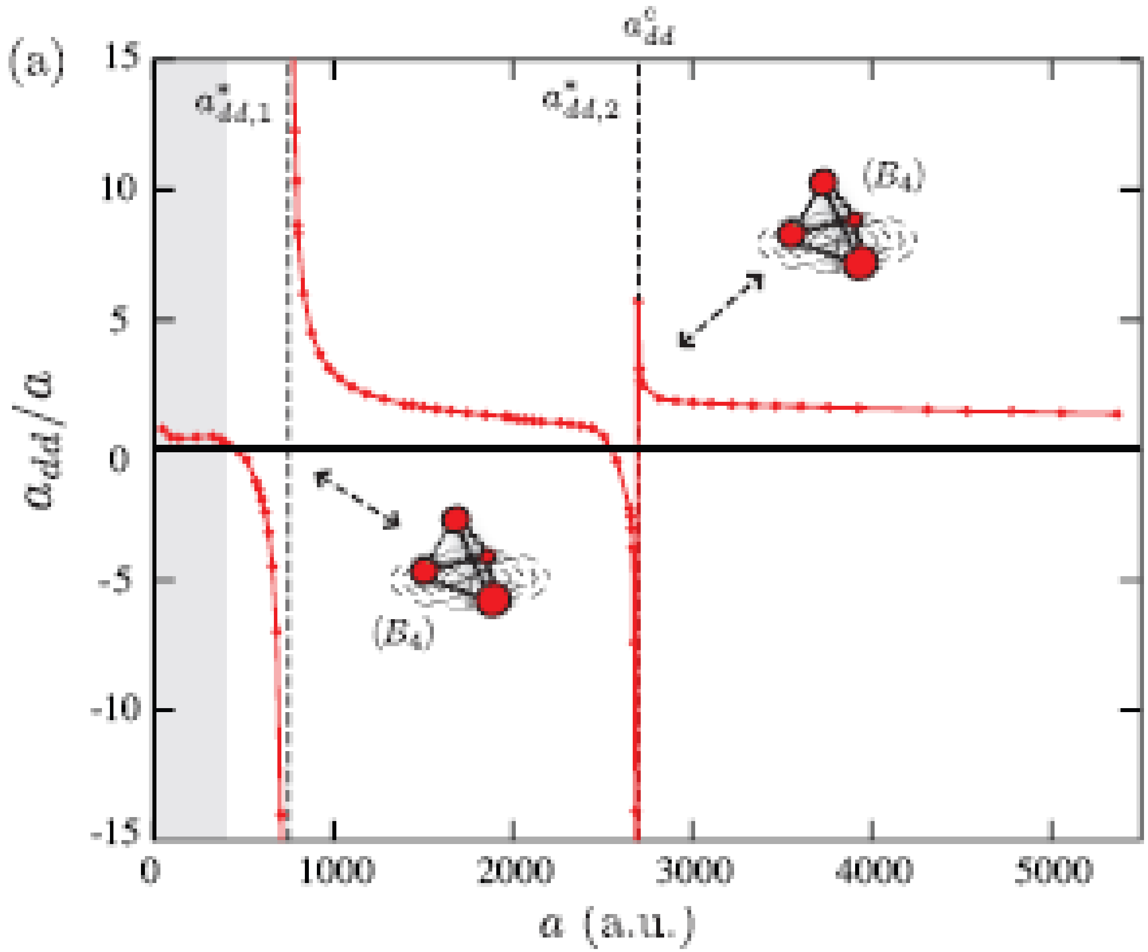
No additional 4-body parameter is needed to describe the 4-boson state energies in the universality regime, supporting the conjecture of Hammer and Platter.

Experimental and theoretical progress in understanding universality is suddenly going forward in leaps and bounds.

Other properties from the 4-boson hyperspherical potential curves and couplings.

