Renormalising nuclear forces

or

How can we build an effective Hamiltonian for nuclear physics?

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"Effective field theories and the many-body problem", April–June 2009

What's the point of an effective (field) theory?

- no model assumptions – just low-energy degrees of freedom and symmetries
- estimates of errors and theory will tell you if it breaks down (no convergence)
- consistency of effective operators and interactions
- effective coupling constants are "universal"

$\longrightarrow$ links between different low-energy phenomena
(c_i's: πN scattering $\leftrightarrow$ two-pion exchange forces)

$\longrightarrow$ bridges between low-energy observables and underlying theory
(scattering lengths: scattering processes $\leftrightarrow$ lattice QCD)

How does it work?

- systematic expansion in powers of ratios of low-energy scales Q
  (momenta, $m_\pi, \ldots \sim 200$ MeV)
  to scales of underlying physics Λ_0
  (m_ρ, M_N, $4\pi F_\pi, \ldots \gtrsim 700$ MeV?)

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- interactions with ranges $\sim 1/\Lambda_0$ not resolved at scales Q

$\longrightarrow$ replaced by contact interactions

- iterations (loop diagrams) usually infinite

$\longrightarrow$ need to renormalise

- works provided we have a consistent expansion
  (otherwise trying to renormalise an infinite number of constants, simultaneously)

Where does it work?

Works well for purely pionic and πN systems

- pions $\sim$ Goldstone bosons of hidden chiral symmetry
- strong interactions weak at low energies

$\longrightarrow$ chiral perturbation theory

- terms organised by naive dimensional analysis
  aka "Weinberg power counting"
  (simply counts powers of low-energy scales – momenta and m_π)

What's the problem with building an EFT for nuclear forces?

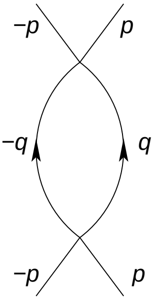
Chiral perturbation theory

- simply counting powers of low-energy scales: perturbative
- works for weakly interacting systems
  (eg pions, photons and ≤ 1 nucleon)
- but nucleons interact strongly at low-energies
- bound states exist (nuclei!)

$\longrightarrow$ need to treat some interactions nonperturbatively

Basic nonrelativistic loop diagram

$$\frac{M}{(2\pi)^3}\int \frac{\mathrm{d}^3q}{p^2-q^2+\mathrm{i}\varepsilon} = -\mathrm{i}\,\frac{M\,p}{4\pi} + \text{analytic}$$



- of order Q [Weinberg (1991)]
- but potential starts at order Q^0
  (OPE and simplest contact interaction)
- each iteration suppressed by power of Q/Λ_0

→ perturbative provided $Q < \Lambda_0$

- integral linearly divergent

→ cut off (or subtract) at $q = \Lambda$

- contributions multiplied by powers of Λ/Λ_0

→ again perturbative provided $\Lambda < \Lambda_0$

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- expand potential to some order in Q
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- resums subset of terms to all orders in Q (and some of these depend on regulator)

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- resums subset of terms to all orders in Q (and some of these depend on regulator)
- not necessarily a problem if these terms are small
- but what if we rely on them to generate bound states?

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  "Let the renormalisation group decide!"

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- Liberal
  "Let the renormalisation group decide!"

and the orthodox party seems to be winning the election, so far...

How can we iterate interactions consistently?

Identify new low-energy scales

- promote leading-order terms to order Q^{-1}
  (cancels Q from loop $\longrightarrow$ iterations not suppressed)
- can, and must, then be iterated to all orders
  (all other terms: perturbations)

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Examples of new scales

- S-wave scattering lengths $1/a \lesssim 40$ MeV
  [van Kolck; Kaplan, Savage and Wise (1998)]

$\longrightarrow$ for $p \ll m_\pi$: "pionless EFT" $\equiv$ effective-range expansion
  [Schwinger (1947); Bethe (1949)]

- also atomic systems with Feshbach resonance close to threshold

One-pion exchange

- important for nuclear physics at energies $\sim$ 100 MeV
- order Q^0 in chiral counting

$\longrightarrow$ treat as a perturbation [Kaplan, Savage and Wise (1998)]

- S waves: series coverges slowly, if at all
- OPE "unnaturally" strong
  (cf success of older phenomenology and Weinberg prescription)
- strength of OPE set by scale

$$\lambda_{NN} = \frac{16\pi F_\pi^2}{g_A^2 M_N} \simeq 290 \text{ MeV}$$

built out of high-energy scales ($4\pi F_\pi$, M_N) but $\sim 2m_\pi$

$\longrightarrow$ another low-energy scale?

How do we analyse scale-dependence of strongly-interacting systems?

