## Many-body quantum mechanics in flatland

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## **Outline**

- General motivation
- Lattice QCD and many-body physics
- The three-dimensional Bose gas
- The two-dimensional Bose gas
- Conclusion

*"...to explore the connections between QCD, cold-atom physics, and few-hadron systems."*

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Ultra-cold atoms: At nano-K temperatures, have a non-relativistic few-body system whose inter-particle interaction can be tuned.

*"...to explore the connections between QCD, cold-atom physics, and few-hadron systems."*

Ultra-cold atoms: At nano-K temperatures, have a non-relativistic few-body system whose inter-particle interaction can be tuned.

It gets better.... consider atoms tightly confined in the z direction:

$$
V_H(z) = \frac{1}{4} m \omega_0^2 z^2 \qquad \qquad \ell_0 = \sqrt{\frac{\hbar}{m \omega_0}}
$$

Can continuously move from 3 to 2 spatial dimensions!



[28] M. Luscher and U. Wolff, Nucl. Phys. B 339, 222 (1990). Controversy: ground state energy of 2-d Bose g [26] R. Jackiw, in \*Jackiw, R.: Diverse topics in theoretical and mathematical physics\*, 35-53.  $\begin{array}{ccc} \n1 & 1 & 1 \end{array}$ Controversy: ground state energy of 2-d Bose gas

[25] D. B. Kaplan, arXiv:nucl-th/0510023; Unpublished Notes.

G.E. Astrakharchik et al., Phys. Rev. A  $79$ ,  $051602(R)$   $(2009)$ 

C. Mora and Y. Castin, Phys. Rev. Lett.  $102$ ,  $180404$   $(2009)$ .

 $\frac{1}{100}$  erg none of the computation, https://mathworld.wolfram.com/  $\sim$  Soft $\sim$  Superince  $\sim$  $\alpha$ pandy in oad roading conceatorio and oile circums. claim discrepancy in sub-leading corrections and cite error in:

[12] S. R. Beane, P. F. Bedaque, A. Parren˜o and M. J. Savage, Phys. Lett. B 585, 106 (2004).

[26] R. Jackiw, in \*Jackiw, R.: Diverse topics in theoretical and mathematical physics\*, 35-53.

J.O. Andersen Eur. Phys. J. **B28**, 389 (2002).

## My interest in many-body QM began with Lattice QCD calculations of multi-pion interactions

Silas Beane



### Consider two boson scattering in QM:  $\bigcap$ enerigienture in energy scattering, in  $\bigcap A$ . and infinite tower of contact operations of contact operators, which contact operators to be determined via th

expression for the isotropic scattering phase shift in any number of dimensions. If one is

### Assume: finite range interaction in d dimensions  $A_{\rm eff}$  few of the contact only a few of the contact operators will be important. This is the essence of the essenc **of effective field the EFT** of the EFT of the Field of Direction in the USION is the field of the field operator  $\frac{1}{2}$ sents the all-orders interaction derived from the  $\sim$

interacting through contact interactions has the following Lagrangian:

FIG. 1: Feynman diagrams that give the exact two-body scattering amplitude. The oval blob repre-

This Lagrangian, constrained by Galilean invariance, parity and time-reversal invariance,

 $\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}) = 2-\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}})$  and  $\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}})$  in this paper is on d  $=3$  denotes in this paper is on d  $=3$ 

our general discussion of two- and three-body interactions, we will keep d arbitrary as this

$$
\mathcal{L} = \psi^{\dagger} \left( i \partial_t + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{4} (\psi^{\dagger} \psi)^2 - \frac{C_2}{8} \nabla (\psi^{\dagger} \psi) \nabla (\psi^{\dagger} \psi) + \dots
$$



 $\mathcal{L}^2$   $=$   $2\pi$   $\mathcal{L}^2$   $=$  d  $2\pi$  d  $\mathcal{L}^2$   $=$   $2\pi$   $\mathcal{L}$ 

 $\overline{\phantom{a}}$   $\overline{\$ 

Im(I0(p))

1

This lagrangian, constrained by Galilean invariance, parity and time-reversal invariance,

$$
\mathcal{A}_2(p) = -\frac{\sum C_{2n} p^{2n}}{1 - I_0(p) \sum C_{2n} p^{2n}} = \frac{-1}{\text{Im}(I_0(p)) [\cot \delta(p) - i]}
$$

#! \$ d<sup>D</sup>−<sup>1</sup>q

diagrams, shown in fig. 1, computed in the EFT gives the two-body scattering amplitude [22–

$$
\frac{2n}{\sum C_{2n} p^{2n}} = \frac{-1}{\text{Im}(I_0(p)) [\cot \delta(p) - i]}
$$

\$C2<sup>n</sup> <sup>p</sup><sup>2</sup><sup>n</sup> <sup>−</sup> Re(I0(p))

. (7)

where N (p) is a normalization factor that depends on d and is fixed by unitarity. Indeed

combining eq. (2) and eq. (2) and eq. (2) and one can parameter  $\alpha$  in the can parameter the can parameter the can parameter the can parameter  $\alpha$ 

"µ

amplitude:

For three spatial dimensions one obtains the usual effective range expansion, For three spatial dimensions one obtains the usual effective range expansion, 2. d = 3 + 1 d=4 Effective range theory: <sup>I</sup>0(p) <sup>=</sup> <sup>M</sup> ange theory:

 $\mathcal{A} = \mathcal{A} \cup \mathcal{A}$ 

and it is understood that the ultraviolet divergences in the EFT are regulated using di-

means generally appropriate [23, 24]. However it is a convenient choice if no assumption is

assume that the S-matrix element for isotropic (s-wave) scattering exists in an arbitrary scattering exists in<br>In an arbitrary scattering exists in an arbitrary scattering exists in an arbitrary scattering exists in an ar

$$
p \cot \delta(p) = -\frac{1}{a_3} + r_3 p^2 + \mathcal{O}(p^4)
$$

1

$$
a_3 = \frac{MC_0}{8\pi} \qquad \qquad r_3 = \frac{16\pi C_2}{MC_0^2}
$$

Neglecting the effective range, for a<sup>3</sup> > 0 there is a bound state with binding momentum

. (14)<br>14) - Andrej Bartham, fransk politik († 14)<br>14) - Andrej Bartham, fransk politik († 14)

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0 (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16<br>19) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) . (16) .

