

Multi-meson systems in LQCD

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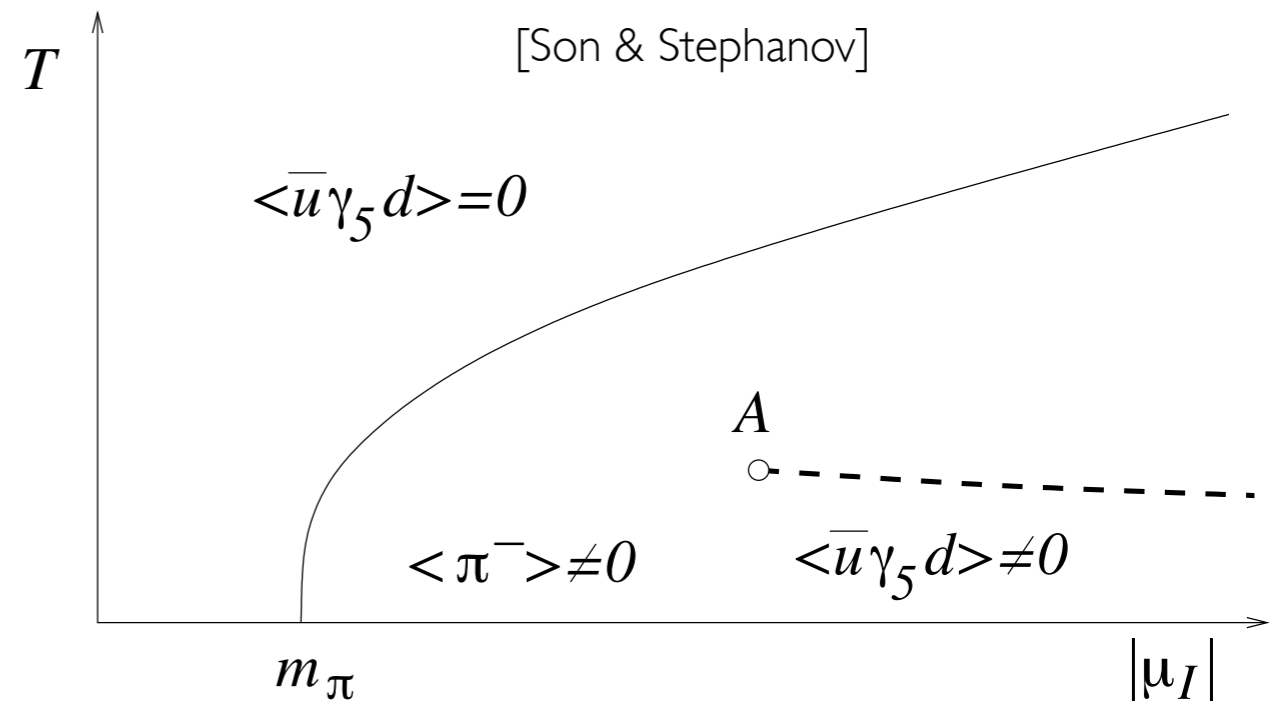
Simulations and Symmetries, INT, May 14th 2010

- $n < 13$ pion or kaon systems
 - 2- and 3- body interactions
 - Meson condensates
 - Screening of static quark potential
- Recent developments
 - Really large systems: $^{238}\pi^+$
 - Mixed species systems

Motivation

- Multi hadron interactions
- Meson condensates: interesting state of matter with a complex phase diagram

- finite μ_I : BEC-BCS crossover
- Spontaneous rotational symmetry breaking for $\mu_I \geq m_\rho$



- Kaon condensation may be phenomenologically relevant in n-stars [Kaplan/Nelson]
- Complexity frontier: precursor to nuclear systems

Few meson systems

Phys. Rev. Lett. 100:082004, 2008

Phys Rev D78:014507, 2008

Phys Rev D78:054514, 2008

Phys. Rev. Lett. 102:032004, 2009

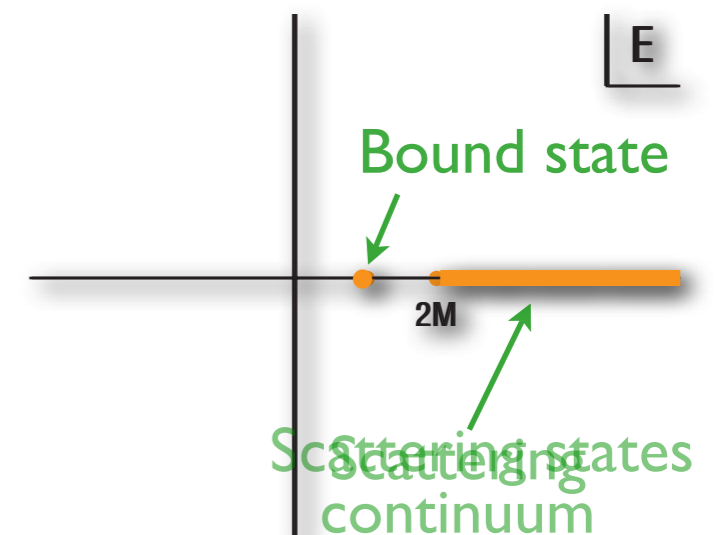
Hadron scattering

- Maiani-Testa: extracting multi-hadron S -matrix elements from Euclidean lattice calculations of Green functions is impossible
- Lüscher: volume dependence of two-particle energy levels \Rightarrow scattering phase-shift up to inelastic threshold

$$\Delta E_{(n)} = \sqrt{|\mathbf{q}_{(n)}|^2 + m_A^2} + \sqrt{|\mathbf{q}_{(n)}|^2 + m_B^2} - m_A - m_B$$

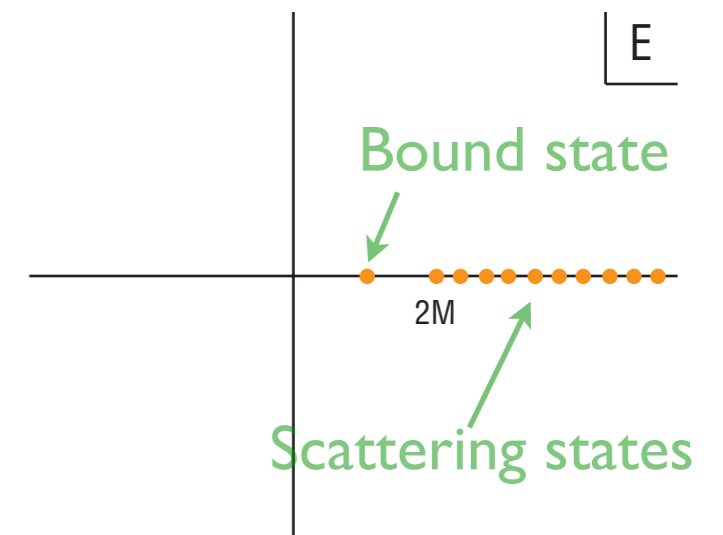
$$q_{(n)} \cot \delta(q_{(n)}) = \frac{1}{\pi L} S \left(\frac{q_{(n)} L}{2\pi} \right)$$

$$S(\eta) = \lim_{\Lambda \rightarrow \infty} \left[\sum_{|\vec{n}| < \Lambda} \frac{1}{|\vec{n}|^2 - \eta^2} - 4\pi\Lambda \right]$$

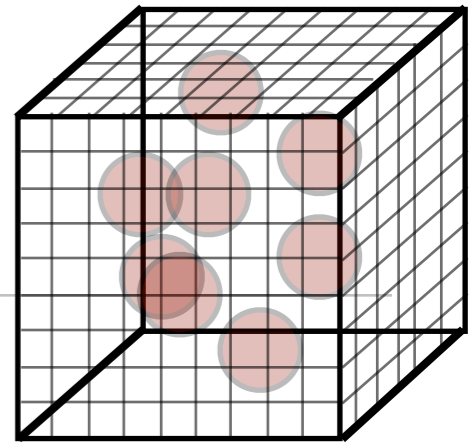


Hadron scattering

- Maiani-Testa: *extracting multi-hadron S-matrix elements from Euclidean lattice calculations of Green functions is impossible*
- Lüscher: volume dependence of two-particle energy levels
⇒ scattering phase-shift up to inelastic threshold
- Exact relation provided $r \ll L$
- Used for $\pi\pi$, KK , NN , ΛN , ...
- What about $n > 2$ hadrons?



Bosons in a box



- Large volume expansion of ground state energy of n meson system to $1/L^7$
- 2 & 3 body interactions (N body: $L^{-3(N-1)}$)
- $n=2$: reproduces expansion of Lüscher

$$\Delta E_n = \frac{4\pi\bar{a}}{M L^3} {}^n C_2 \left\{ 1 - \left(\frac{\bar{a}}{\pi L} \right) \mathcal{I} + \left(\frac{\bar{a}}{\pi L} \right)^2 [\mathcal{I}^2 + (2n - 5)\mathcal{J}] \right. \\ \left. - \left(\frac{\bar{a}}{\pi L} \right)^3 [\mathcal{I}^3 + (2n - 7)\mathcal{I}\mathcal{J} + (5n^2 - 41n + 63)\mathcal{K}] \right\} \\ + {}^n C_3 \frac{1}{L^6} \hat{\eta}_3^L + \mathcal{O}(L^{-7})$$

Scattering length

Geometric coefficients

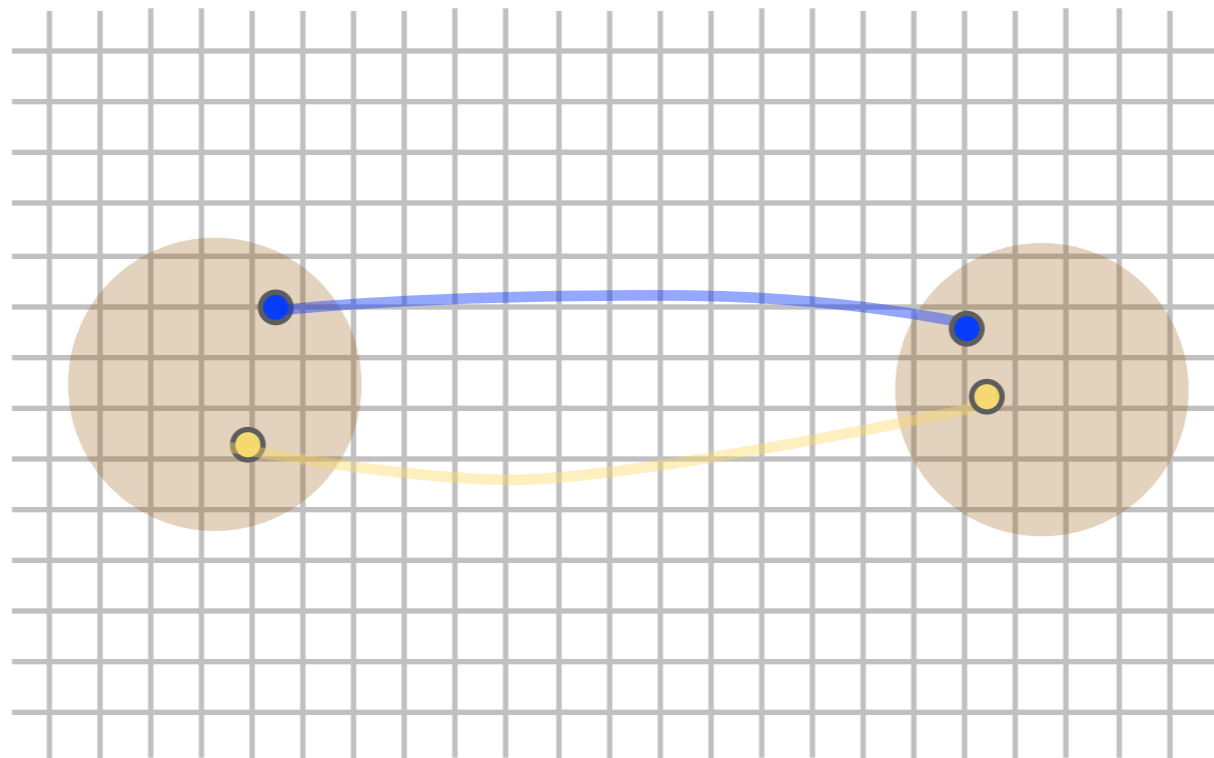
Three body Interaction

Many mesons in LQCD

- Consider π^+ correlator ($m_u=m_d$)

$$C(t) = \left\langle 0 \left| \left[\sum_{\mathbf{x}} \bar{d}\gamma_5 u(\mathbf{x}, t) \bar{u}\gamma_5 d(\mathbf{0}, 0) \right] \right| 0 \right\rangle$$