General tool for this: the renormalisation group

- scattering by contact interactions is ill-defined in QM
- couple to virtual states with arbitrarily high momenta
- example: basic loop diagram for S waves behaves as

$$\frac{M}{(2\pi)^3}\int \frac{\mathrm{d}^3 q}{p^2 - q^2 + \mathrm{i}\varepsilon} \sim -\frac{M}{2\pi^2}\int \mathrm{d}q \quad \text{for large } q$$

(linear divergence)

$\longrightarrow$ need to renormalise

Procedure

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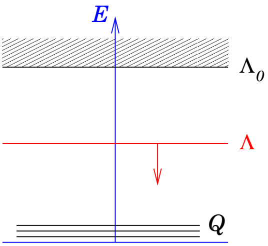

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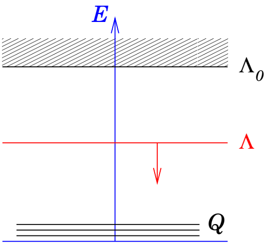

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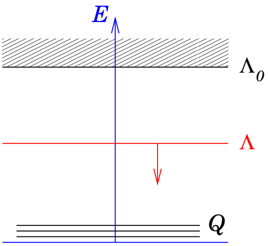

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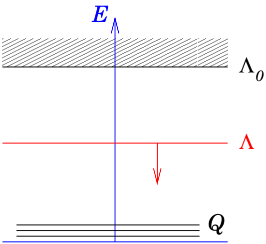

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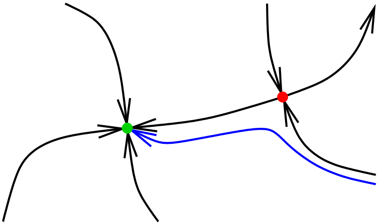
- rescale: express all dimensioned quantities in units of Λ (potential and all low-energy scales)

Follow flow of effective potential as $\Lambda \to 0$

$\longrightarrow$ look for fixed points

- rescaled theories independent of Λ
- correspond to scale-free systems
- endpoints of RG flow



● stable fixed point ● unstable fixed point

Expand around fixed point using perturbations that scale like Λ^ν

- $\nu < 0$ relevant or superrenormalisable
  (unstable; eg masses in QFTs)
- $\nu > 0$ irrelevant or nonrenormalisable
  (stable; eg mesonic ChPT)
- $\nu = 0$ marginal or renormalisable
  ($\rightarrow \ln\Lambda$ scale dependence; eg couplings in QED, QCD)

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Λ is highest acceptable low-energy scale

- order Q
- rescaling $\rightarrow$ power of Λ counts low-energy scales

What does the RG tell us about short-range potentials?

RG equation for $\hat{V}(\hat{k}', \hat{k}, \hat{p}; \Lambda)$ (rescaled)

$$\Lambda \frac{\partial \hat{V}}{\partial \Lambda} = \hat{p} \frac{\partial \hat{V}}{\partial \hat{p}} + \hat{k}' \frac{\partial \hat{V}}{\partial \hat{k}'} + \hat{k} \frac{\partial \hat{V}}{\partial \hat{k}} + \hat{V} + \hat{V}(\hat{k}', 1, \hat{p}; \Lambda) \frac{1}{1 - \hat{p}^2} \hat{V}(1, \hat{k}, \hat{p}; \Lambda)$$

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Two fixed points

- trivial $V_0 = 0 \longrightarrow$ free particles
- nontrivial [Birse, McGovern, Richardson (1998)]
  $\longrightarrow$ "unitary limit" (bound state at threshold, $a \longrightarrow \infty$)
- both scale-free systems

Trivial fixed point

Expansion around $V_0 = 0$ in powers of momenta

$$V(p) = C_0 + C_2\,p^2 + C_4\,p^4 + \cdots$$

- p^{2n} are RG eigenfunctions
- orders given by naive (Weinberg) counting: Q^0, Q^2, Q^4, ...
- coefficients C_{2n} related to expansion of on-shell K matrix
  (like T matrix but standing-wave bc's)
- perturbative
- appropriate EFT for thermal *np* scattering
  and other systems without low-energy bound/virtual states

Nontrivial fixed point

$$V_0(p,\Lambda) = -\frac{2\pi^2}{M\Lambda}\left[1 - \frac{p}{2\Lambda}\ln\frac{\Lambda+p}{\Lambda-p}\right]^{-1} \quad \text{(sharp cutoff)}$$

- order Q^{-1} (so must be iterated)
- exactly cancels basic loop integral in LS equation

$\rightarrow$ $T(p) = \mathrm{i}\,\frac{4\pi}{Mp}$ (unitary limit)

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Expanding around this point

$$V(p,\Lambda) = V_0(p,\Lambda) + V_0(p,\Lambda)^2\,\frac{M}{4\pi}\left(-\frac{1}{a} + \frac{1}{2}\,r_e\,p^2 + \cdots\right)$$

- factor $V_0^2 \propto \Lambda^{-2}$ promotes terms by two orders compared to naive expectation: Q^{-2}, Q^0, ...
- coefficients of perturbations directly related to observables: effective-range expansion

Enhancement follows from form of wave functions as $r \to 0$

Two particles in unitary limit

- irregular solutions: $\psi(r) \propto r^{-1}$ (S wave)
- cutoff smears contact interaction over range $R \sim \Lambda^{-1}$

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Other partial waves

- wave functions $\psi(r) \propto r^L$ for small r
  (assuming no low-energy bound state – regular solution)
- extra factor Λ^{2L} needed in potential