) (15)

Neglecting the effective range, for a<sup>3</sup> > 0 there is a bound state with binding momentum  $\mathsf N$ o running couplings in  $\overline{MS}$ 

Neglecting the effective range, for a<sup>3</sup> > 0 there is a bound state with binding momentum

$$
\mathcal{A}_2(p) = \frac{8\pi}{M} \frac{1}{p \cot \delta(p) - ip}
$$

log

5

α2(µ) = −MC0(µ) = −MC0(}<br>= −MC0(}) = −MC0(})

with dimensionless coupling and scattering and scattering area, respectively. The coupling area, respectively,

"µ

$$
a_3 \to \infty \qquad r_3 \to 0 \qquad \qquad \text{unitarity limit}
$$

 $\frac{1}{\sqrt{2}}$ 

8 **x**  $\frac{1}{2}$  ; r2  $\frac{1}{$ 

<sup>α</sup>2(µ) <sup>=</sup> <sup>−</sup>MC0(µ)

 $\frac{1}{2}$ 

2 = 8C<br>2 = 8C2(1)<br>2 = 8C2(1)

Maria Barat

2 = 8C2(2)

 $\frac{1}{\sqrt{2}}$ 

2 p2 + O(p4 <br>2 p2 + O(p4 + O(p) + O(p)

number of spacetime dimensions. We then have generally dimensions. We then have generally dimensions. We then <br>We then have generally dimensions. We then have generally dimensions. We then have generally dimensions. We th

 $2.3 \pm 3.3$ 

 $2.3 \pm 3.3$ 

3. d = 2 + 1

κ = 1/a3.

3. d = 2 + 1

#### What about two bosons in a confined geometry? (e.g. a lattice) eigenvalue et al.  $\sqrt{p}$  $E, \mathcal{G}$ . c a lattice)  $\frac{1}{2}$ sons in a conimed geometry :<br>a a lattica)  $\mathcal{G}$ . diditive) A. Eigenvalue equation

the integer sums using an integer cutoff. We then have via eq. (7) our general form for the

in a confined geometry with periodic boundary with periodic boundary conditions. Specifically, we will consider

where we have chosen to define the sum with a sharp cutoff (d = 2 is ultraviolet finite). The

 $\overline{\phantom{0}}$ 

 $\mathcal{O}(\mathcal{O}_\mathcal{P})$ 

 $(2)$ 

 $\mathcal{O}(\mathcal{O}_\mathcal{D})$  and  $\mathcal{O}(\mathcal{O}_\mathcal{D})$  and  $\mathcal{O}(\mathcal{O}_\mathcal{D})$  and  $\mathcal{O}(\mathcal{O}_\mathcal{D})$ 

 $p_{\alpha}$  ,  $p_{\alpha}$  ,

#C2n p2n + Re(I{DR) + R<br>(I{DR) + Re(I{DR) + R<br>

we can write the eigenvalue equation as well as the eigenvalue equation as well as well as well as the eigenva<br>We can write the eigenvalue equation as well a

 $\overline{\phantom{a}}$  ,  $\overline{\phantom{a}}$ 

 $\begin{pmatrix} 1 & 1 \end{pmatrix}$  $(d \cdot$ - 1)-di  $\overline{\phantom{a}}$ mensional torus the contract of the contract o  $(d - 1)$ -dimensional torus

With the scattering theory that we have developed we may now find the eigenvalue equation

in a confined geometry with periodic boundary conditions. Specifically, we will consider

(d=1). In the confined geometry, all bound and scattering states appear as poles of the S-10 million states appear as<br>In the S-10 million states appear as poles of the S-10 million states appear as poles of the S-10 milli

matrix, or scattering amplitude, A2(p). Hence, from eq. (2) we have the eigenvalue equation

$$
q^{d-3} \cot \delta(p) = \Gamma\left(\frac{d-1}{2}\right) \pi^{-\frac{d+1}{2}} \sum_{\mathbf{n} \in \mathbb{Z}^{d-1}}^{\Lambda_n} \frac{1}{\mathbf{n}^2 - q^2} + \frac{2\Lambda_n^{d-1}}{\pi (d-1)q^2} \text{Re}\left[2\mathcal{F}_1\left(1, \frac{d-1}{2}, \frac{d+1}{2}; \frac{\Lambda_n^2}{q^2}\right)\right]
$$

gives energy levels:  $q \equiv pL/2\pi$  $q - \mu$  is dimensions up to corrections theory that are exponentially suppressed in the exponential value in  $q - \mu$  $q \equiv pL/2\pi$ 

where we have chosen to define the sum with a sharp cutoff (d = 2 is ultraviolet finite). The

 $\frac{1}{\sqrt{2}}$ 

#C2<sup>n</sup> <sup>p</sup><sup>2</sup><sup>n</sup> <sup>=</sup> <sup>I</sup><sup>L</sup>

therefore convenient to write

II. SCATTERING IN A CONFINED GEOMETRY OF THE STATE OF THE

## <sup>p</sup> cot <sup>δ</sup>(p) <sup>=</sup> <sup>1</sup> Finite Volume

$$
p \cot \delta(p) = \frac{1}{\pi L} \mathcal{S}_3\left(\frac{pL}{2\pi}\right) \qquad \mathcal{S}_3(\eta) \equiv \sum_{n=1}^{\Lambda_n} \frac{1}{n^2 - \eta^2} - 4\pi \Lambda_n
$$

!pL

"

, (36)  $\sim$ 

 $T_h$ Weak coupling expansion:

$$
E_0(2,L) = \frac{4\pi a_3}{ML^3} \left\{ 1 - \left(\frac{a_3}{\pi L}\right) Q_2 + \left(\frac{a_3}{\pi L}\right)^2 \left[Q_2^2 - Q_4\right] + \left(\frac{a_3}{\pi L}\right)^3 \left[ -Q_2^3 + 3Q_2Q_4 - Q_6 \right] \right\}
$$
  
+ 
$$
\frac{8\pi^2 a_3^3}{ML^6} r_3 + \mathcal{O}\left(L^{-7}\right)
$$

$$
Q_2 = \lim_{\Lambda_n \to \infty} \sum_{n \neq 0}^{\lfloor n \rfloor \le \Lambda_n} \frac{1}{n^2} - 4\pi \Lambda_n = -8.91363291781
$$
\n
$$
Q_4 = \sum_{n \neq 0} \frac{1}{n^4} = 16.532315959
$$
\n
$$
Q_6 = \sum_{n \neq 0} \frac{1}{n^6} = 8.401923974433
$$

 $n{\neq}0$ 

 $\mathbf{n}^6$ 

This case is the best known. For completeness we give result.

### What about N bosons in a confined geometry? ot obout NL booopo in a confined geometry?

energy-shift of two particles in finite volume for a2  $\mu$  a2 is the two-body scattering for a2 is the two-body scatter

length. I extend this to include the effective range also. I then compute the shift in the

" + λ<sup>3</sup>|n(3)

<sup>|</sup>V<sup>ˆ</sup> <sup>|</sup>n(j−1)

|Vnk|

<sup>E</sup>(0) <sup>n</sup> <sup>−</sup> <sup>E</sup>(0)

VnpVpsVskVkn

<sup>k</sup> )(E(0) <sup>n</sup> <sup>−</sup> <sup>E</sup>(0)

<sup>k</sup> )<sup>2</sup>(E(0) <sup>n</sup> <sup>−</sup> <sup>E</sup>(0)

<sup>p</sup> ) <sup>−</sup> <sup>V</sup>nn

|
|-<br>| Vnk|

<sup>k</sup> )(E(0) <sup>n</sup> <sup>−</sup> <sup>E</sup>(0)

2

Vincent Company

#### where  $\blacksquare$  is an eigenstate of Home energy example in the state of the state is an energy example. Endyleight-ochliddingen i i. Rayleigh-Schrodinger PT: Lets remind ourselves what we do in grad school! The full Hamiltonian is written as dinger PT:  $\blacksquare$