$\rightarrow A e^{-E t}$

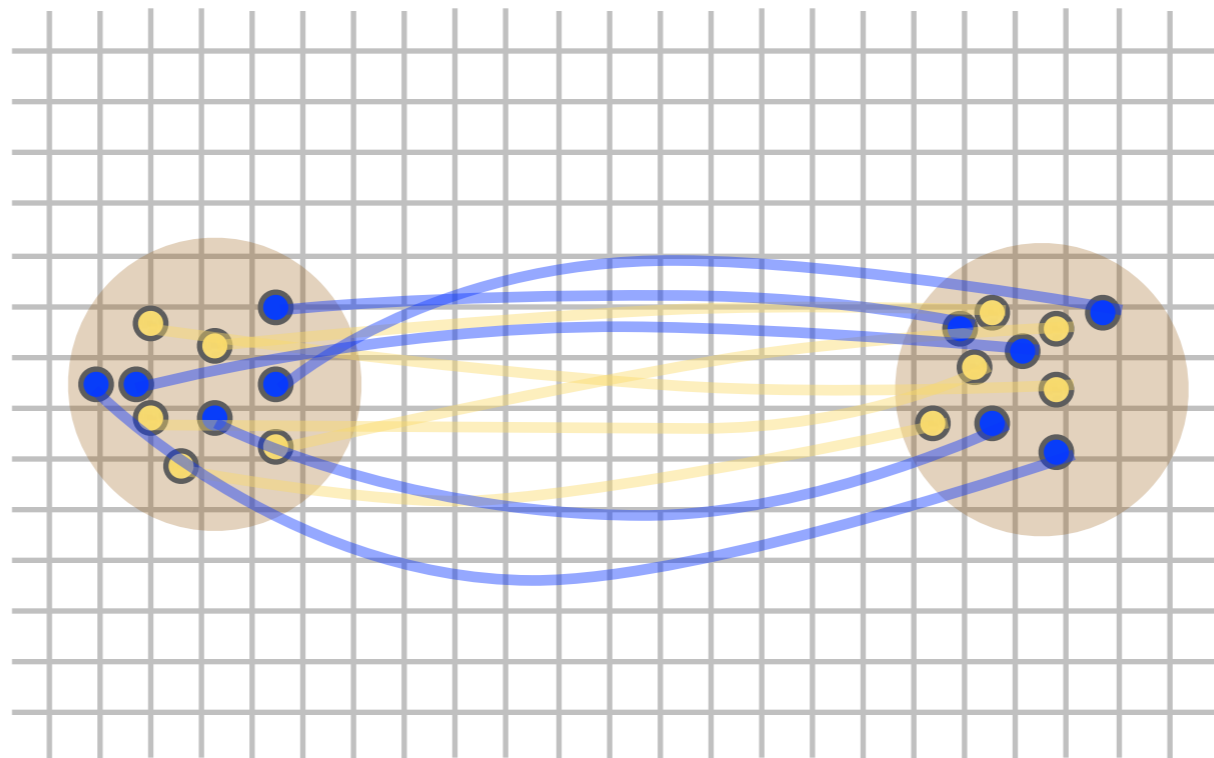


Many mesons in LQCD

- Consider n π^+ correlator ($m_u=m_d$)

$$C_n(t) = \left\langle 0 \left| \left[\sum_{\mathbf{x}} \bar{d}\gamma_5 u(\mathbf{x}, t) \bar{u}\gamma_5 d(\mathbf{0}, 0) \right]^n \right| 0 \right\rangle$$

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Many mesons in LQCD

- Consider $n \pi^+$ correlator ($m_u = m_d$)

$$C_n(t) = \left\langle 0 \left| \left[\sum_{\mathbf{x}} \bar{d} \gamma_5 u(\mathbf{x}, t) \bar{u} \gamma_5 d(\mathbf{0}, 0) \right]^n \right| 0 \right\rangle$$

$\rightarrow A e^{-E_n t}$

- $n!^2$ Wick contractions: $(12!)^2 \sim 10^{17}$

$$C_3(t) = \text{tr} [\Pi]^3 - 3 \text{tr} [\Pi] \text{tr} [\Pi^2] + 2 \text{tr} [\Pi^3]$$

$$\Pi = \sum_{\mathbf{x}} \gamma_5 S(\mathbf{x}, t; 0) \gamma_5 S^\dagger(\mathbf{x}, t; 0)$$



- Maximal isospin: only a single quark propagator

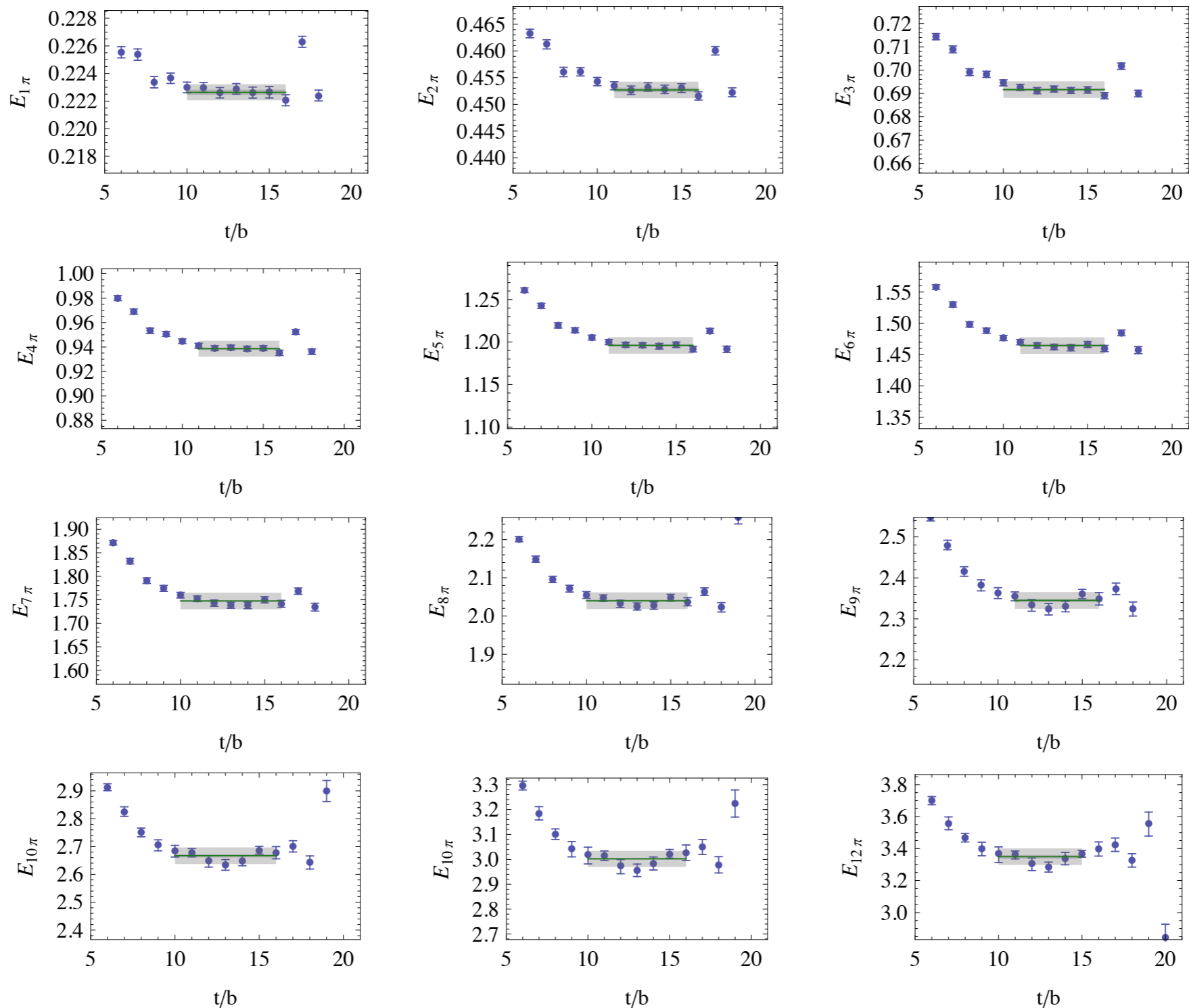


Lattice details

- Calculations use MILC gauge configurations
 - $L=2.5$ fm, $a=0.12$ fm, *rooted* staggered
 - also $L=3.5$ fm and $a=0.09$ fm
- NPLQCD: domain-wall quark propagators
 - $m_\pi \sim 291, 318, 352, 358, 491, 591$ MeV
 - 24 propagators / lattice in best case
- $I_z=n=1, \dots, 12$ pion and ($S=n$) kaon systems

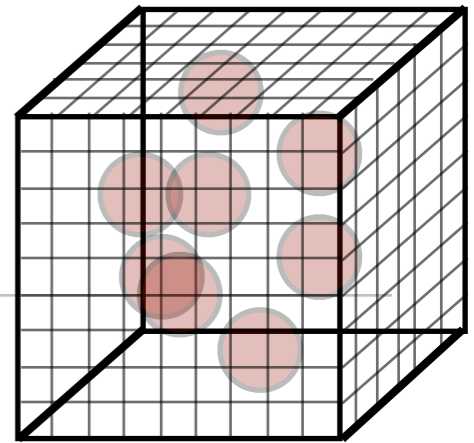
n-meson energies

- Effective energy plots: $\log[C_n(t)/C_n(t+1)]$



$m_\pi = 352$ MeV

Bosons in a box



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- 2 & 3 body interactions (N body: $L^{-3(N-1)}$)
- n=2: reproduces expansion of Lüscher