$\longrightarrow$ leading term in L-th partial wave of order Q^{2L}
(Weinberg counting: powers of Q from derivatives of δ-function)

Three-body systems

Attractive: 3 bosons or 3 distinct fermions in unitary limit (triton)

- naive dimensional analysis $\longrightarrow$ leading contact term of order Q^3
- as hyperradius $R \longrightarrow 0$ wave functions behave like

$$\psi(R) \propto R^{-2 \pm i s_0} \qquad s_0 \simeq 1.006$$ [Efimov (1971)]

$\longrightarrow$ leading three-body force promoted to order Q^{-1}

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Repulsive: 1 distinct and 2 identical fermions in unitary limit
(alkali atoms or neutrons)

- hyperradial wave functions $\psi(R) \propto R^{-2+2.1662}$

$\longrightarrow$ leading three-body force of noninteger order $Q^{3.3324}$

How do pion-exchange forces affect the power counting?

Treat λ_{NN} as low-energy scale $\longrightarrow$ iterate OPE

Central OPE (spin-singlet waves)

- $1/r$ singularity – not enough to alter power-law forms of wave functions at small r, even if iterated
- $L \geq 1$ waves: weak scattering $\longrightarrow$ Weinberg power counting
- ${}^1S_0$: similar to expansion around unitary fixed point

$\longrightarrow$ KSW-like power counting

Tensor OPE (spin-triplet waves)

- $1/r^3$ singularity
- but higher partial waves protected by centrifugal barrier
- above critical momentum waves resolve singularity
  $\longrightarrow$ OPE not perturbative
- $L \geq 3$: $p_c \gtrsim 2$ GeV $\longrightarrow$ Weinberg counting OK
- $L \leq 2$: $p_c \lesssim 3m_\pi$ $\longrightarrow$ new counting needed
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  [Nogga, Timmermans and van Kolck (2005)]
- wave functions $\psi(r) \propto r^{-1/4}$ multiplied by either sine or exponential function of $1/\sqrt{\lambda_{NN} r}$

$\longrightarrow$ leading contact interaction of order $Q^{-1/2}$ in P, D waves
(very weakly irrelevant)

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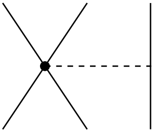
One-pion exchange ("c_D")

- contains two-body contact vertices like $(N^\dagger N)(N^\dagger \sigma\tau N)\cdot\nabla\pi$
- promoted in same way as contact interactions for $L \leq 2$



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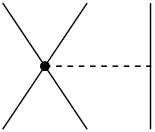

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Contact interaction (“c_E”)

- counting still not known: need to solve 3-body problem with $1/r^3$ potentials [L Platter]
- expect to be promoted $\rightarrow$ order Q^d, $-1 < d < 3$?

So, how should we build an effective Hamiltonian?

Order	NN	NNN
Q^{-1}	$^1S_0$, $^3S_1$ C_0's, LO OPE	
$Q^{-1/2}$	^3P_J, ^3D_J C_0's	
Q^0	$^1S_0$ C_2	
$Q^{1/2}$	$^3S_1$ C_2	
$Q^{5/4}$		$^1S_0$–$^3S_1$ C_{D0} OPE
$Q^{3/2}$	^3P_J, ^3D_J C_2's	
$Q^{7/4}$		$^3P_0$–$^1S_0$ C_{D0} OPE
Q^2	$^1S_0$ C_4, $^1P_1$ C_0, NLO OPE, LO TPE	$^3P_1$–$^3S_1$ C_{D0} OPE
$Q^{5/2}$	$^3S_1$ C_4	^3P_J, ^3D_J C_{D0}'s OPE
Q^3	NLO TPE	LO 3N TPE
$Q^?$		C_E

- orange terms absent from "N2LO chiral potential" (Weinberg Q^3)
- red terms absent from "N3LO" (Weinberg Q^4)
- order Q^{-1}: have to iterate; order $Q^{-1/2}$: may be better to

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Yes, provided you are careful . . .

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- without the counterterms needed to renormalise them
- dangerous: can alter form of short-distance wave functions and destroy power counting (or, at best, change it)
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- but problems don't arise, provided higher-order terms are small
- general way to ensure this: keep cutoff small, $\Lambda < \Lambda_0$
- introduces artefacts $\propto (Q/\Lambda)^n \longrightarrow$ radius of convergence Λ not Λ_0

$\longrightarrow$ leaves only a narrow window: Λ just below Λ_0

Where does all this leave us?

Renormalisation group

$\longrightarrow$ clear power counting rules for most partial waves

- controlled by forms of wave functions as $r \longrightarrow 0$
- in general, not naive dimensional analysis!
- two-body couplings directly related to observables
  (DWBA or DW effective-range expansion)
- enhancements of other effective operators
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Open questions

- counting for 3-body forces in presence of tensor OPE?
- critical momenta for tensor OPE in ${}^3P_J$, ${}^3D_J$ waves with $m_\pi \neq 0$?
- same counting for waves where tensor OPE is repulsive?