 $\mathcal{L}$  $\hat{H} = \hat{H}_0 + \lambda \hat{V}$   $E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots$  $\mathcal{L}(\mathbf{z})$ where  $\mathcal{N}(\mathcal{N})$  is an eigenstate of Hˆ0 with energy eigenvalue E(0) n . The energy eigenvalue  $\mathcal{N}(\mathcal{N})$  $\nabla$   $n^{(0)}$  is  $n^{(1)}$  is  $2n^{(2)}$ that are not correctly normed (simplicity) normed (simplicity)  $\alpha$  simplicity)  $\alpha$ 

$$
E_n^{(1)} = \langle n^{(0)} | \hat{V} | n^{(0)} \rangle = V_{nn}
$$
  
\n
$$
E_n^{(2)} = \sum_{k \neq n} \frac{|\langle n^{(0)} | \hat{V} | k^{(0)} \rangle|^2}{E_n^{(0)} - E_n^{(0)}} = \sum_{k \neq n} \frac{|V_{nk}|^2}{E_n^{(0)} - E_n^{(0)}}
$$
  
\n
$$
E_n^{(3)} = \sum_{k \neq n} \sum_{p \neq n} \frac{V_{np} V_{pk} V_{kn}}{(E_n^{(0)} - E_n^{(0)}) (E_n^{(0)} - E_p^{(0)})} - V_{nn} \sum_{k \neq n} \frac{|V_{nk}|^2}{(E_n^{(0)} - E_n^{(0)})^2}
$$
  
\n
$$
E_n^{(4)} = \sum_{k \neq n} \sum_{p \neq n} \sum_{s \neq n} \frac{V_{np} V_{ps} V_{sk} V_{kn}}{(E_n^{(0)} - E_n^{(0)}) (E_n^{(0)} - E_p^{(0)}) (E_n^{(0)} - E_s^{(0)})}
$$
  
\n
$$
- V_{nn} \sum_{k \neq n} \sum_{p \neq n} \frac{V_{np} V_{pk} V_{kn}}{(E_n^{(0)} - E_n^{(0)})^2 (E_n^{(0)} - E_p^{(0)})}
$$
  
\n
$$
- E_n^{(1)} \left( \sum_{k \neq n} \sum_{p \neq n} \frac{V_{np} V_{pk} V_{kn}}{(E_n^{(0)} - E_n^{(0)}) (E_n^{(0)} - E_p^{(0)})^2} - V_{nn} \sum_{k \neq n} \frac{|V_{nk}|^2}{(E_n^{(0)} - E_k^{(0)})^3} \right)
$$
  
\n
$$
- E_n^{(2)} \sum_{k \neq n} \frac{|V_{nk}|^2}{(E_n^{(0)} - E_k^{(0)})^2},
$$

Vincent Vincent

−<br>− E(1) n<br>− E(1) n

!

!

 $\overline{\phantom{a}}$ 

#### N bosons in a finite volume to the pseudo-potential is given by <sup>η</sup> <sup>=</sup> <sup>−</sup><sup>4</sup><sup>π</sup> *<sup>M</sup> <sup>p</sup>*−<sup>1</sup> tan <sup>δ</sup>(*p*) <sup>=</sup> <sup>4</sup><sup>π</sup> *<sup>M</sup> <sup>a</sup>*<sup>+</sup> <sup>π</sup> *<sup>M</sup> a*<sup>2</sup>*r*(*p* <sup>→</sup><sup>2</sup> + *p* <sup>←</sup><sup>2</sup>)+*. . .*, keeping only the contributions **from and** *r*, *a* **and** *r***,** *a* **and** *r***,** *a* **and** *r***,** *r***,** *a* **and** *r***,** *a*potential, is more up to the discussion of the discussion has been phrased in terms of  $\sigma$  multiple in the discussion  $\pi$

where the ellipsis denote higher-body interactions that do not contribute at the order to which we work (in ge

pseudo-potentials, the modern language with which to describe these interactions and calculations is that of the *Detmold et al, (2007) Tan, (2007)*

## Two low-density regimes:

#### $I \gg N_{d_{\Omega}}$ perturbative expansion the perturbative expansion of EFT( $n_{\rm s}$ ), the ground-state energy of  $n_{\rm s}$  $L \gg \mathbf{N}a_3$

$$
E_0(\mathbf{N}, L) = \frac{4\pi a_3}{M L^3} \left\{ \begin{pmatrix} \mathbf{N} \\ 2 \end{pmatrix} - \left(\frac{a_3}{\pi L}\right) \begin{pmatrix} \mathbf{N} \\ 2 \end{pmatrix} Q_2 + \left(\frac{a_3}{\pi L}\right)^2 \left\{ \begin{pmatrix} \mathbf{N} \\ 2 \end{pmatrix} Q_2^2 - \left[ \begin{pmatrix} \mathbf{N} \\ 2 \end{pmatrix}^2 - 12 \begin{pmatrix} \mathbf{N} \\ 3 \end{pmatrix} - 6 \begin{pmatrix} \mathbf{N} \\ 4 \end{pmatrix} \right] Q_4 \right\} + \left(\frac{a_3}{\pi L}\right)^3 \left[ -\left(\frac{\mathbf{N}}{2}\right) Q_2^3 + 3 \begin{pmatrix} \mathbf{N} \\ 2 \end{pmatrix}^2 Q_2 Q_4 - \left(\frac{\mathbf{N}}{2}\right)^3 Q_6 - 24 \begin{pmatrix} \mathbf{N} \\ 3 \end{pmatrix} \left( Q_2 Q_4 + 2Q + \mathcal{R} - Q_6 \begin{pmatrix} \mathbf{N} \\ 2 \end{pmatrix} \right) \right. - 6 \begin{pmatrix} \mathbf{N} \\ 4 \end{pmatrix} \left( 3Q_2 Q_4 + 51Q_6 - 2 \begin{pmatrix} \mathbf{N} \\ 2 \end{pmatrix} Q_6 \right) - 300 \begin{pmatrix} \mathbf{N} \\ 5 \end{pmatrix} Q_6 - 90 \begin{pmatrix} \mathbf{N} \\ 6 \end{pmatrix} Q_6 \right] + \left(\frac{\mathbf{N}}{3}\right) \frac{64\pi a_3^4}{M L^6} (3\sqrt{3} - 4\pi) \log(\mu L) + \left(\frac{\mathbf{N}}{2}\right) \frac{8\pi^2 a_3^3}{M L^6} r_3 + \left(\frac{\mathbf{N}}{3}\right) \frac{\eta(\mu)}{L^6} + \mathcal{O} (L^{-7})
$$

 ${\bf N}^{\frac{1}{3}} a_3 \ll L \ll {\bf N} a_3$  BEC: thermodynamic limit  $\frac{1}{3} a_3 \ll L \ll \mathbf{N} a_3$  BEC: thermodynamic limit

$$
\frac{E_0}{N} = \frac{2\pi\rho a_3}{M} \left( 1 + \frac{128}{15\sqrt{\pi}} (\rho a_3^3)^{1/2} + \frac{8}{3} (4\pi - 3\sqrt{3})\rho a_3^3 \ln(\rho a_3^3) + \rho a_3^3 \eta' + \ldots \right) + \ldots
$$

### How does one calculate in BEC regime? <sup>ρ</sup>(*µ*) <sup>=</sup> <sup>−</sup>d*<sup>F</sup>* d<sub>1</sub><br>d<sub>1</sub><br>
d<sub>1</sub><br>

be expressed as the expression of the e

from the energy density.