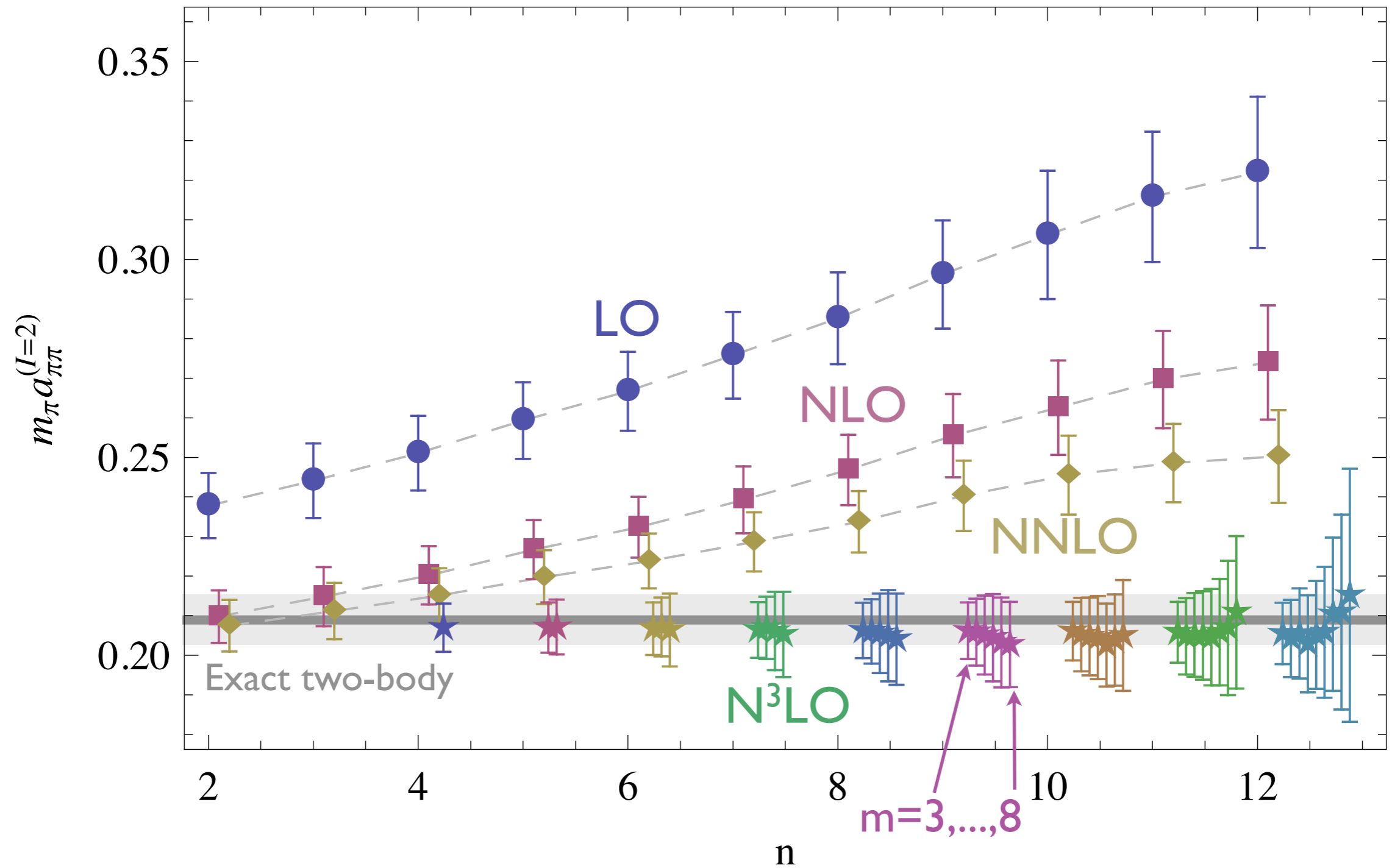
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Scattering length

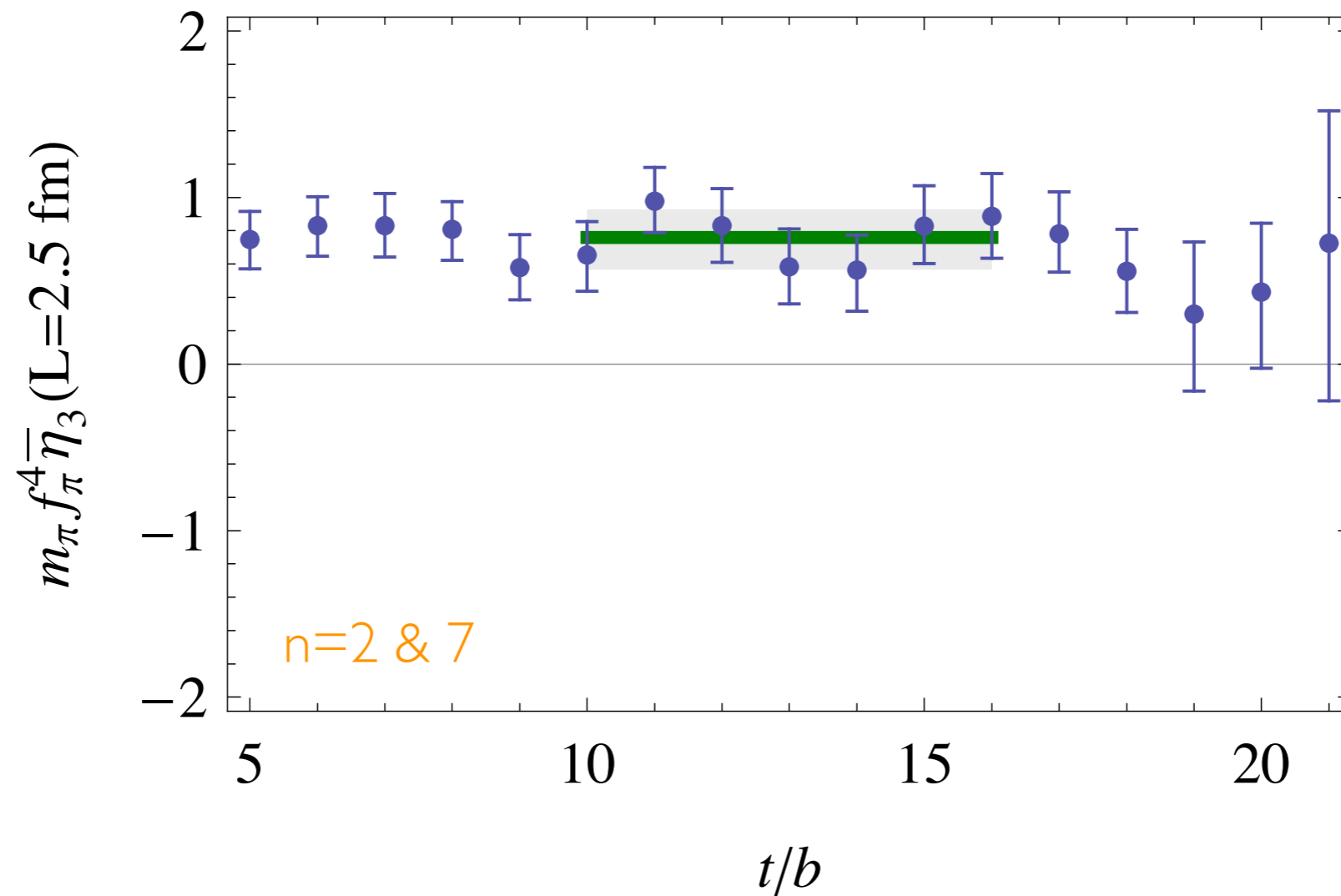
Geometric coefficients

Three body Interaction

Pion scattering



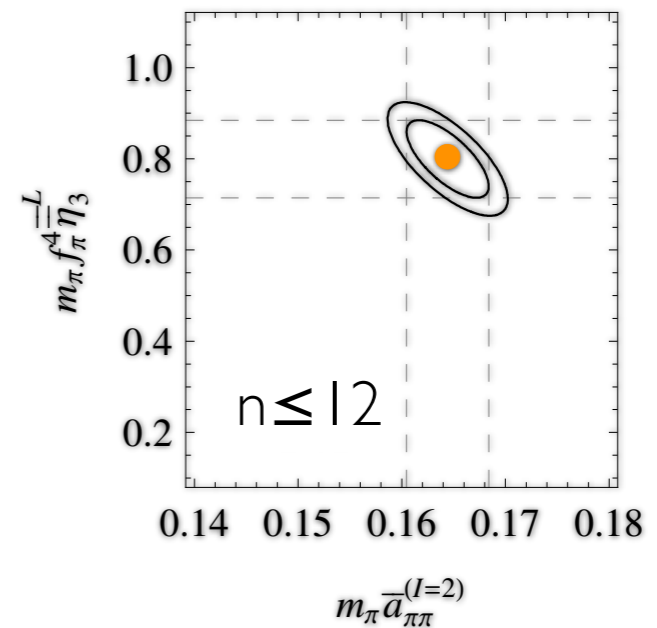
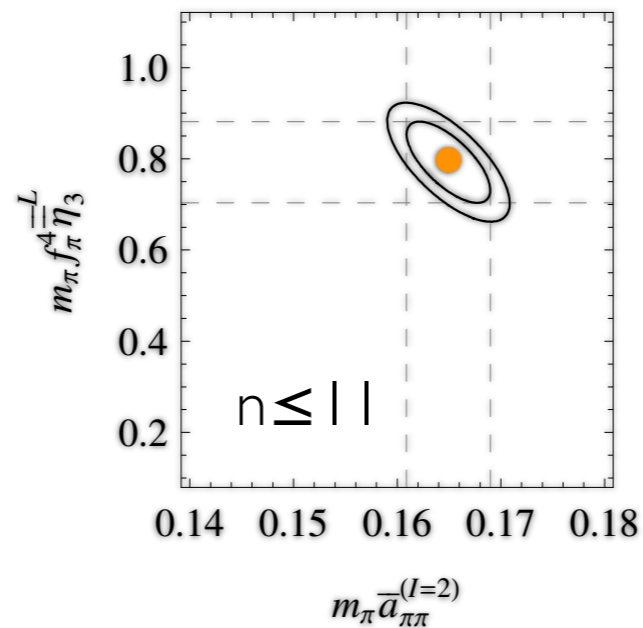
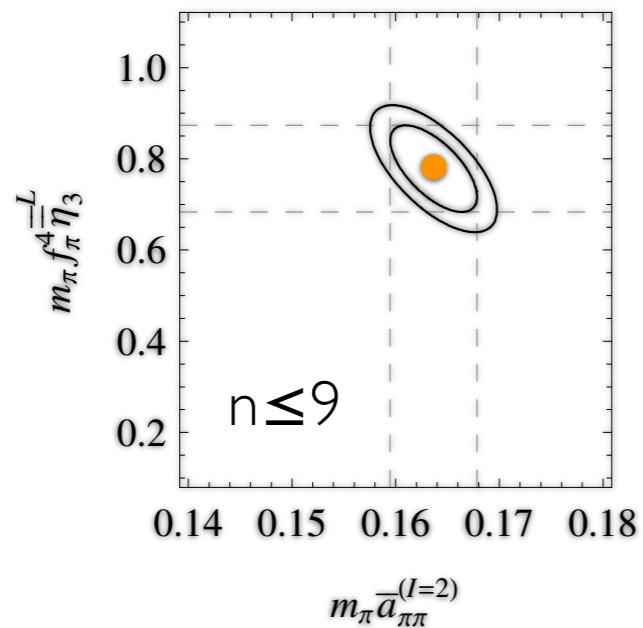
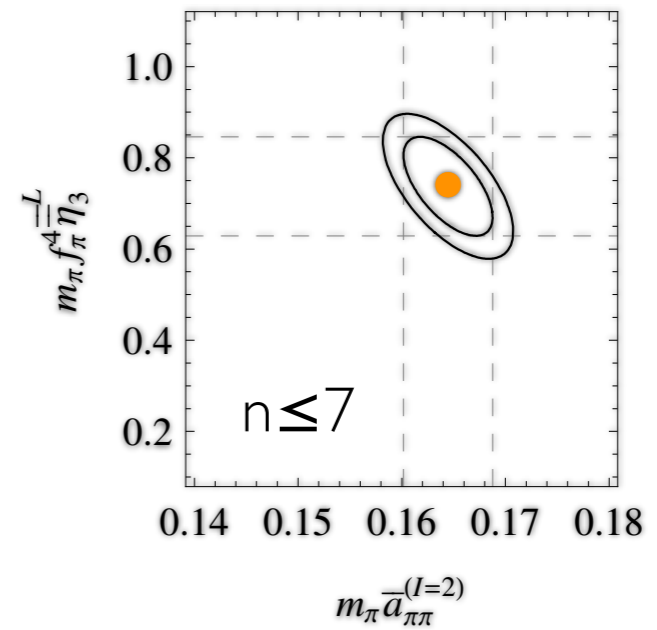
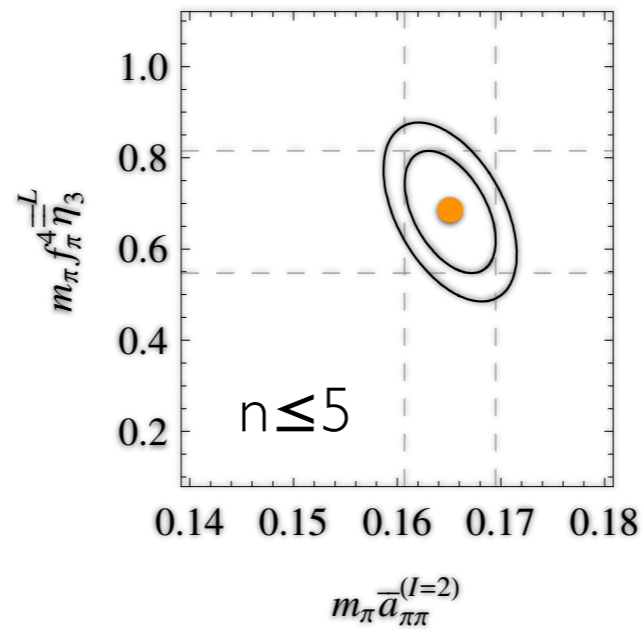
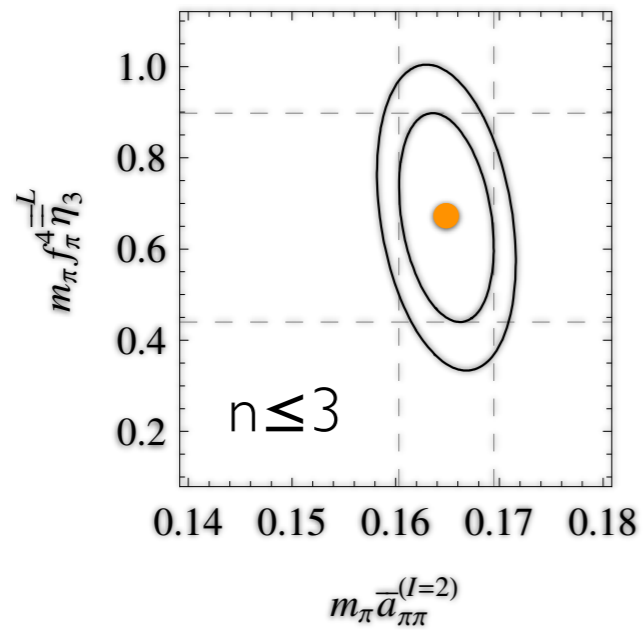
$\pi^+\pi^+\pi^+$ interaction