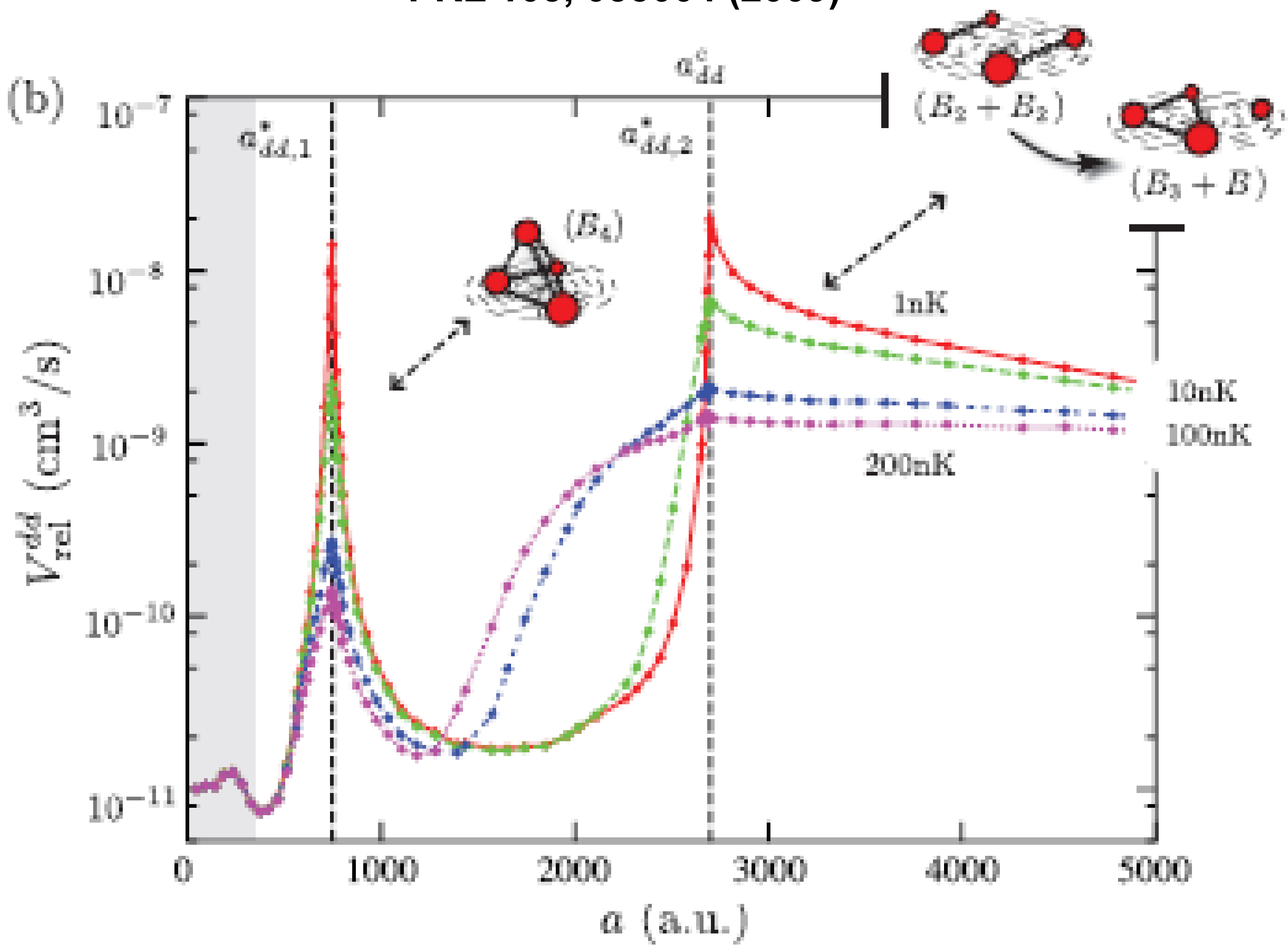
Predicted scattering length between two bosonic dimers a_{dd}, versus the atom-atom scattering length a.

PRL 103, 033004 (2009)