3.1 Perturbative framework

<sup>ρ</sup>(*µ*) <sup>=</sup> <sup>−</sup>∂Ω

 $\mathcal{L}_\text{max}$  set up a perturbative framework. We first set up a perturbative framework

for carrying out calculations in the presence of a nonzero

chemical potential. We use the framework to calculate the

energy density to second order in the quantum correc-

tions. We then carry out the renormalizations of *µ* and *g*

that are necessary to remove power ultraviolet divergences

 $W_{\rm eff}$  describe a Bose gas with nonzero density  $\rho$ 

 $\mathcal{O}(\log n)$  with an appropriately chosen value of the theory chosen value of the theo

chemical potential. For simplicity, we set *g*<sup>3</sup> = 0 and omit

all terms in (10) that are higher order in  $\mathcal{M}(\mathcal{M})$ 

ignore for the moment the moment the counterterms associated with  $\alpha$ 

renormalization, so the parameters *µ* and *g* should be re-

garded as bare parameters. Inserting the field parameter-

**Braaten and Nieto, (1999)** 

$$
\mathcal{L} = \psi^{\dagger} \left( i \partial_t + \frac{\nabla^2}{2M} + \mu \right) \psi - \frac{C_0}{4} (\psi^{\dagger} \psi)^2 - \frac{C_2}{8} \nabla (\psi^{\dagger} \psi) \nabla (\psi^{\dagger} \psi) - \frac{D_0}{36} (\psi^{\dagger} \psi)^3 + \dots
$$

$$
\rho(\mu)=<\psi^\dagger\psi>_\mu
$$

### ψ(r*,t*) in terms of two real-valued quantum fields ξ and quantum fluctuations around a mean field

$$
\psi(\mathbf{r},t) = v + \frac{\xi(\mathbf{r},t) + i\eta(\mathbf{r},t)}{\sqrt{2}}
$$

mean field theory: 
$$
\rho_0 = v^2
$$

## Example: multi-pion interactions from Lattice QCD:



$$
\chi-\mathrm{PT}: \qquad \rho_I \quad = \quad \frac{1}{2} f_\pi^2 \mu_I \left(1-\frac{m_\pi^4}{\mu_I^4}\right)
$$

## Can one take the thermodynamic limit directly?

(How does the Bose gas lose knowledge of its container?)

$$
\mathbf{N} \to \infty \qquad V \to \infty \qquad \rho \equiv \frac{\mathbf{N}}{V} \text{ fixed}
$$

leading order is trivial

corrections look mysterious...

Seems to require understanding of:

$$
Q_{2s} \equiv \sum_{\mathbf{n} \in \mathbb{Z}^3 \neq 0}^{\infty} \frac{1}{(\mathbf{n}^2)^s}
$$



Carl Gustav Jacob Jacobi

## Relations among elliptic integrals gives:

$$
\sum_{\mathbf{n}\in\mathbb{Z}^2\neq 0}^{\infty}\frac{1}{(\mathbf{n}^2)^s} = 4\zeta(s)\beta(s)
$$

$$
\zeta(s) \equiv \sum_{m=0}^{\infty} \frac{1}{(m+1)^s}
$$

$$
\frac{1}{(m+1)^s} \qquad \beta(s) \equiv \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^s}
$$

Riemann zeta function **Dirichlet beta function** 

$$
\zeta(s) = \sum_{m=0}^{\infty} \frac{1}{(m+1)^s}
$$
\n
$$
\beta(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^s}
$$
\nis an example of the equation

\n
$$
\sum_{\mathbf{n} \in \mathbb{Z}^4 \neq 0} \frac{1}{(\mathbf{n}^2)^s} = 8(1 - 2^{2-2s})\zeta(s)\zeta(s-1)
$$

## As usual, three spatial dimensions is a pain in the..



# Let's consider flatland

### d=3 Effective range theory: In this section we consider the case d = 3 in some detail. This case is particularly interesting interesting i<br>In some details in some detail. This case is particularly interesting interesting in the case of the case is p  $m = 3$  Fffective range theory: is clear that classical parameter and independent of the classical parameter and independent  $\mathsf{d}$  dimensions  $\mathsf{d}$ P2 − include the contract of the contract of

FIG. 2: Feynman diagram that gives the leading contribution to the three-body scattering amplitude.

a non-relativistic conformal invariance (Schrüssen invariance)  $\mathcal{L}(\mathcal{L})$ 

$$
\cot \delta(p) = \frac{1}{\pi} \log \left( \frac{p^2}{\mu^2} \right) - \frac{1}{\alpha_2(\mu)} + \sigma_2 p^2 + \mathcal{O}(p^4)
$$

because of its analogy with renormalizable quantum field theories, and QCD in particular [25,

$$
\alpha_2(\mu) = \frac{MC_0(\mu)}{8} \qquad \sigma_2 = \frac{8C_2(\mu)}{MC_0^2(\mu)}
$$

which may be integrated to give the familiar renormalization group evolution equation

$$
\overline{MS} \qquad \alpha_2(\mu) = \frac{\alpha_2(\nu)}{1 - \frac{2}{\pi} \alpha_2(\nu) \log\left(\frac{\mu}{\nu}\right)}
$$

\$

d2

 $\ddot{\phantom{0}}$ 

|
|
|

2∇ψ†

26]. From our general formula, eq. (10), we find

### effective potential which distances at long distribution does be will see at long distribution of the second second the second seco repulsive case, this pole is not physical. Asymptotically free for attractive case! Asymptotically free for att

function interaction, quantum effects generate an attractive logarithmic contribution to the

Many interesting properties in two spatial dimensions may be traced to scale invariance.