$m_\pi = 352$ MeV

n correlations

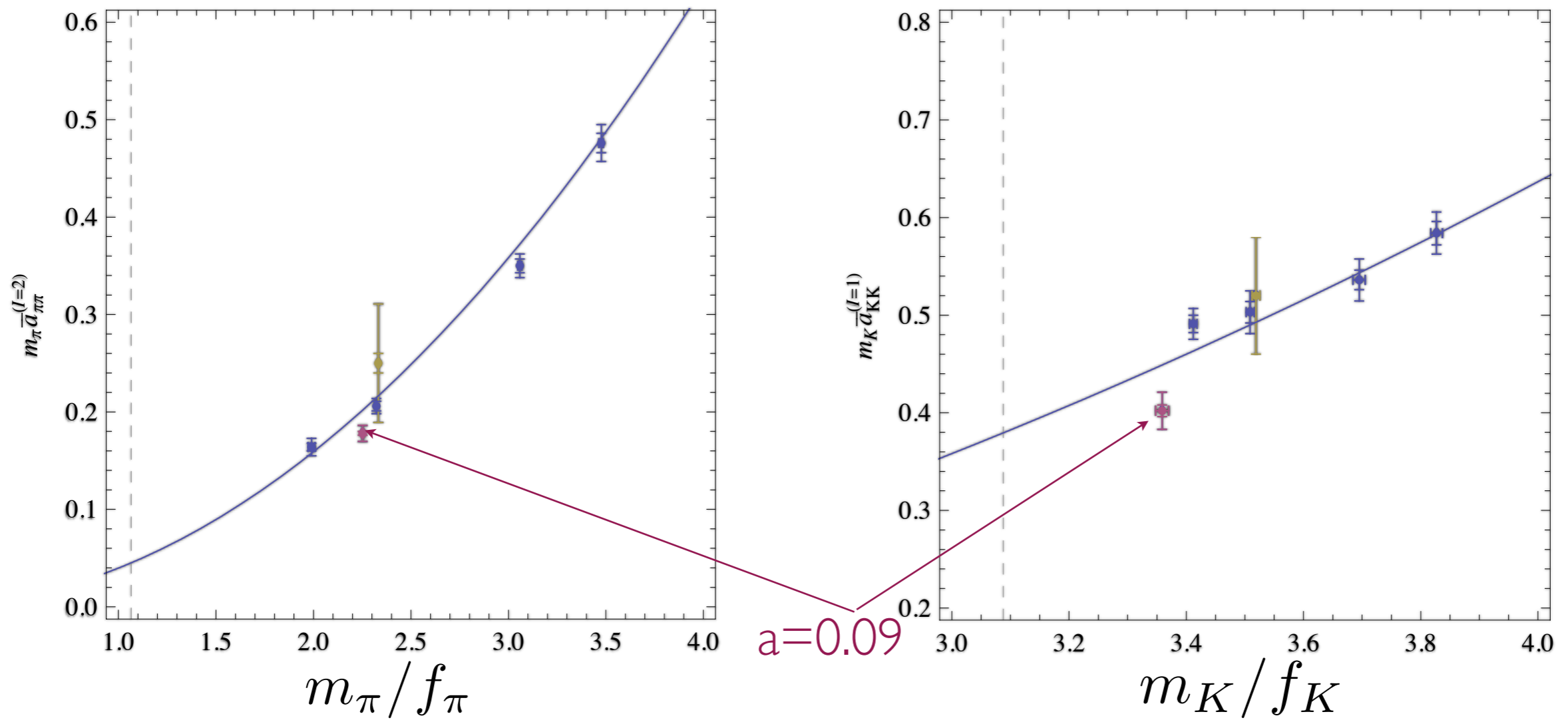
Three-body



Two body

$2\pi^+$ and $2K^-$ interaction

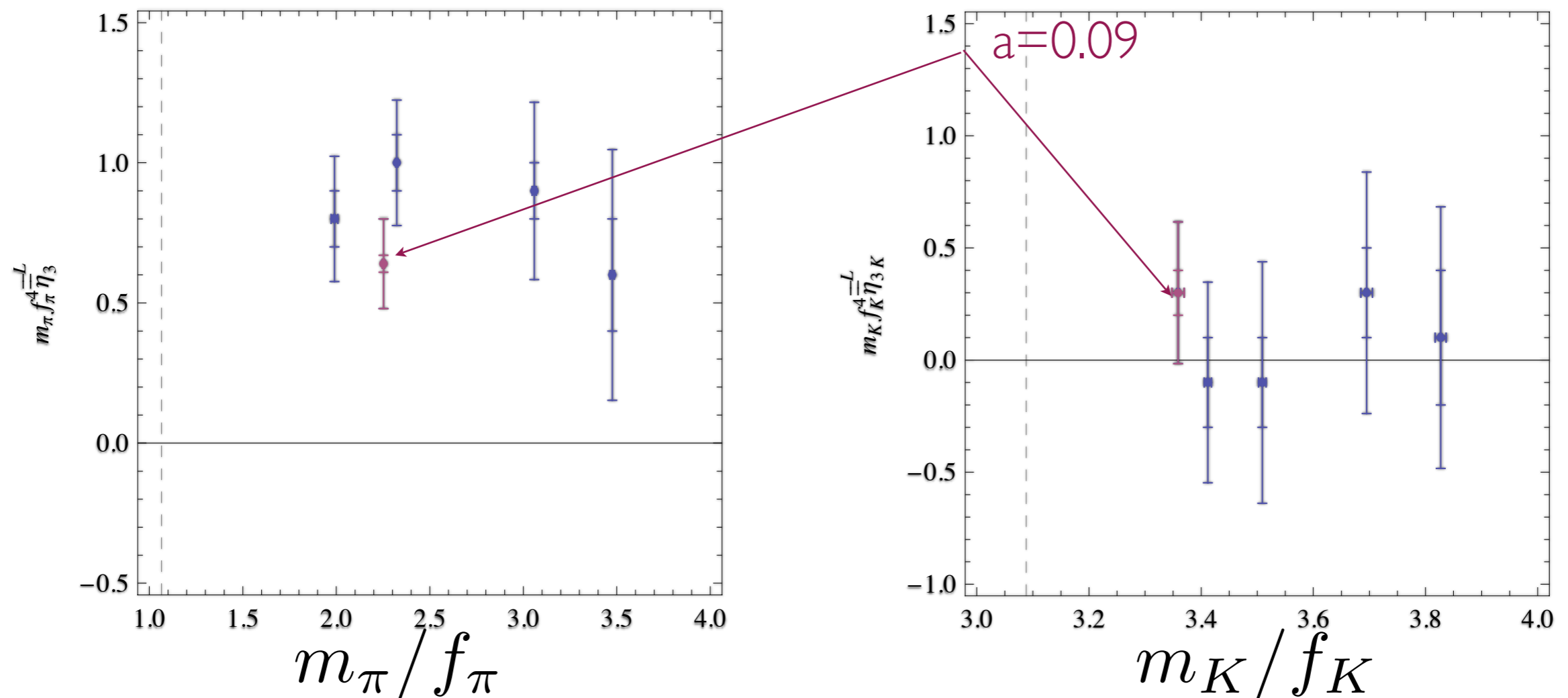
- Scattering lengths



curves: Weinberg

$3\pi^+$ and $3K^-$ interaction

- First QCD three body interaction



Naïve dimension analysis: I

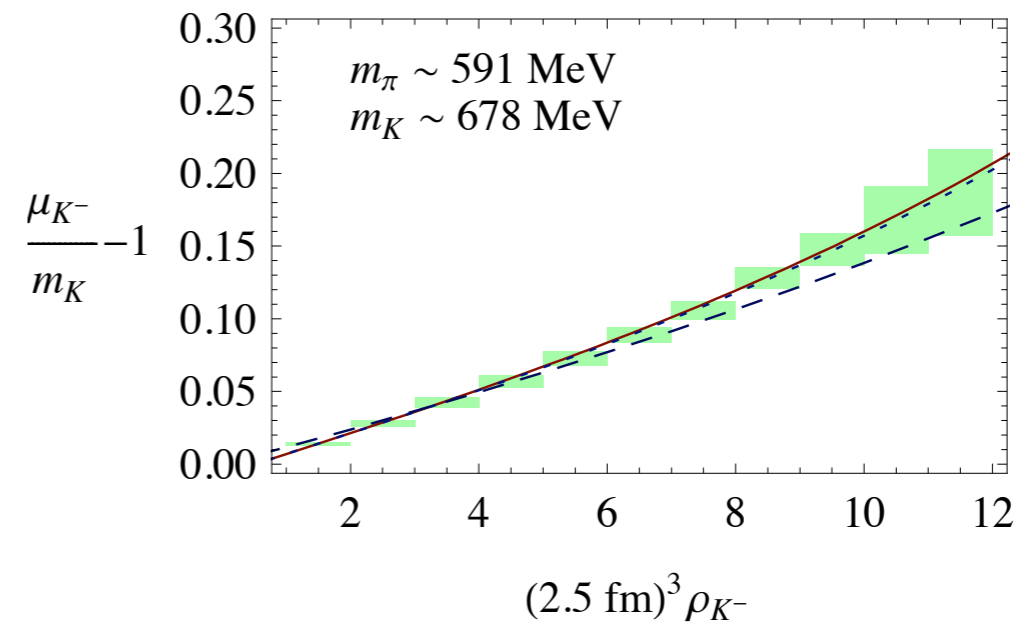
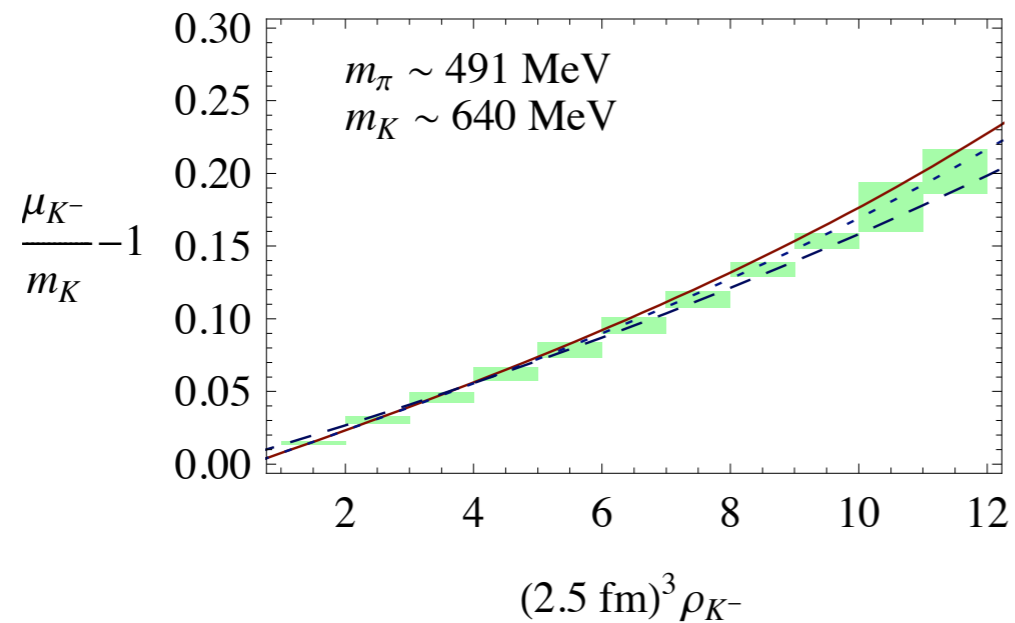
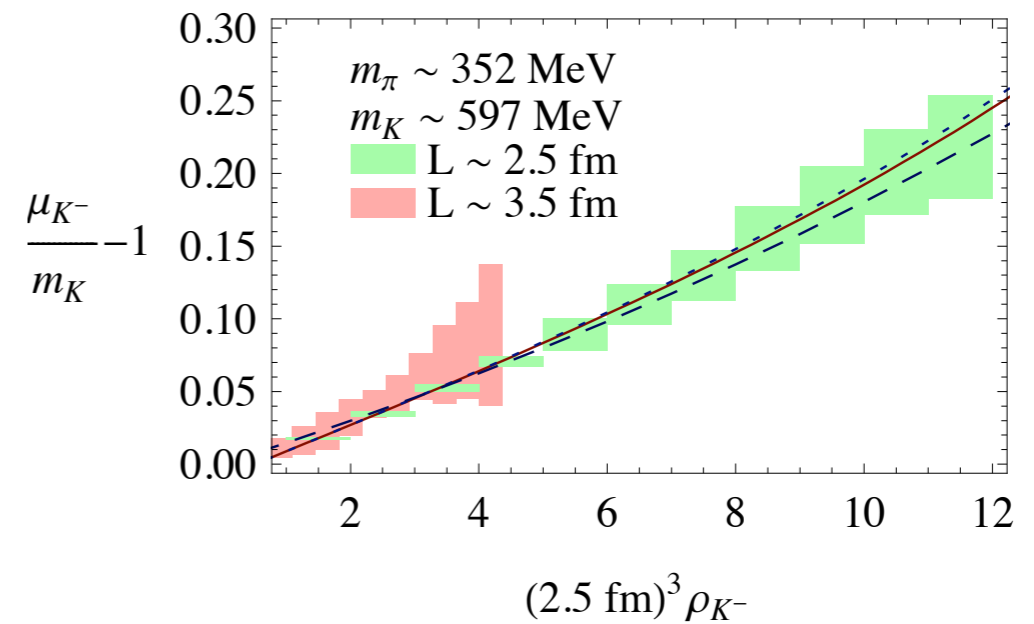
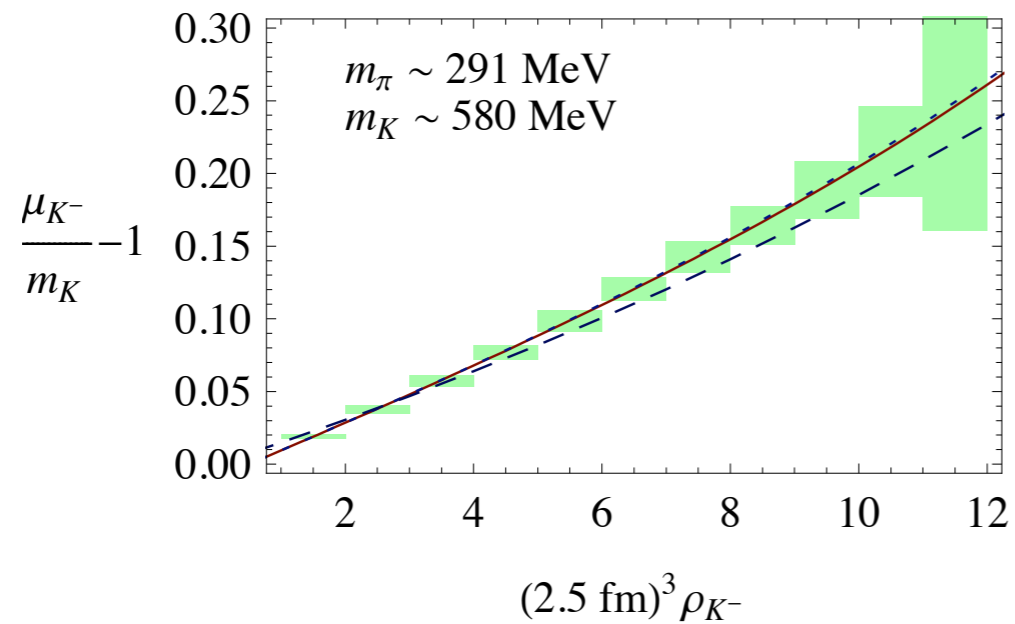