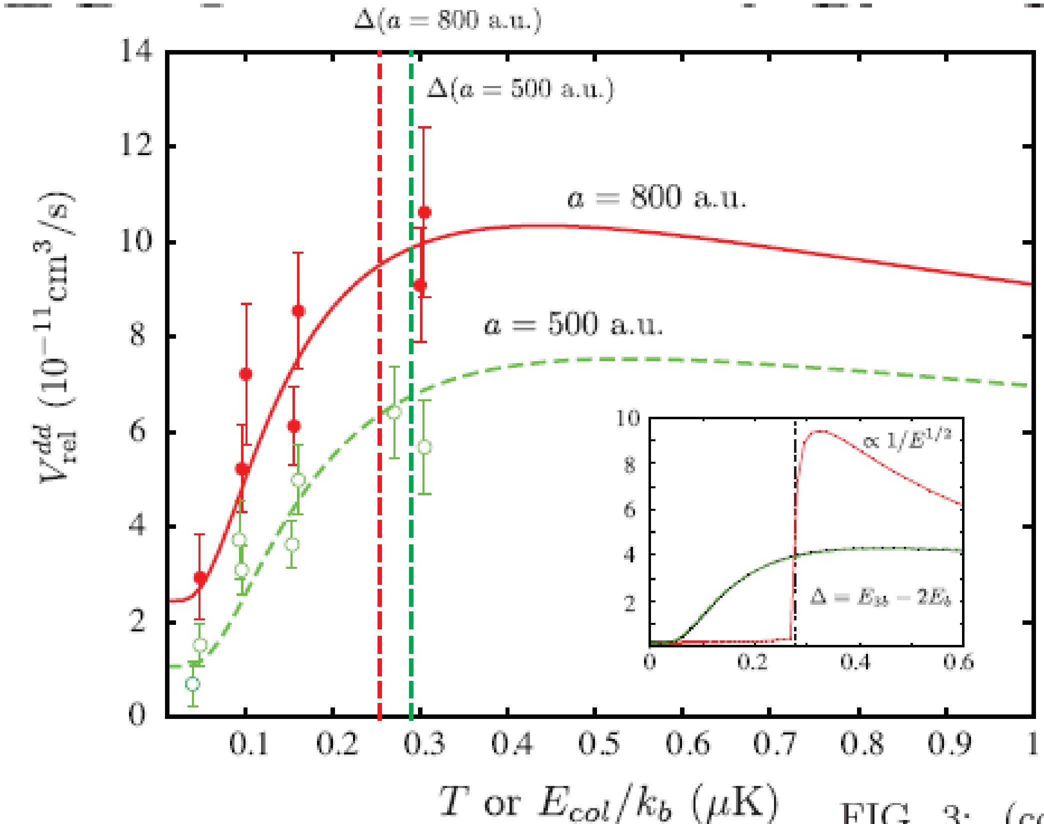
Predicted thermally averaged inelastic collision rates for two bosonic dimers, versus the atom-atom scattering length *a*.

PRL 103, 033004 (2009)



Comparison between theory and a recent experiment from the Grimm group in Innsbruck:

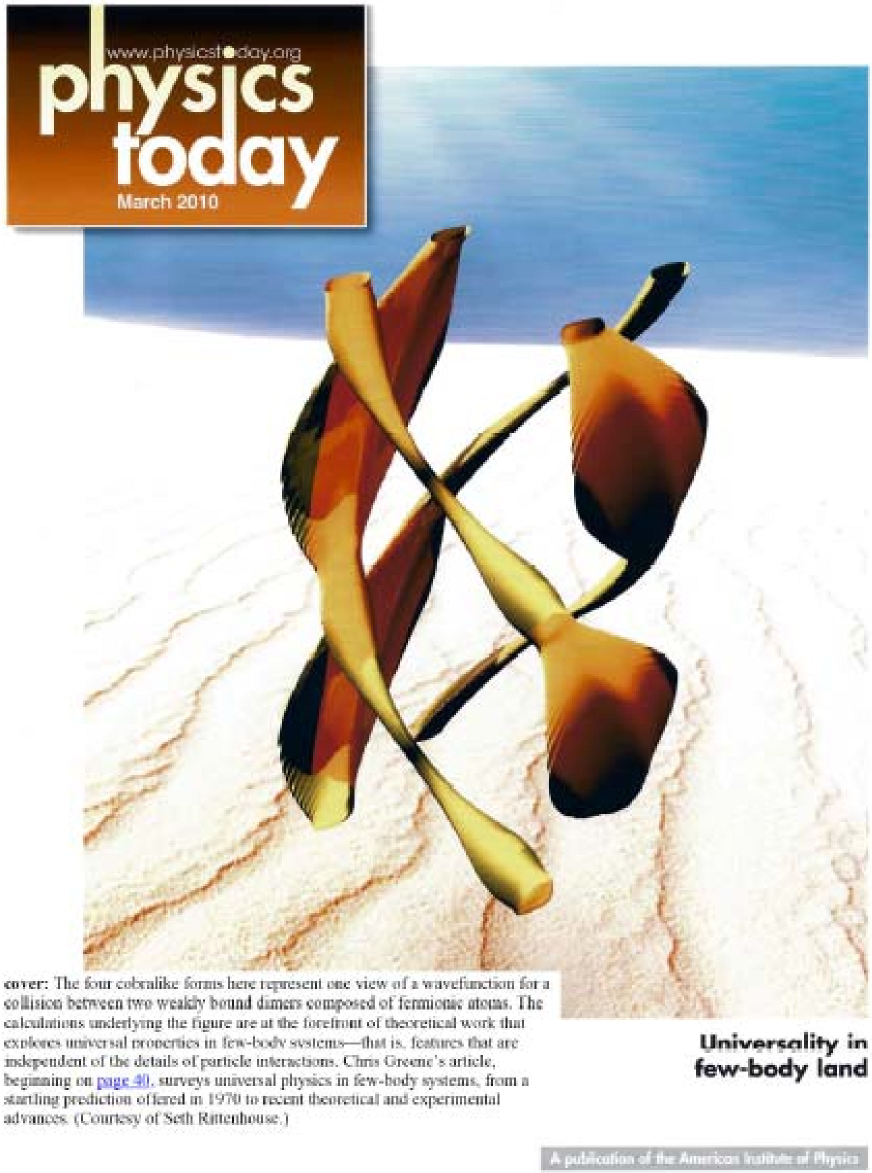
F. Ferlaino *et al.*, Phys. Rev. Lett. **101**, 023201 (2008)