Note that α<sup>2</sup> is a dimensionless coupling, and #|σ2| is the effective range. Neglecting

Many interesting properties in two spatial dimensions may be traced to scale invariance. Earlie only the leading only the East of the East of the East of Tepulsive Case znalou polo for ropuloivo oo , (15), (15 where the field and spatial condition  $\frac{1}{2}$ Landau pole for repulsive case

where the field and spatial coordinates have been rescaled by  $\mathbb{R}^n$   $\mathbb{R}^n$   $\mathbb{R}^n$ 

<del>Ω</del>

2

&

 $e^{2i\pi\left(\frac{1}{2}\right)}$  ,  $e^{2i\pi\left(\frac{1}{2}\right)}$  ,

 $\overline{\phantom{a}}$ 

Now we should relate the scattering amplitude in the EFT, A2(p), whose normalization is

assume that the S-matrix element for isotropic (s-wave) scattering exists in an arbitrary exists in an arbitrary

made about the relative size of the renormalized EFT coefficients.

number of spacetime dimensions. We then have generally

% 1

p2 + is in the internal

%3 − D

## Weirdness of two spatial dimensions:

<sup>α</sup>2(µ) <sup>=</sup> MC0(µ)

range corrections, for α2(µ) of either sign, there is a bound state with binding momentum

function interaction, quantum effects generate an attractive logarithmic contribution to the

effective potential which dominates at long distances. However, as we will see below, in the

Keeping only the leading EFT operator, the leading EFT operator, the Hamiltonian may be written as the Hamiltonian may be written as the Hamiltonian may be written as well as the Hamiltonian may be written as well as the H

Id state for attractive *and* repulsive coupling. Bound state for attractive *and* repulsive coupling:

 $\gamma = \mu \exp(\pi/2\alpha_2(\mu))$ 

epulsive case corresponds to Landau pole! In repulsive case corresponds to Landau pole!

### cutoff of EFT cutoff of EFT

### Here will focus on repulsive case <sup>2</sup><sup>C</sup> <sup>f</sup> C ρ versal properties. Namely, if one denotes the size of the  $T_{\rm eff}$  is f $T_{\rm eff}$ Here will locus on repulsive case

to exhibit universal properties pertaining to large finite

We shall show that the system possesses surprising uni-

systems, which are not in the thermodynamic limit.

 $s_{\rm eff}$  amplitude depicted in Fig. 2. The single (double) in Fig. 2. The single (double) in Fig.

line indicates the boson (full dimeron) propagators, re-

 $s_{\rm B}$  spectrum state amplitude amplitude. While the bound state amplitude amplitude amplitude.

It depends on the total energy E and the total energy  $E$  and the relative moment-

tum of the boson and the dimeron. The dimeron and the dimeron. The three-body bind-body bind-body bind-body bind-

ing energies are given by the to-separate  $\alpha$ 

 $\mathcal{L}_{\mathcal{B}}$  =  $\mathcal{L}_{\mathcal{B}}$  =  $\mathcal{L}_{\mathcal{B}}$  for which the homogeneous integral  $\mathcal{B}$ 

equation shown in Fig. 2 has a nontrivial solution. The

case [11]. There are only bound states if the dimeron

and the third boson are in a relative  $S_{\rm eff}$  wave. The for-wave. The for-wave.

mation of bound states in the higher partial waves in the higher partial waves is  $\mathcal{L}^{\text{max}}$ 

prevented by the angular momentum barrier. Projecting

Equation (17) resembles the Hartree equation for the

Three-body bound state.—We next describe our compu-

ning coupling is used instead of the bare one.

#### Many-boson state with attraction  $\frac{1}{\sqrt{2}}$  shape given by the function  $\frac{1}{\sqrt{2}}$  solving  $\frac{1}{\sqrt{2}}$ ,  $\frac{1}{\sqrt{2}}$ derivation of the integral equation using the Feynman rules given above proceeds as in the three-dimensional a: The integral equation for the three-body and the three-body and the three-body and the three-body and the t<br>Meanwhile and the three-body and t<br> The single (double) line indicates the boson (full dimeron) l

**Hammer and Son, (2004)** The size of the bound state decreases exponentially with N: adding a boson into an existing N-boson droplet re-

$$
B_N = c_1 B_2 c^{N-2}
$$
  $c \approx 8.567$ 

$$
B_{N+1}/B_N\approx 8.567
$$

This implies that the energy required to remove one par-

A = <sup>1</sup>

arXiv:cond-mat/0405206v2 [cond-mat.other] 25 Nov 2004

where  $\alpha$  is the result of the result of

tion (16) now can be written as

The solution can be found numerically by using, e.g.,  $\mu$  using, e.g.,  $\mu$  using,  $\mu$  using,  $\mu$ 

the shooting method. The solution, shown in Fig. 1,

### Note: in CM literature  $N_{\text{left}}$  is  $C_{\text{right}}$  which sets the cutoff scale of the EFT. This state is state is state is state in the EFT. This state is stated in the EFT. This state is stated in the EFT. This state is stated in the EFT. This stat Note: in CM literature  $\frac{113000}{2}$ . (17) that the attractive case,  $\alpha$

Below we will also make use of a more conventional

Below we will also make use of a more conventional

III. SCATTERING IN A CONFINED GEOMETRY

$$
\cot \delta(p) = \frac{1}{\pi} \log (p^2 a_2^2) + \sigma_2 p^2 + \mathcal{O}(p^4)
$$

 $a_2$  is the scattering length or...  $a_2 = ae^{\gamma}/2$ Hence, in the republic case, a−1<br>Hence, a−1 in the republic case, <sup>2</sup> is the momentum cutoff scale. Therefore, from the point  $a<sub>2</sub>$  is the scattering Below we will also make use of a more conventional  $a_2 = a e^{\gamma}/2$ 

course, while the parameter a<sup>2</sup> is expected to be very small as compared to physical scales,

its effect is enhanced as it appears in the argument of the logarithm.

Here a<sup>2</sup> is the scattering length in two spatial dimensions. By matching with eq. 13, one

where  $3.$  Note that the position of the position of the position of the position of the repulsive case coincides  $\alpha$ 

 $\mathcal{A} = \{ \mathcal{A} \mid \mathcal{A} \in \mathcal{A} \}$  , where shifts  $\mathcal{A} = \{ \mathcal{A} \mid \mathcal{A} \in \mathcal{A} \}$  , where shifts  $\mathcal{A} = \{ \mathcal{A} \mid \mathcal{A} \in \mathcal{A} \}$ 

 $\mathcal{L} = \mathcal{L} \left( \mathcal{L} \right)$ 

 $2 \cdot \frac{1}{2}$  is the momentum cutoff scale. The momentum cutoff scale. The point scale  $\mathcal{L}$ 

 $\mathcal{L} = \{ \mathbf{0}, \$ 

with the position of the Landau pole, which sets the cutoff scale of the EFT. This state is

 $4$  parametrization of the phase shift:

of the EFT, and the EFT, an course, which parameter a<br>compared to be very small as compared to physical scales, which is expected to phy its enhanced is enhanced as it appears in the endanglement of the argument of the logarithm. ct!

its effect is enhanced as it appears in the argument of the logarithm.

$$
a_2^{-1} = \mu \exp(\pi/2\alpha_2(\mu))
$$

A. Eigenvalue equation is the position of the Landau pole! of view of the EFT, a2 is a most unsuitable parameter for describing low-energy physics. Of the EFT, a2 is a most unsure physics. Of the EFT, a2 is a most unsure parameter for describing low-energy physics. Of the EFT, a2 is the position of the Landau pole!

therefore unphysical.