Equation of State

- For large n : Bose-Einstein condensate
- $1/L$ expansion: analytic form of EOS
- Chemical potential $\mu(\rho)$ numerically using finite difference

$$\mu = \left. \frac{d E}{d n} \right|_{V \text{ const}}$$

- Compare with LO χ PT [Son & Stephanov]

Kaon Chemical Potential

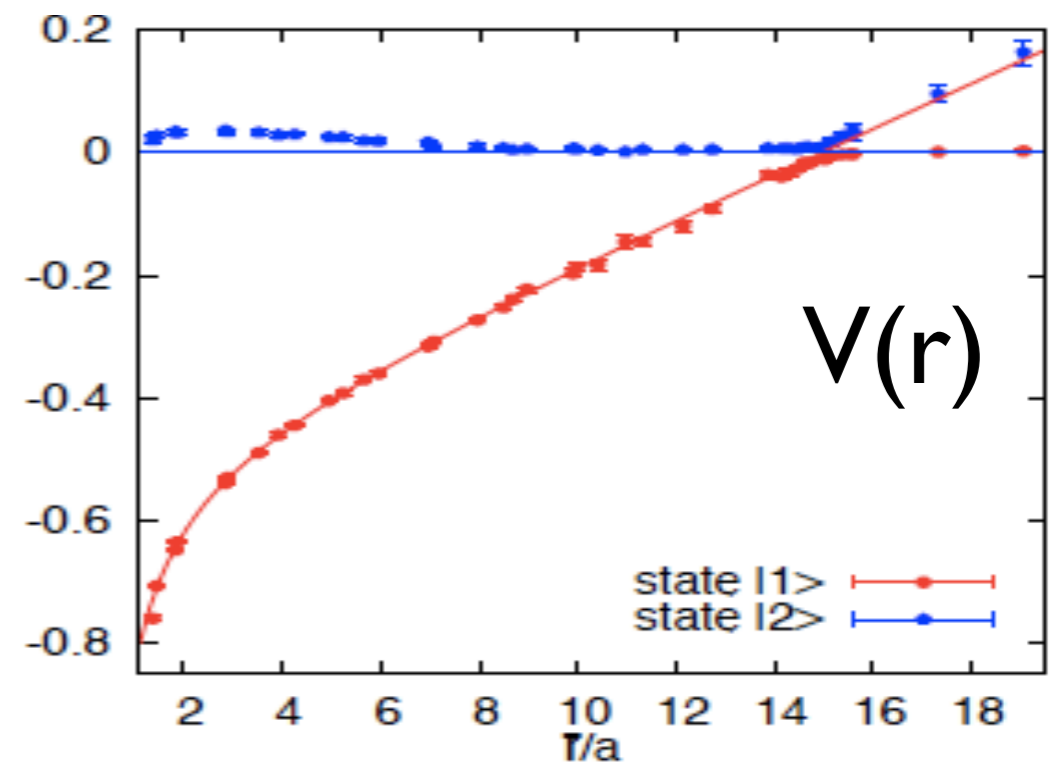
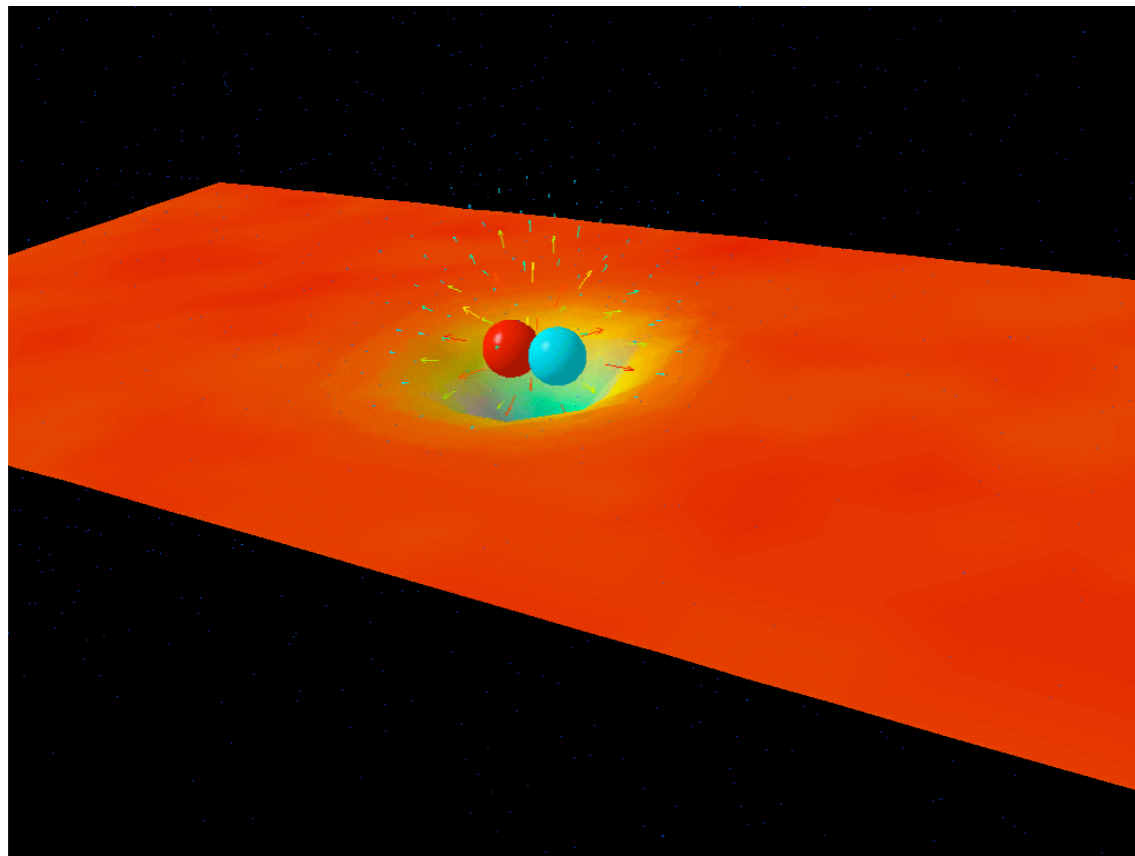


— 2+3 body fit No 3 body - - - LO χ PT

Colour screening by pions

[WD+ M Savage PRL 09]