Temperature dependence of the inelastic dimer-dimer collision rate, at two different scattering lengths a.

FIG. 3: (color online). Comparison between experimental data [10] (filled circles: $a = 800$ a.u.; open circle: $a = 500$ a.u.) and our model [Eq. (4)] for the temperature dependence of V^{dd}_{rel}. In our model, the apparent deviation from the Wigner threshold law is caused by the presence of a trimer state just above the dimer-dimer threshold, as indicated by the vertical lines. Inset: Comparison of our model (black-solid line) with numerical rates (the red-solid line is our total rate as a function of E_{col}, and the green-dashed line shows the thermally averaged results as a function of T).

And now, some applications and questions about interacting Fermi gases...



cover: The four cobralike forms here represent one view of a wavefunction for a collision between two weakly bound dimers composed of fermionic atoms. The calculations underlying the figure are at the forefront of theoretical work that explores universal properties in few-body systems—that is, features that are independent of the details of particle interactions. Chris Greene's article, beginning on page 40, surveys universal physics in few-body systems, from a startling prediction offered in 1970 to recent theoretical and experimental advances. (Courtesy of Seth Rittenhouse.)

Some highlights from our recent extensions to many-fermion ultracold gases (with D. Blume, J. von Stecher, S. Rittenhouse)

1. **Collapse is predicted to occur for 3-component and 4-component fermi gases with attraction. (also for more than 4 components)**

PHYSICAL REVIEW A 77, 033627 (2008)

Stability of Inhomogeneous Multi-Component Fermi Gases

D. Blume,[1,2] Seth T. Rittenhouse,[3] J. von Stecher,[3] and Chris H. Greene[3]

[1] *Department of Physics and Astronomy, Washington State University, Pullman, Washington 99164-2814*
[2] *JILA, University of Colorado, Boulder, CO 80309-0440*
[3] *Department of Physics and JILA, University of Colorado, Boulder, CO 80309-0440*

(Dated: January 12, 2008)

Two-component equal-mass Fermi gases, in which unlike atoms interact through a short-range two-body potential and like atoms do not interact, are stable even when the interspecies s-wave scattering length becomes infinitely large. Solving the many-body Schrödinger equation within a hyperspherical framework and by Monte Carlo techniques, this paper investigates how the properties of trapped two-component gases change if a third or fourth component are added. If all

The Hamiltonian H for an atomic Fermi gas with χ components under external spherically symmetric harmonic confinement is given by

$$H = \sum_{\alpha=1}^{\chi}\sum_{i=1}^{N_\alpha}\left(-\frac{\hbar^2}{2m_\alpha}\nabla^2_{\vec{r}_{\alpha i}} + \frac{1}{2}m_\alpha\omega_\alpha^2\vec{r}^{\,2}_{\alpha i}\right) + \sum_{\alpha<\beta}^{\chi}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}V_{\alpha\beta}(|\vec{r}_{\alpha i} - \vec{r}_{\beta j}|). \quad (1)$$

Qualitative reason why the 3-component (or more) fermi gas with attraction might be unstable, whereas the 2-component gas is stable at all negative scattering lengths. Counting argument:

TABLE I: Number N_{att} of attractive interactions, number N_{rep} of effectively repulsive interactions and ratio N_{rep}/N_{att} for finite and infinite N for a χ-component Fermi gas ($\chi = 2$ through 4) in which all interspecies interactions are equal (or resonant).