#### Finite Area that is a size, L. One readily checks that experience the familiar eigenvalue equations of the fami  $\mathcal{L}_{\mathcal{M}}$  and dimensions up to corrections up to corrections up to corrections that are exponentially suppressed in for d = 2 [28] and d = 2 [28] and d = 4 [28] and d = 4 [10–12] and is in agreement with Ref. [17] for d = 3.<br>The first state of the 3.1 million Ref. [17] for d = 3.1 million Ref. [17] for d = 3.1 million Ref. [17] for d  $\mathbf{r}$  $\frac{1}{2}$  $\Box$  $\frac{1}{2}$  can now computed the eigenvalue equation, eq. (13), with the eigenvalue equation, equation, equation, equation,  $\frac{1}{2}$ <u>Area</u>  $\frac{2}{2}$

 $\gamma$  as L  $\gamma$  as L  $\gamma$  as L  $\gamma$  above is valid within the radius of convergence of

the non-relativistic EFT, this eigenvalue equation is eigenvalue equation is expected to be valid for an arbitrary

In a finite area, the energy levels for the two-particle system follow from the eigenvalue

Using the results derived in Appendix II, this integer sum can be expressed as

and the attractive theory is at strong coupling.

, (25)

. (28)

 $\frac{1}{2}$ 

<sup>1</sup> <sup>−</sup> <sup>η</sup><sup>2</sup>

, and the set of  $\mathcal{L}^2$  , and the set of  $\mathcal{L}^2$  , and the set of  $\mathcal{L}^2$  , and the set of  $\mathcal{L}^2$ 

 $\mathcal{L}(\mathcal{D}) = \mathcal{L}(\mathcal{D})$  . The  $\mathcal{L}(\mathcal{D}) = \mathcal{L}(\mathcal{D})$  and  $\mathcal{L}(\mathcal{D}) = \mathcal{L}(\mathcal{D})$ 

(2% + 1)

#### a mode removedl Using the renormalization group equation, eq. 17, we then have the renormalization, eq. 17, we then have the n<br>In the renormalization, eq. 17, we then have the renormalization, eq. 17, we have the results of the results o zero mode removed! π<sup>2</sup>S<sup>2</sup>  $\overline{a}$  (28)  $\overline{a}$  (2 zero mode removed!

$$
\cot \delta'(p) = \frac{1}{\pi^2} \mathcal{S}_2 \left( \frac{pL}{2\pi} \right)
$$

$$
\cot \delta'(p) \equiv -\frac{1}{\alpha_2} + \sigma_2 p^2 + \mathcal{O}(p^4) \qquad \qquad \mathcal{S}_2(\eta) \equiv \sum_{n=1}^{\Lambda_n} \frac{1}{n^2 - \eta^2} - 2\pi \log \Lambda_n
$$

$$
\alpha_2 \equiv \alpha_2(2\pi/L)
$$

and  $\alpha$   $\alpha$   $\beta$   $\beta$   $\gamma$  and in the eigenvalue equation, the eigenvalue equation, the energy of the energ

cancel, and the scale of the scale of the scale of the coupling is fixed to 2 $\mu$ , the most infrared scale in the EFT 5.

#### $\overline{ }$ Therefore as one approaches the continuum limit, the repulsive theory is at weak coupling and the attractive coupling is we **C. Weak coupling the reduction of the strategy of the strateg** repulsive coupling is weak in the infrared!

 $\mu$ Wednoeder Mari $12, 2010$  $\epsilon$  energy eigenvalues in the energy eigenvalues in the purpose of obtaining  $\epsilon$ Wednesday, May 12, 2010

and the attractive theory is at strong coupling.

eq. (25), to find

<sup>−</sup> <sup>1</sup>

<sup>α</sup>2(µ) <sup>−</sup> <sup>2</sup>

### N bosons in a finite area dimensionality resides entirely in the coupling constant and the geometric constants.

two spatial dimensions is essentially identical, as the combinatoric factors for the ground-compinatoric factor

### $|\alpha_2|$ **N**  $\ll 1$

$$
E_0 = \frac{4 \alpha_2}{ML^2} \left[ \binom{\mathbf{N}}{2} - \left(\frac{\alpha_2}{\pi^2}\right) \binom{\mathbf{N}}{2} \mathcal{P}_2 + \left(\frac{\alpha_2}{\pi^2}\right)^2 \left( \binom{\mathbf{N}}{2} \mathcal{P}_2^2 - \left[ \binom{\mathbf{N}}{2}^2 - 12 \binom{\mathbf{N}}{3} - 6 \binom{\mathbf{N}}{4} \right] \mathcal{P}_4 \right) + \left. \left( \frac{\alpha_2}{\pi^2} \right)^3 \left( - \binom{\mathbf{N}}{2} \mathcal{P}_2^3 + 3 \binom{\mathbf{N}}{2}^2 \mathcal{P}_2 \mathcal{P}_4 - \binom{\mathbf{N}}{2}^3 \mathcal{P}_6 - 24 \binom{\mathbf{N}}{3} \left( \mathcal{P}_2 \mathcal{P}_4 + 2 \mathcal{Q}_0 + \mathcal{R}_0 - \mathcal{P}_6 \binom{\mathbf{N}}{2} \right) \right) - 6 \binom{\mathbf{N}}{4} \left( 3 \mathcal{P}_2 \mathcal{P}_4 + 51 \mathcal{P}_6 - 2 \binom{\mathbf{N}}{2} \mathcal{P}_6 \right) - 300 \binom{\mathbf{N}}{5} \mathcal{P}_6 - 90 \binom{\mathbf{N}}{6} \mathcal{P}_6 \right) + \mathcal{O}(\alpha_2^4) + \frac{16 \alpha_2^3 \sigma_2}{ML^4} \binom{\mathbf{N}}{2}
$$

$$
|\alpha_2| \ln |\alpha_2| \ll 1
$$
 BEC: thermodynamic limit  
\n
$$
\frac{E_0}{N} = \frac{2\alpha'_2 \rho}{M} \left[ 1 + \mathcal{O}(\alpha'_2 \ln \alpha'_2) \right]
$$
 ?? CM I and CM II

Can one take the thermodynamic limit directly? rdvnai mic limit direc **ec**  $\overline{?}$ u lerri logynamichin  $\mathsf{a}$ directly<sup>®</sup>  $\mathsf{y}$  :

P3 (Particularly 1997)

π2

−

 $\alpha_2 (2\pi/L)$  iil defined in thermo limit cancel, and the scale of the scale of the scale of the coupling is fixed to 2 $\mu$ , the most infrared scale in the EFT 5. ill defined in thermo limit  $\alpha_2(2\pi/L)$  $2P_{\rm eff} = 2P_{\rm eff} = 2P_{\$ 

Therefore as one approaches the continuum limit, the repulsive theory is at weak coupling and the attractive the attractive theory is at strong the overlation. So what? Choose new scale using RG evolution  $\frac{1}{2}$ 

$$
\nu = 2\pi \sqrt{\rho \lambda} \qquad \qquad \lambda \quad \text{is scale ambiguity!}
$$

$$
\alpha_2' \equiv \alpha_2(\nu)
$$

coupling expansion of the energy eigenvalues in the energy eigenvalues in the coupling a2. For the purpose of<br>The energy eigenvalues in the purpose of obtaining a2. For the purpose of obtaining a2. For the purpose of obt

strategy is to rearrange the expansion according to the maximum powers of N that appear

<sup>2</sup>. We can then re-write eq. 49 as the energy-per-particle:

 $2 \times 2$  +  $2$  +  $2$  +  $2$  +  $2$  +  $2$  +  $2$  +  $2$  +  $2$  +  $2$  +  $2$ 

Rewrite finite-area energy:  $\mathsf{R}$  awrita finita aras anarov. **REWILLE IIIIIIIE-area energy:**  $\overline{a}$ !α! 2  $\overline{a}$  $\overline{P}$ 

!α!