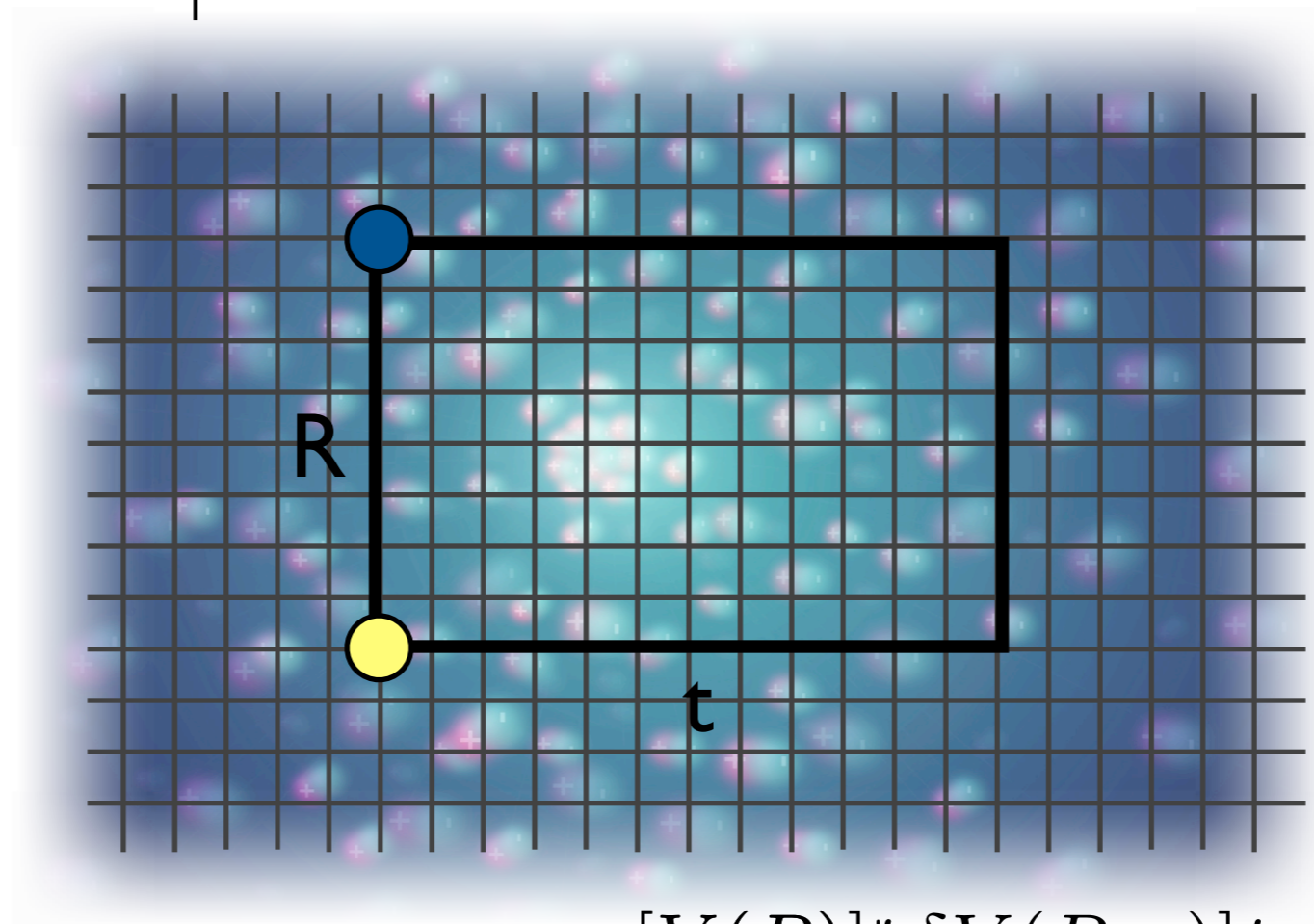
- Static quark potential



- Screening: evidence for quark-gluon plasma

Color screening

- Static quark potential

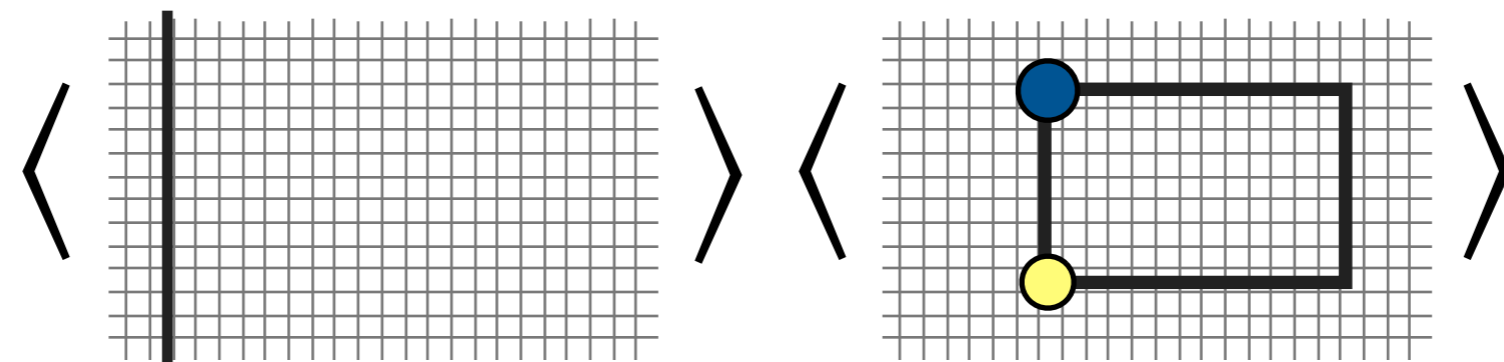
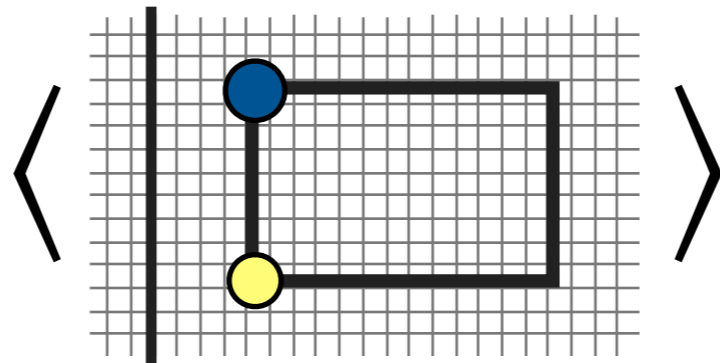


$$\rightarrow \# e^{-[V(R)]\# \delta V(R,n)]t}$$

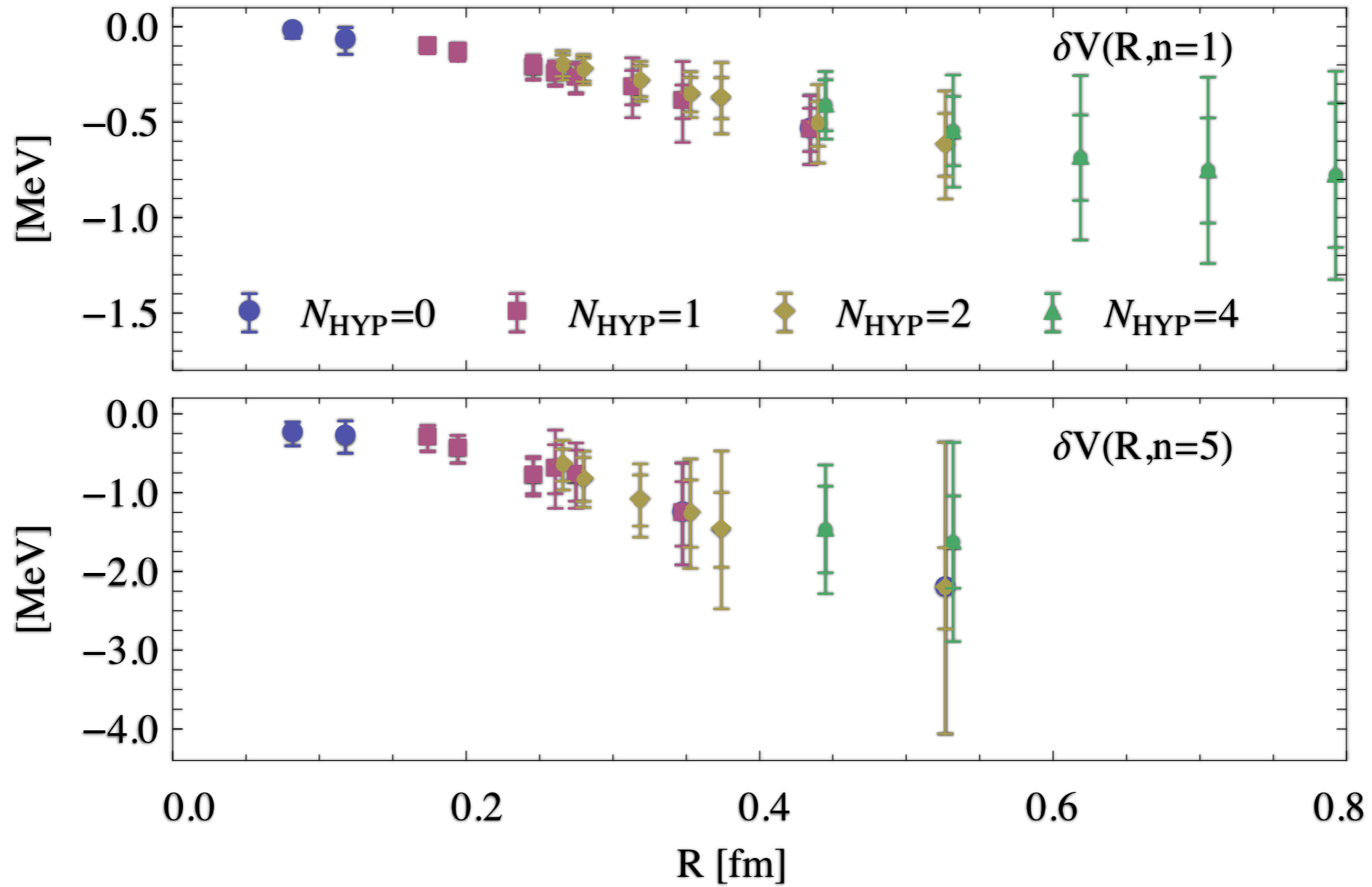
- Modified by condensate? Hadronic screening?

In medium effects

$$G_{n,W}(R, t_\pi, t_W, t) = \frac{C_{n,W}(R, t_\pi, t_W, t)}{C_n(t_\pi, t)C_W(R, t_w, t)}$$
$$\longrightarrow \# \exp[-\delta V(R, n)(t - t_w)]$$



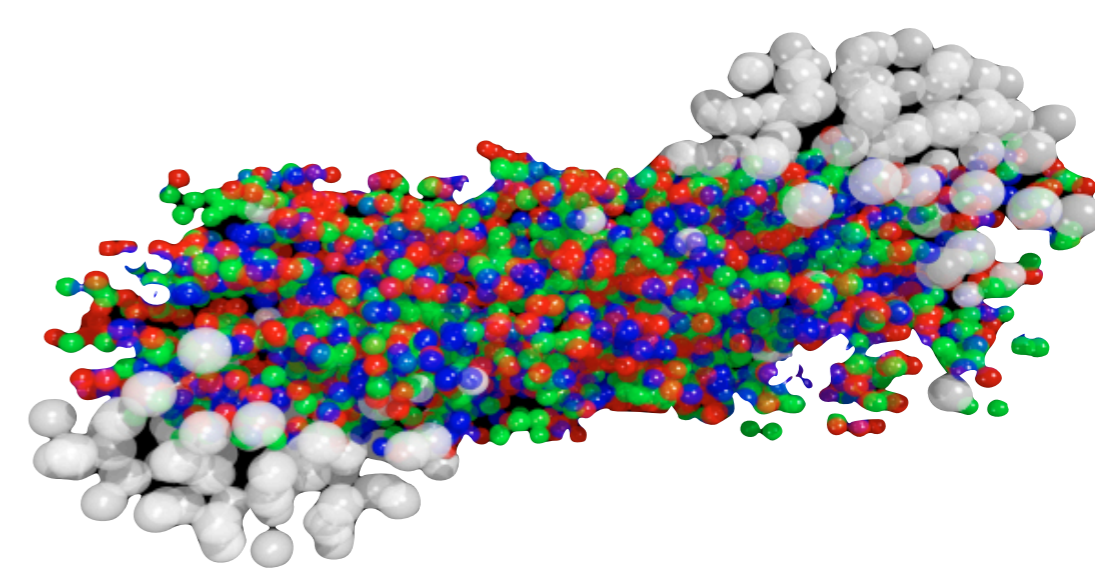
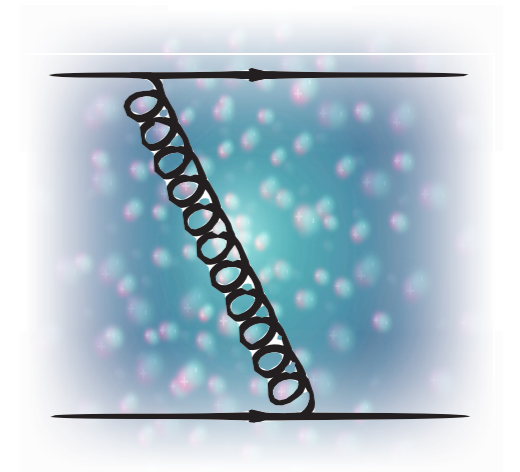
$\delta V(R, n=1 \text{ \& } 5)$



DWF on MILC: $a=0.09$ fm, $28^3 \times 96$, $m_\pi=318$ MeV

Pion screening

- r independent shift in $Q\bar{Q}$ force
- Dielectric medium inside flux tube
- Small effect: $\delta F(n=1)/F = 0.002$ at large R
- Hadronic medium effect
- Relevance to J/ψ suppression @ SPS/RHIC?