	$\chi = 2$	$\chi = 3$	$\chi = 4$
N_{att}	$\frac{1}{4}N^2$	$\frac{1}{3}N^2$	$\frac{3}{8}N^2$
N_{rep}	$\frac{N}{2}\left(\frac{N}{2}-1\right)$	$\frac{N}{2}\left(\frac{N}{3}-1\right)$	$\frac{N}{2}\left(\frac{N}{4}-1\right)$
N_{rep}/N_{att} (N finite)	$\frac{N-2}{N}$	$\frac{N-3}{2N}$	$\frac{N-4}{3N}$
N_{rep}/N_{att} ($N \to \infty$)	1	$\frac{1}{2}$	$\frac{1}{3}$

Illustrations of why we expect collapse of the 3-component or 4-component degenerate Fermi gas:

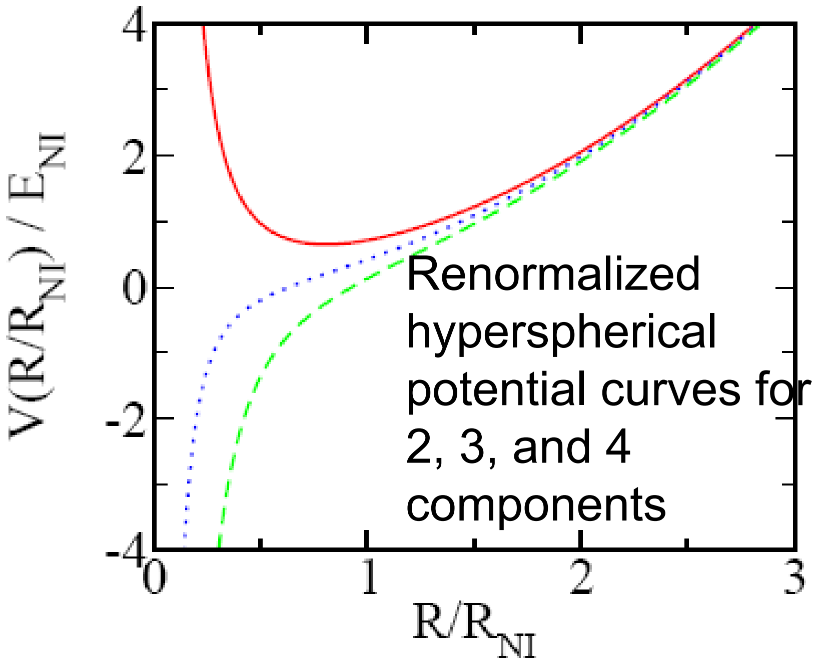


FIG. 1: (Color online) Hyperradial potential curve $V_0(R) + V_{trap}(R)$ as a function of the hyperradius R for $\chi = 2$ (solid line), $\chi = 3$ (dotted line) and $\chi = 4$ (dashed line) in the large N limit. All interactions between unlike atoms are characterized by an infinite scattering length, and the coefficient C_0 is taken to be $(\chi - 1)\beta$ with $\beta = -0.58$. Both length and energy are scaled by the corresponding values of the non-interacting system (see text).

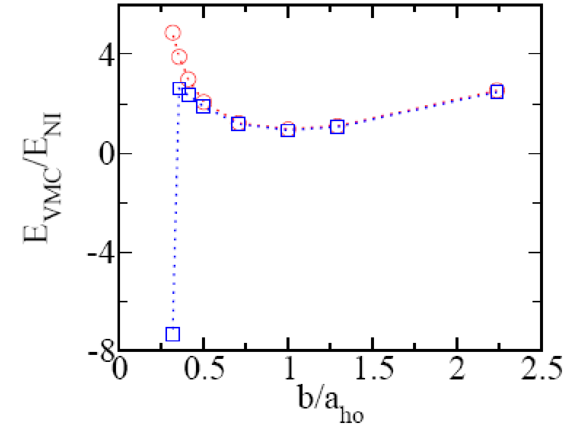


FIG. 6: (Color online) Circles and squares show the variational energy E_{VMC} for a three-component Fermi gas with $N = 12$ atoms interacting through a square well potential (range $R_0 = 0.01a_{ho}$) with $a_s = -0.05a_{ho}$ and $-0.1a_{ho}$ (all interspecies scattering lengths are equal), respectively, as a function of the variational parameter b. E_{VMC} is scaled by the energy E_{NI} of the non-interacting system, $E_{NI} = 27\hbar\omega$. Dotted lines connect data points for ease of viewing.