"4

$$
\frac{E_0}{\mathbf{N}} = \frac{2\alpha_2'}{M} \left( \rho + \frac{1}{L^2} \right) \left[ 1 + \frac{1}{\mathbf{N}} \mathcal{G} + \frac{1}{\mathbf{N}^2} \left( \pi \log \left( \mathbf{N} \lambda \right) \mathcal{H} + \mathcal{I} \right) \right. \\ \left. - \left( \frac{\alpha_2'}{\pi^2} \right) \left( \mathcal{P}_2 + \pi \log \left( \mathbf{N} \lambda \right) \right) \\ \left. + \left( \frac{\alpha_2'}{\pi^2} \right)^2 \left( \mathcal{P}_2^2 - 5\mathcal{P}_4 + \pi \log \left( \mathbf{N} \lambda \right) \left( 2\mathcal{P}_2 + \pi \log \left( \mathbf{N} \lambda \right) \right) \right) + \mathcal{O}(\alpha_2'^3) \right]
$$

−

π2

P2 + π log (Na) + π

 $2$ 

 $427$ 

coefficients of the first two sums are related to the Catalan numbers. We will postpone till

$$
z\,\equiv\,{\bf N}\alpha_2^\prime/\pi^2
$$

$$
\begin{aligned}\n\mathcal{G}(z) &= 2z^2 \mathcal{P}_4 - 5z^3 \mathcal{P}_6 + 14z^4 \mathcal{P}_8 + \mathcal{O}(z^5) \\
\mathcal{H}(z) &= -6z^3 \mathcal{P}_4 + 20z^4 \mathcal{P}_6 - 70z^5 \mathcal{P}_8 + \mathcal{O}(z^6) \\
\mathcal{I}(z) &= -z^3 \Big( 2\mathcal{P}_2 \mathcal{P}_4 - 41 \mathcal{P}_6 + 8(2\mathcal{Q}_0 + \mathcal{R}_0) \Big) \\
&\quad + z^4 \left( 4\mathcal{P}_4 \mathcal{P}_6 - \mathcal{P}_4^2 + 227 \mathcal{P}_8 + \ldots \right) + \mathcal{O}(z^5)\n\end{aligned}
$$

$$
\mathcal{G}\left(z\right) = \sum_{n=2}^{\infty} (-1)^n C(n) z^n \mathcal{P}_{2n}
$$

## The Catalan Numbers



"In how many ways can a regular n-gon be divided into n-2 triangles if different orientations are counted separately?"

$$
C(n-2)
$$

where now α! 2 ≈ Le Jierre de la consecuencia en la component of lineau. Scale invariance constrains the thermo limit.. aing  $\mathbb{R}^2$  $\frac{1}{2}$  th Scale invariance constrains the thermo limit.. **and L are large but finite we then have but finite** we then have been the set of the set of the set of the set of th

2

!

ρ +

z2 H(z) in limited in<br>→ h(z) in limited in l

1

" #

1 +

h + 2P2 +

1

<sup>N</sup> <sup>G</sup> <sup>+</sup>

1

coefficients of the first two sums are related to the Catalan numbers. We will postpone till

later discussion of the evaluation of these sums, in order to focus on obtaining the form of

<sup>+</sup> <sup>O</sup>(α#

2

)

.(56)

\$

 $Z^2$  if  $Z^2$ 

 $\mathcal{L}(\mathcal{A})$  H  $\mathcal{L}(\mathcal{A})$  H  $\mathcal{L}(\mathcal{A})$ 

 $\overline{\phantom{a}}$ 

.(56)

' <sup>+</sup> <sup>O</sup>(z<sup>5</sup>

<sup>N</sup> <sup>=</sup> <sup>2</sup>α!

E<sup>0</sup>

 $G(\mathbb{Z})$   $=$   $G(\mathbb{Z})$ 

$$
\frac{E_0}{\mathbf{N}} = \frac{2\alpha_2'}{M} \left( \rho + \frac{1}{L^2} \right) \left[ 1 + \frac{1}{\mathbf{N}} \mathcal{G} + \frac{1}{\mathbf{N}^2} \left( \pi \log \left( \mathbf{N} \lambda \right) \mathcal{H} + \mathcal{I} \right) \right. \left. - \left( \frac{\alpha_2'}{\pi^2} \right) \left( \mathcal{P}_2 + \pi \log \left( \mathbf{N} \lambda \right) \right) \left. + \left( \frac{\alpha_2'}{\pi^2} \right)^2 \left( \mathcal{P}_2^2 - 5\mathcal{P}_4 + \pi \log \left( \mathbf{N} \lambda \right) \left( 2\mathcal{P}_2 + \pi \log \left( \mathbf{N} \lambda \right) \right) \right) + \mathcal{O}(\alpha_2'^3) \right]
$$

the low-density expansion which is based purely on general considerations. It is clear from

 $z \equiv \mathbf{N}\alpha'_2/\pi^2$  $\frac{2}{2}/\pi^2$ . The mathematically-inclined reader with the that the theorem is no the that the theorem is no the that the theorem is no the theorem is no the theorem i  $z\,\equiv\,{\bf N}\alpha'_2/\pi^2$ 

2 − 5P4 + m log (NA)<br>2 + 5P4 + m log (NA)<br>3 + 5P4 + m log (NA)

 $\begin{array}{ccc} 1 & 1 & 1 \end{array}$  $\frac{1}{z}\mathcal{G}(z) = \mathfrak{g}$  $(2)$ ;  $\lim_{z \to \infty} \frac{1}{z^2} \pi(z) = 0$  $(2)$ ;  $\lim_{z \to \infty} \frac{1}{z^2}$  $\lim_{z\to z_0} \frac{1}{z^2} \mathcal{I}(z)$  $\lim_{z\to\infty}$ 1  $\frac{1}{z}\mathcal{G}(z) \equiv \mathfrak{g}(z) ; \lim_{z\to\infty}$ 1  $\frac{1}{z^2}$  H(z) =  $\mathfrak{h}(z)$  ;  $\lim_{z\to\infty}$ 1  $\frac{1}{z^2} \mathcal{I}(z) \equiv \mathfrak{i}(z)$ d  $\begin{bmatrix} 0 & z \end{bmatrix}$  $\mathcal{G}(z) \;\equiv\; \mathfrak{g}(z) \; ; \; \; \lim_{z \to \infty} \frac{1}{z^2} \, \mathcal{H}(z) \;\; \equiv \; \; \; \;$  $\ddot{=}$  $\phi(z)$ ;  $\lim_{z\to\infty} \frac{1}{z^2} \mathcal{I}(z) \equiv$  $\ddot{i}$  $\big)$ 

$$
\begin{array}{lcl}\n\mathfrak{g}(z) & = & \pi \log z + \bar{g} ; \\
\mathfrak{h}(z) & = & -2\pi \log z + \bar{h} ; \\
\mathfrak{i}(z) & = & \pi^2 \log^2 z - \pi \left( \bar{h} + 2\mathcal{P}_2 \right) \log z + \bar{i}\n\end{array}
$$

where  $g_{\alpha\beta}$  are integration constants. Plugging these functions into eq. 56 we may take functions in the set