Complex systems

[WD, Savage 1001.2768]

Large systems

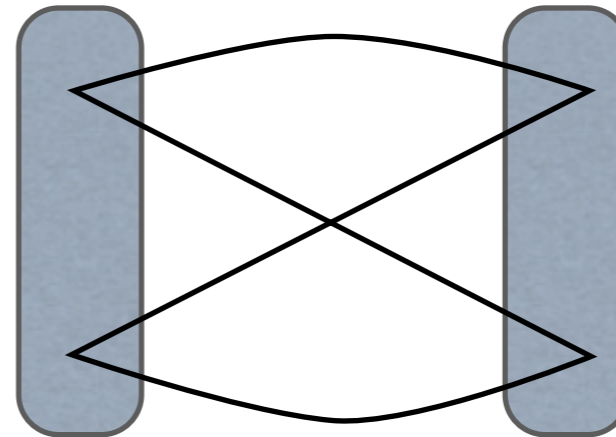
- How do we deal with complexity of contractions?
 - One species: $N_{\text{terms}} \sim e^{\pi\sqrt{2n/3}} / \sqrt{n}$ [Ramanujan & Hardy]
 - Two-species is harder, more is unfeasible
- How do we go beyond $n=12$?
 - Previous method fails because of Pauli principle
 - Avoid by using multiple propagator sources but this leads to contraction complexity

Few pion contractions

$$C_{1\pi}(t) = \text{Diagram 1}$$

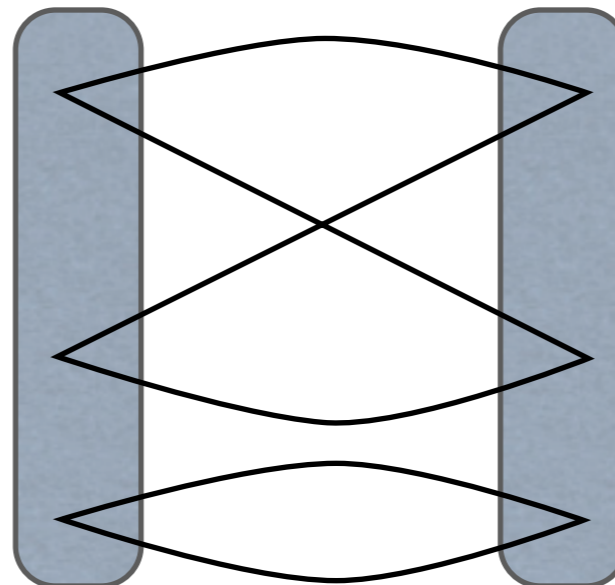
$$C_{2\pi}(t) = \text{Diagram 2} - \text{Diagram 3}$$

-

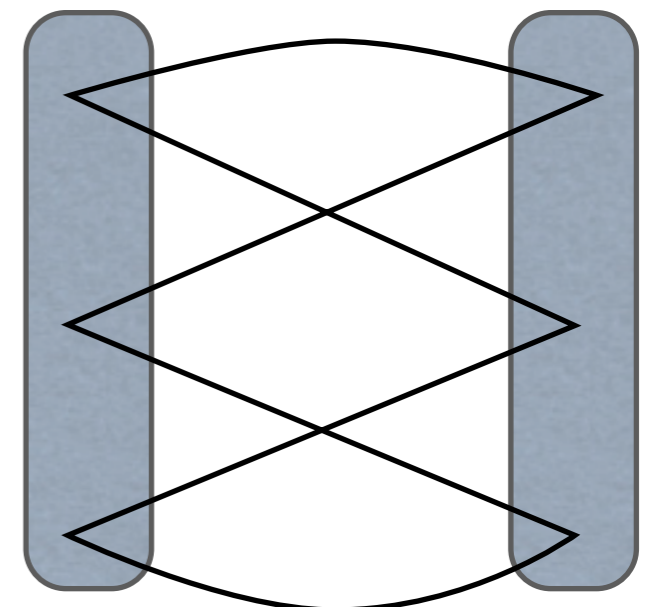


$$C_{3\pi}(t) = \text{Diagram 4} - 3 \text{Diagram 5} - 2 \text{Diagram 6}$$

-3



-2

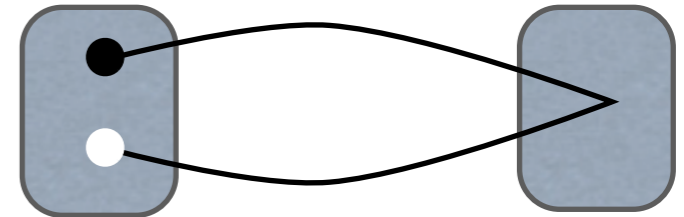


Blocks

- Define a partly contracted pion correlator

$$\Pi \equiv R_1 = \sum_{\mathbf{x}} S_u(\mathbf{x}, t; x_0) \gamma_5 S_d(x_0; \mathbf{x}, t) \gamma_5 = \sum_{\mathbf{x}} S_u(\mathbf{x}, t; x_0) S_d^\dagger(\mathbf{x}, t; x_0)$$

- Time-dependent 12×12 matrix (spin-colour indices)



- Correlators

$$C_1(t) = \langle \Pi \rangle, \quad C_2(t) = \langle \Pi \rangle^2 - \langle \Pi^2 \rangle, \dots$$

- Functional definition

$$\Pi_{ij} = \bar{u}_i(x) u_k(x_0) \frac{\delta}{\delta \bar{u}_j(x) \delta u_k(x_0)} C_1(t)$$

- Generalises to

$$(R_n)_{ij} \equiv \bar{u}_i(x) u_k(x_0) \frac{\delta}{\delta \bar{u}_j(x) \delta u_k(x_0)} C_n(t)$$

Recursion relation

[WD, Savage 1001.2768]

- The block objects are simply related
- Recursion relation

$$R_{n+1} = \langle R_n \rangle R_1 - n R_n R_1$$

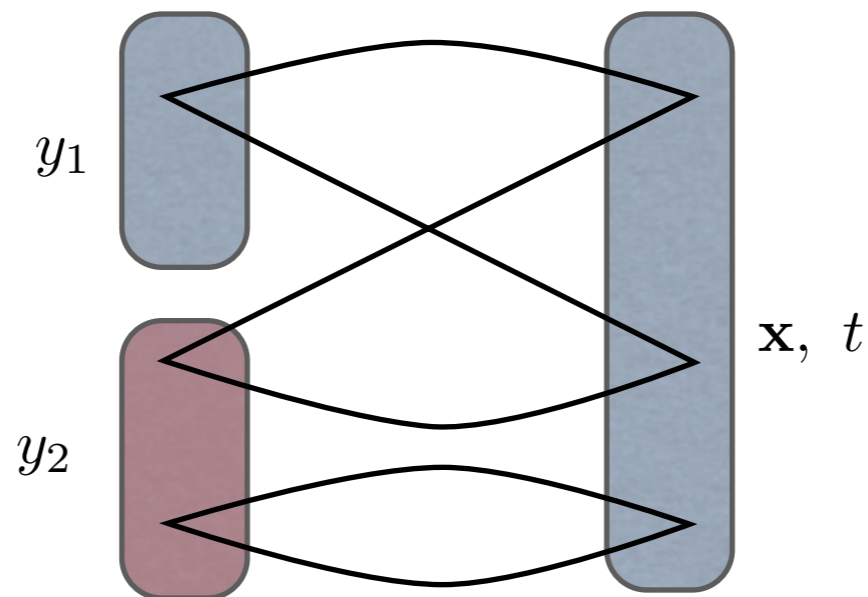
- Initial condition is that $R_1 = \Pi$, $R_j = 0, \forall j < 1$
- Can also construct a descending recursion as we know that $R_{13}=0$

Multi-source systems

- To get beyond $n=1, 2$, need to consider multi-source systems
- Consider two sources first

$$C_{(n_1 \pi_1^+, n_2 \pi_2^+)}(t) = \left\langle \left(\sum_{\mathbf{x}} \pi^+(\mathbf{x}, t) \right)^{n_1+n_2} \left(\pi^-(\mathbf{y}_1, 0) \right)^{n_1} \left(\pi^-(\mathbf{y}_2, 0) \right)^{n_2} \right\rangle$$

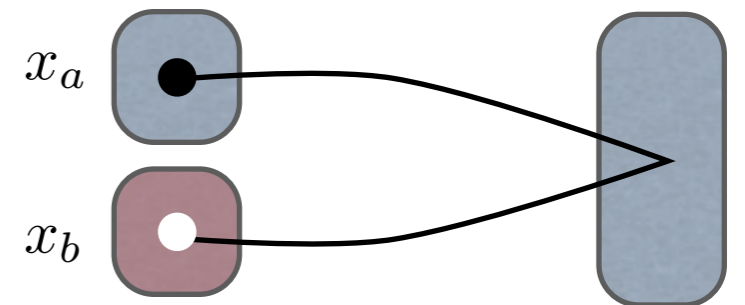
- $C_{(2,1)}(t)$ contains contractions like



Multi-source systems

- Multiple types of blocks needed

$$A_{ab} = \sum_{\mathbf{x}} S_u(\mathbf{x}, t; x_a) S_d^\dagger(\mathbf{x}, t; x_b)$$



- Two species case has a simple recursion relation:
First define

$$P_1 = \left(\begin{array}{c|c} A_{11}(t) & A_{12}(t) \\ \hline 0 & 0 \end{array} \right), \quad P_2 = \left(\begin{array}{c|c} 0 & 0 \\ \hline A_{21}(t) & A_{22}(t) \end{array} \right)$$

Then the generalisations of the R_n satisfy

$$Q_{(n_1+1, n_2)} = \langle Q_{(n_1, n_2)} \rangle P_1 - (n_1 + n_2) Q_{(n_1, n_2)} P_1 \\ + \langle Q_{(n_1+1, n_2-1)} \rangle P_2 - (n_1 + n_2) Q_{(n_1+1, n_2-1)} P_2$$

Extensions

- Recursions also constructed for
 - m -source systems
 - k -species systems: π 's, K 's, D 's, B 's, ...
 - m -source, k -species systems

$$T_{\mathbf{n}+\mathbf{1}_{rs}} = \sum_{i=1}^k \sum_{j=1}^m \langle T_{\mathbf{n}+\mathbf{1}_{rs}-\mathbf{1}_{ij}} \rangle P_{ij} - \bar{N} T_{\mathbf{n}+\mathbf{1}_{rs}-\mathbf{1}_{ij}} P_{ij}$$

where subscripts are matrices

- Implemented as matrix multiplications - computationally tractable

Extensions

- Factorial cost reduced to a linear cost with N mesons
- Each iteration involves essentially two-body contractions
- Recursions also exist for baryon contractions but are messier
- Different choices of basis objects for recursion
- Should allow calculations of $B=4,5,\dots$ systems

Mixed species systems

[work in preparation with B Smigielski (W&M/UW)]

n pions and m kaons

- Weakly interacting two species systems: pions and kaons
- $E_{n,m}$ of n pions and m kaons depends on three 2-body and four 3-body interaction parameters
- Analytic form is known for weak interactions [Smigielski & Wasem '08] but fairly ugly (only $1/L^6$)
- Matching to lattice energies allows for extraction of interaction parameters
- Reduced symmetry: contractions significantly more complex – $n=6$ pions, $m=6$ kaons: 1500 terms!