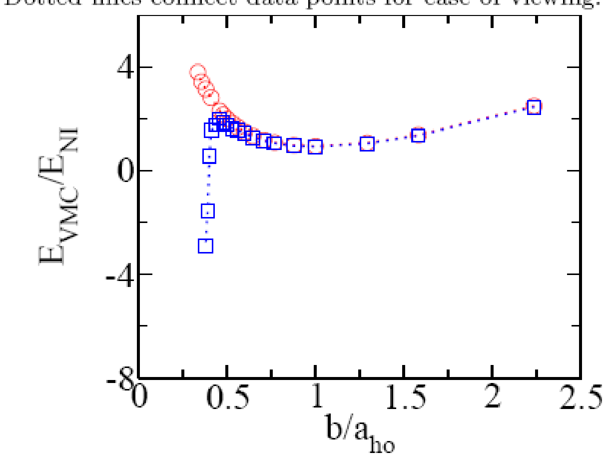


FIG. 7: (Color online) Circles and squares show the variational energy E_{VMC} for a four-component Fermi gas with $N = 16$ atoms interacting through a square well potential (range $R_0 = 0.01a_{ho}$) with $a_s = -0.05a_{ho}$ and $-0.07a_{ho}$, respectively, as a function of the variational parameter b. E_{VMC} is scaled by the energy E_{NI} of the non-interacting system, $E_{NI} = 36\hbar\omega$. Dotted lines connect data points for ease of viewing.

2. Behavior of energies at unitarity, Castin's "universal $1/R^2$ potential curve", excitation frequencies "exactly" 2 hbar omega.

PHYSICAL REVIEW A **74**, 053604 (2006)

Unitary gas in an isotropic harmonic trap: Symmetry properties and applications

Félix Werner and Yvan Castin
Laboratoire Kastler Brossel, École Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex 05, France
(Received 31 July 2006; revised manuscript received 15 September 2006; published 6 November 2006)

We consider N atoms trapped in an isotropic harmonic potential, with s-wave interactions of infinite scattering length. In the zero-range limit, we obtain several exact analytical results: mapping between the trapped problem and the free-space zero-energy problem, separability in hyperspherical coordinates, SO(2,1) hidden symmetry, existence of a decoupled bosonic degree of freedom, and relations between the moments of the trapping potential energy and the moments of the total energy.

In our language, the hyperspherical potential curves for an N-fermion system at unitarity, in a trap, have the form:

$$V_{s_\nu}(R) = \frac{\hbar^2 s_\nu(s_\nu + 1)}{2\mu_N R^2}. \qquad (26)$$

The eigen energies of Eq. (25) are then given by

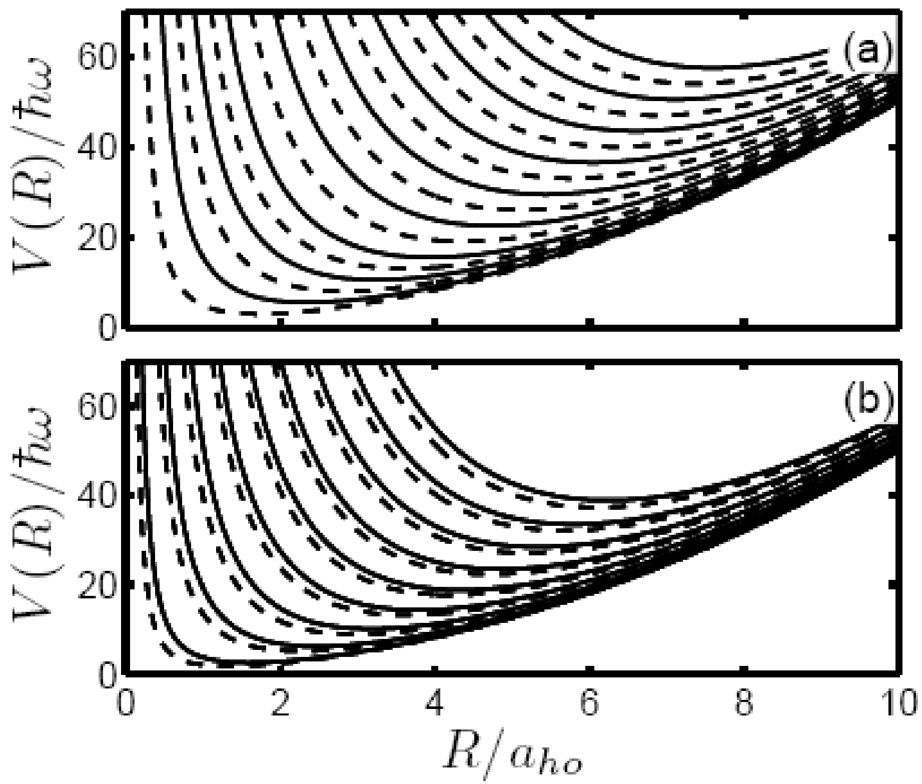
$$E^{rel}_{\nu n} = \left(s_\nu + 2n + \frac{3}{2}\right)\hbar\omega, \qquad (27)$$