LQCD calculations

- One ensemble of anisotropic clover lattices
 - Dynamical $N_f=2+1$ lattices from JLab
 - $m_\pi=390$ MeV, $a_s=0.123$ fm, $\xi=3.5$, $20^3 \times 128$
 - ~ 30 K measurements: ~ 75 sources on ~ 400 cfgs
- Anti-periodic BCs for quarks (periodic for mesons)
 - Correlators have complicated time dependence
- Correlators for all sets of $\{n,m\}$ with $n+m < 13$

 Jefferson Lab



LQCD correlators

- Extend single species construction

$$C_{N\pi, MK}(t) = \left\langle \left(\sum_{\mathbf{x}} \mathcal{O}_d^\dagger(\mathbf{x}) \right)^N \left(\sum_{\mathbf{x}} \mathcal{O}_s^\dagger(\mathbf{x}) \right)^M \left(\mathcal{O}_d(0) \right)^N \left(\mathcal{O}_s(0) \right)^M \right\rangle$$

where

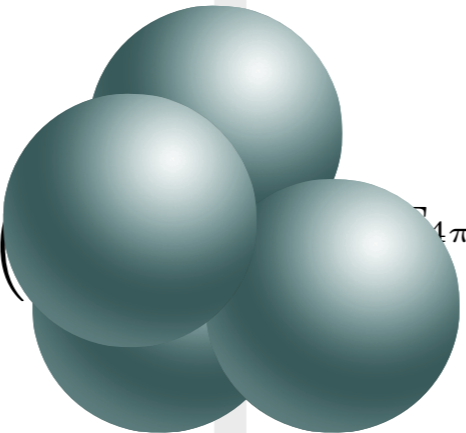
$$\mathcal{O}_q(x) = \bar{u}(x) \gamma_5 q(x) \text{ and } x = (\mathbf{x}, t)$$

- Can show the expected behaviour is

$$C_{N\pi, MK}(t) = \frac{1}{2} \sum_{m=0}^M \sum_{n=0}^N Z_{n,m}^{N-n, M-m} e^{-(E_{N-n, M-m} + E_{n,m}) T/2} \times \quad t_T = t - T$$

$$\cosh \left(\left(E_{N-n, M-m} - E_{n,m} \right) t_T \right) + \frac{1}{2} Z_{\frac{N}{2}, \frac{M}{2}}^{\frac{N}{2}, \frac{M}{2}} e^{E_{N/2, M/2} T/2} \delta_{N,2l} \delta_{M,2k}$$

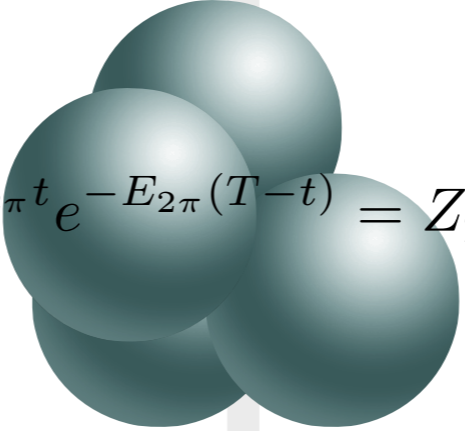
Four pion correlation



$Z_{4\pi} (e^{-E_{4\pi}(T-t)})$



$Z_{3/1\pi} (e^{-E_{3\pi}(T-t)} e^{-E_{1\pi}t})$



$Z_{2/2\pi} e^{-E_{2\pi}t} e^{-E_{2\pi}(T-t)} = Z_{2/2\pi} e^{-E_{2\pi}T}$

$t=0$

Analysis

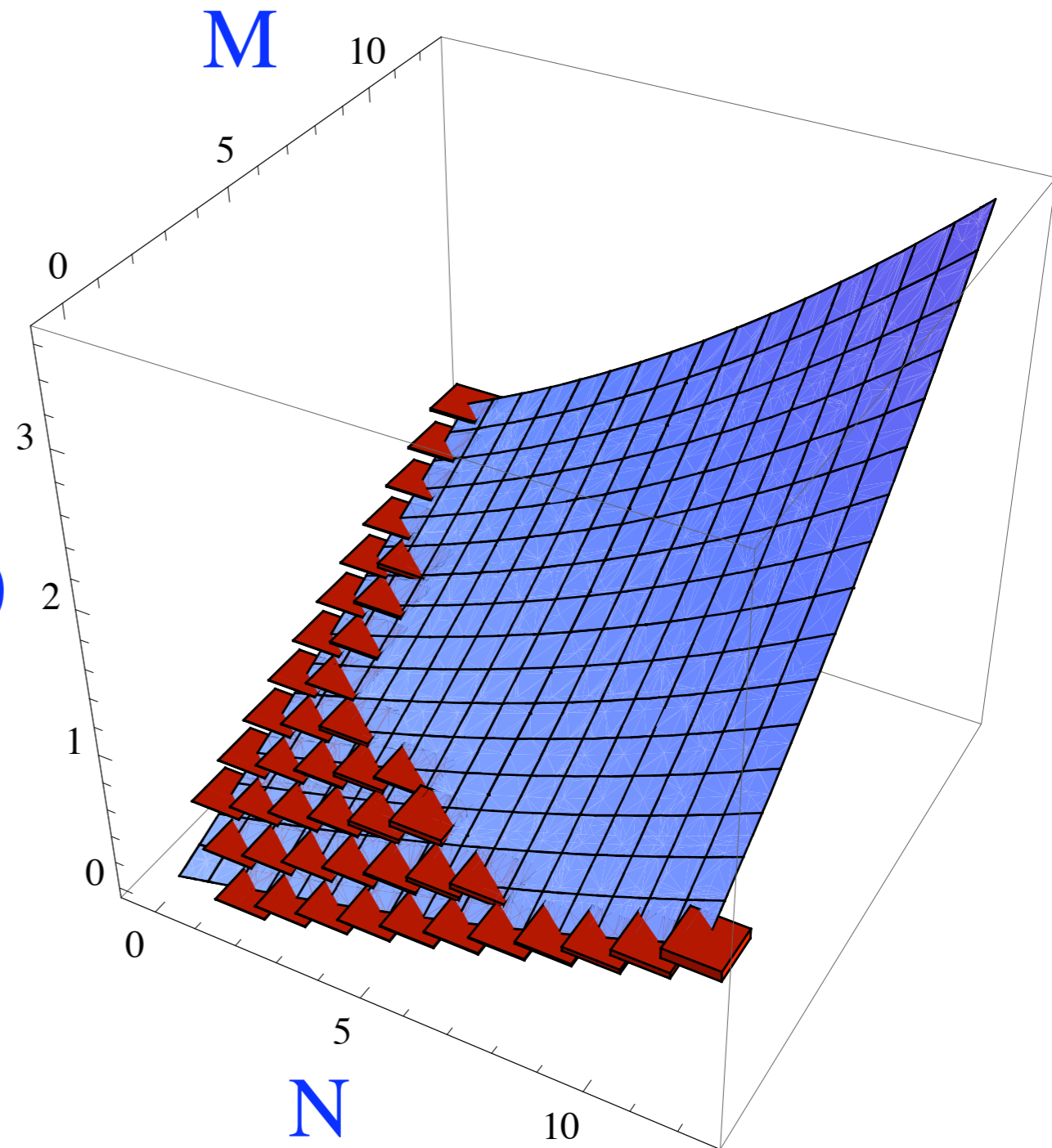
- Extracting the eigen-energies (or interaction parameters) from these correlators is difficult
- Correlations between different $\{n,m\}$
- Huge parameter space, $O(90)$ observables
 - $C_{4\pi,2K}$ involves 18 parameters
 - *Variable projection*: minimize the χ^2 function analytically for analytic parameters
 - Augment χ^2 via Bayesian priors



Extracted energies

- Boxes correspond to extracted energies
- Surface is fitted dependence on 2- and 3-body parameters

$$\Delta E(N, M)$$



Interaction parameters



- Extracted interaction parameters

$$m_K \bar{a}_{KK} = 0.444 \pm 0.0123 \pm 0.005$$

$$m_\pi \bar{a}_{\pi\pi} = 0.243 \pm 0.016 \pm 0.007$$

$$m_{\pi K} \bar{a}_{\pi K} = 0.136 \pm 0.015 \pm 0.007$$

$$m_\pi \bar{\eta}_{3\pi} f_\pi^4 = 1.002 \pm 0.303 \pm 0.149$$

$$m_K \bar{\eta}_{3K} f_K^4 = 0.800 \pm 0.314 \pm 0.127$$

$$\left(\frac{m_K m_\pi}{m_K + 2m_\pi} \right) \bar{\eta}_{3,\pi KK} f_{\pi KK}^4 = 1.123 \pm 0.379 \pm 0.162$$

$$\left(\frac{m_K m_\pi}{m_\pi + 2m_K} \right) \bar{\eta}_{3,\pi\pi K} f_{\pi\pi K}^4 = 0.769 \pm 0.348 \pm 0.150$$

- Single species parameters consistent with previous calculation but with better precision

Summary

- LQCD is making progress in many-body systems
- Explore novel types of QCD matter
- Properties and effects of meson condensates
- New algorithms to deal with contraction complexity
- (Light) Nuclei?



[FIN]

Mixed species case

$$\begin{aligned}
 \Delta \tilde{E}_{\pi K}(n, m, L) &= \frac{2\pi \bar{a}_{\pi K} m n}{m_{\pi K} L^3} \left[1 - \left(\frac{\bar{a}_{\pi K}}{\pi L} \right) \mathcal{I} + \left(\frac{\bar{a}_{\pi K}}{\pi L} \right)^2 \right. \\
 &\quad \times \left(\mathcal{I}^2 + \mathcal{J} \left[-1 + \frac{\bar{a}_{\pi}}{\bar{a}_{\pi K}} (n-1) \left(\frac{1}{m_{\pi K}} + \frac{2}{m_{\pi}} \right) \right. \right. \\
 &\quad \left. \left. + \frac{\bar{a}_K}{\bar{a}_{\pi K}} (m-1) \left(\frac{1}{m_{\pi K}} + \frac{2}{m_K} \right) \right] \right) + \left(\frac{\bar{a}_{\pi K}}{\pi L} \right)^3 \\
 &\quad \left(-\mathcal{I}^3 + f^{\mathcal{K}, \pi K} \left(\frac{\bar{a}_{\pi} \bar{a}_K}{\bar{a}_{\pi K}^2} \right) \mathcal{K} \right. \\
 &\quad \left. + \sum_{i=0}^2 \sum_{p=\pi, K} \left(f_i^{\mathcal{I}\mathcal{J}, p} \mathcal{I}\mathcal{J} + f_i^{\mathcal{K}, p} \mathcal{K} \right) \left(\frac{\bar{a}_p}{\bar{a}_{\pi K}} \right)^i \right) \left. \right] \\
 &\quad + \frac{nm(n-1) \bar{\eta}_{3, \pi \pi K}(L)}{2L^6} + \frac{nm(m-1) \bar{\eta}_{3, \pi K K}(L)}{2L^6} + \mathcal{O}(L^{-7}).
 \end{aligned}$$