where n is a non-negative integer, and the hyperradial wave functions $F_{\nu n}(R)$ (not normalized) by

$$F_{\nu n}(R) = R^{s_\nu+1} L_n^{(s_\nu+1/2)}(R^2/\mathcal{L}^2) \exp\left(-\frac{R^2}{2\mathcal{L}^2}\right), \qquad (28)$$

where $\mathcal{L}$ denotes the oscillator length associated with μ_N, $\mathcal{L} = \sqrt{\hbar/(\mu_N\omega)}$, and $L_n^{(s_\nu+1/2)}$ the Laguerre polynomial.

There are thus families of excitation frequencies exactly equal to 2 hbar omega, but others as well that are different.



(a) **Noninteracting**

(b) **Infinitely interacting (unitarity)**

FIG. 8: Hyperradial potential curves $V(R)$ for equal-mass two-component Fermi systems with (a) vanishing interactions and (b) infinitely strong interactions as a function of R. The hyperradial potential curves naturally appear ordered as N increases: Solid lines correspond, from bottom to top, to $N = 4 - 20$ (N even), while dashed lines correspond, from bottom to top, to $N = 3 - 19$ (N odd).

Test of the universal behavior predicted by Werner and Castin

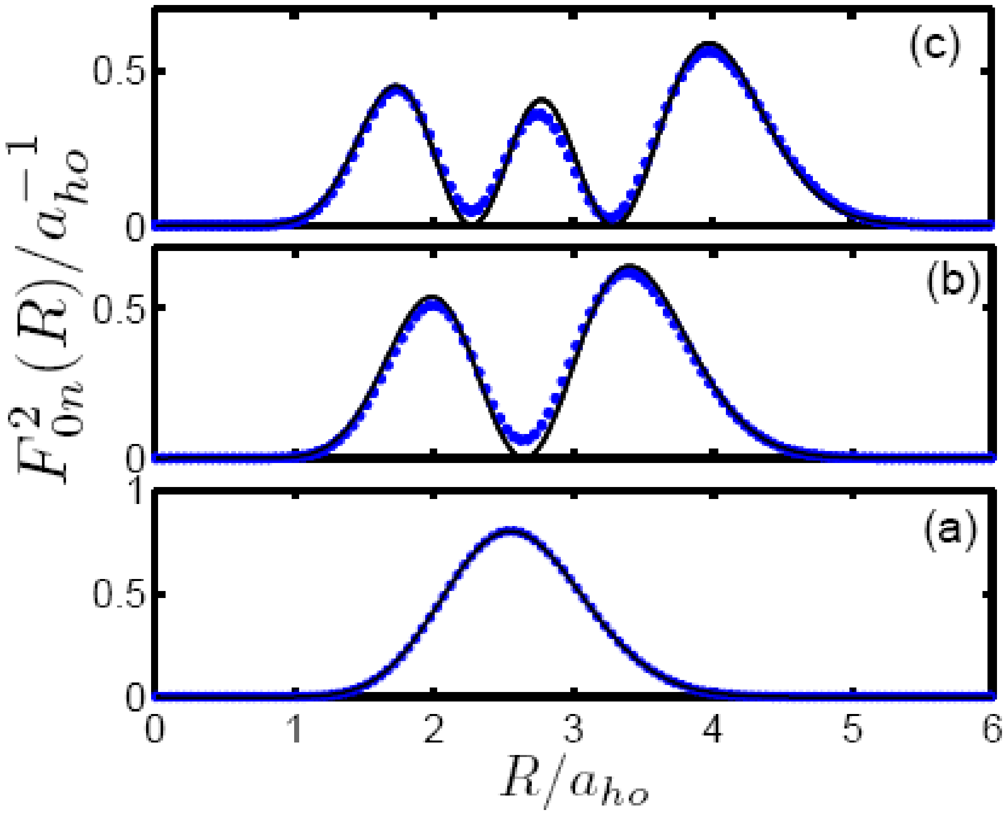


FIG. 21: (Color Online) Hyperradial density $F_{0n}^2(R)$ for $n = 1, 2$ and 3. Here, we choose $\mu_N = m$ ($\mathcal{L} = a_{ho}$). The solid lines show the analytical solutions while the circles show the numerical results obtained by integrating $(\Psi^{rel})^2$ calculated by the CG method over all coordinates but the hyperradius R.

Microscopic calculation of the pairing gap at unitarity for a 2-component equal-mass fermionic gas, compared with Bulgac's very recent density-functional theory description.

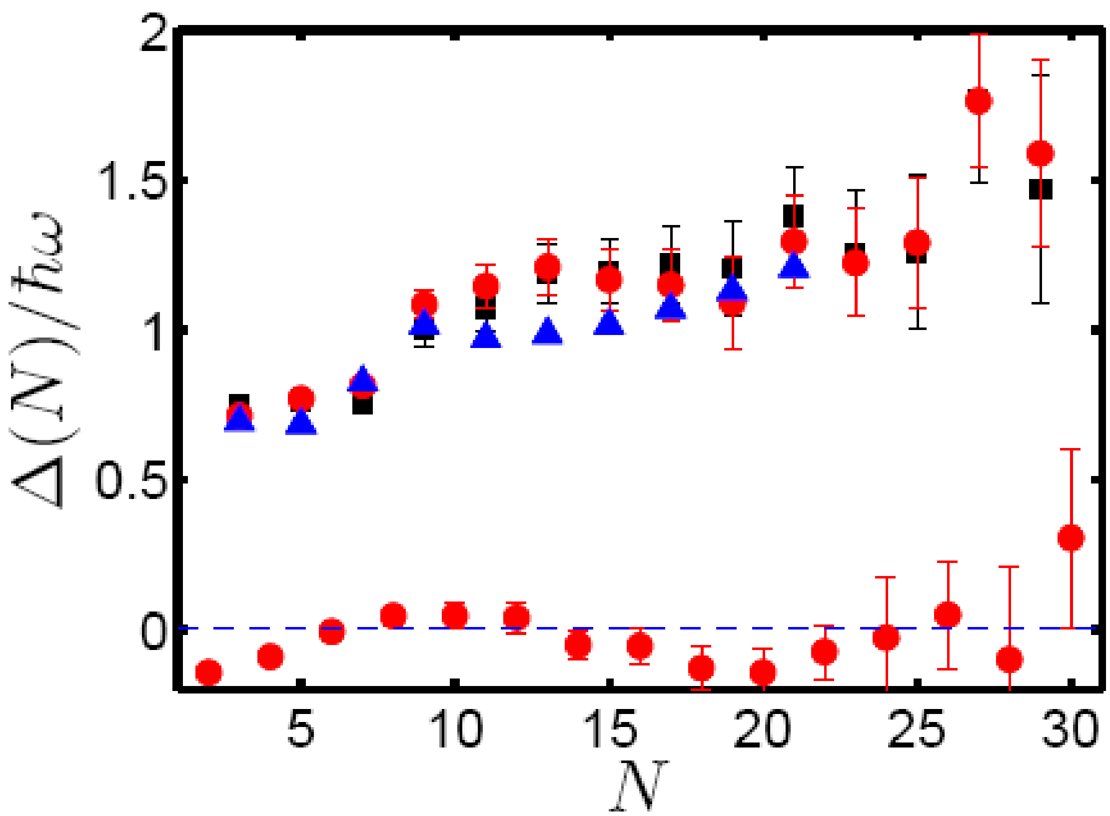


FIG. 7: (Color Online) Excitation gap $\Delta(N)$ (squares) and residual energy $E_{00}(N) - E_{fit}$ (circles) for equal-mass Fermi systems at unitarity as a function of N calculated from the FN-DMC energies. Triangles show $\Delta(N)$ calculated using density functional theory [75].