3 ):

where g(z), h(z) and i(z) have, at most, logarithmic dependence on z. In the limit that N

i + P2<br>i + P2<br>i + P2<br>i + P2

limz→∞

$$
\frac{E_0}{N} = \frac{2\alpha'_2 \rho}{M} \left[ 1 + \left( \frac{\alpha'_2}{\pi} \right) \left( \log \alpha'_2 - \log \lambda \pi^2 - \frac{1}{\pi} (\mathcal{P}_2 - \bar{g}) \right) \right.\n+ \left( \frac{\alpha'_2}{\pi} \right)^2 \left( \log^2 \alpha'_2 - \left( 2 \log \lambda \pi^2 + \frac{1}{\pi} (2\mathcal{P}_2 + \bar{h}) \right) \log \alpha'_2 \right.\n+ \log \lambda \pi^2 \frac{1}{\pi} (2\mathcal{P}_2 + \bar{h}) + \log^2 \lambda \pi^2 + \frac{1}{\pi^2} (\mathcal{P}_2^2 - 5\mathcal{P}_4 + \bar{i}) \right)\n+ \mathcal{O} \left( \alpha'_2{}^3 \right)
$$

 $2\bar{g} + \bar{h} + \pi = 0$ 

and therefore there is a further relation between the integration constants,

Note that in eq. 61, the energy is completely determined to O(α!

removed by a change of scale. (We will do this explicitly below.)

## $\tau$  that the coefficients of the leading logarithms of the form  $\tau$ To go further must evaluate the sums

the form log N, and as the form log N, and as the form of the density expansion had to be such as to cancel the

divergent terms, it is not surprising that the leading logarithms in the expansion can be

<sup>2</sup>). Indeed we see

<sup>2</sup>) are fixed. The

$$
\mathcal{G}\left(z\right) = \sum_{n=2}^{\infty} (-1)^n C(n) z^n \mathcal{P}_{2n}
$$

 $\mathcal{L} = \mathcal{L} = \mathcal$ 

$$
\mathcal{G}\left(z\right) \,=\, \frac{16}{\pi} \int_{0}^{\infty} \frac{d\omega \,\omega^{2}}{(1+\omega^{2})^{2}} \sum_{n=2}^{\infty} \bar{z}^{n} \xi(n) \beta(n)
$$

(−1)

<sup>G</sup> (z) <sup>=</sup> !<sup>∞</sup>



. (A-11)

nC(n)znP2n , (A-8)  $\sim$  (A-8)  $\sim$  (A-8)  $\sim$  (A-8)  $\sim$  (A-8)  $\sim$  (A-8)  $\sim$  (A-8)  $\sim$ 

#### **z** Can ev valuate at larg large a <u>irgume</u> (2% + 1)  $\mathbb{R}^{\mathbb{Z}}$ Can evaluate at large argument!  $\overline{\phantom{a}}$  $\mathbf{1}$  =  $\mathbf{1}$ ate  $\int$

Using the asymptotic form of the digamma function for large argument as well as the Dirich-

$$
\mathcal{G}\left(z\right) \,=\, \pi z \left(\log z + \frac{1}{2} + \frac{\mathcal{P}_{2}}{\pi}\right) \; + \; 1 \; + \; \mathcal{O}(z^{-1})
$$

es therr  $\gamma$  limi  $\overline{\phantom{a}}$  are  $\overline{z}$  $\cup$ # nite-size corrections!  $G(1)$   $G(2)$   $G(1)$   $G(2)$   $G(3)$   $G(4)$ Gives thermo limit and finite-size corrections!  $\frac{1}{2}$   $\frac{1}{2}$ 

π

And finally, matching to eq. 69,

Using eq. A-1 we can write

forward to find

<sup>β</sup>(0) <sup>=</sup> <sup>1</sup>

forward to find

let sums [41]

let sums [41]

one finds

And finally, matching to eq. 69,

$$
\frac{E_0}{N} = \frac{2\alpha'_2 \rho}{M} \left[ 1 + \left( \frac{\alpha'_2}{\pi} \right) \left( \log \alpha'_2 - \log \lambda \pi^2 + \frac{1}{2} \right) \right.
$$
  
+ 
$$
\left( \frac{\alpha'_2}{\pi} \right)^2 \left( \log^2 \alpha'_2 + 2(1 - \log \lambda \pi^2) \log \alpha'_2 + \log \lambda \pi^2 \left( \log \lambda \pi^2 - 2 \right) - 1 - 2C \right)
$$
  
+ 
$$
\mathcal{O}\left(\alpha'_2{}^3\right) \left[ C \right] = -\frac{1}{2} \left( 1 + \frac{1}{\pi^2} \left( \mathcal{P}_2^2 - 5\mathcal{P}_4 + i \right) \right)
$$

π

log and a set of the s

 $2 - \frac{1}{2}$   $\frac{1}{2}$   $\frac{1}{2}$ 

2

## Energy density has scale ambiguity

 $\mathbb{E} \left[ \mathcal{L} \left( \mathcal{L} \right) \right] = \mathcal{L} \left( \mathcal{L} \right)$  ,  $\mathcal{L} \left( \mathcal{L} \right) = \mathcal{L} \left( \mathcal{L} \right)$ 

 $\mathcal{L} = \mathcal{L} \left( \mathcal{L} \right)$ 

M

1 +

$$
\frac{E_0}{N} = \frac{2\alpha'_2 \rho}{M} \left[ 1 + \left( \frac{\alpha'_2}{\pi} \right) \left( \log \alpha'_2 - \log \lambda \pi^2 + \frac{1}{2} \right) \right.
$$
  
+ 
$$
\left( \frac{\alpha'_2}{\pi} \right)^2 \left( \log^2 \alpha'_2 + 2(1 - \log \lambda \pi^2) \log \alpha'_2 + \log \lambda \pi^2 \left( \log \lambda \pi^2 - 2 \right) - 1 - 2C \right)
$$
  
+ 
$$
\mathcal{O}\left(\alpha'_2{}^3\right) \left[ C \right] = -\frac{1}{2} \left( 1 + \frac{1}{\pi^2} \left( \mathcal{P}_2^2 - 5\mathcal{P}_4 + i \right) \right)
$$

π

log and a set of the s

 $2 - \frac{1}{2}$   $\frac{1}{2}$   $\frac{1}{2}$ 

2

### One of the most serious difficulties preventing precise tests of QCD is the scale ambiguity Energy density has scale ambiguity

Consider QCD process:  $\rho = R_{e^+ e^-}(s) - 3\Sigma e_q^2$  $\text{Consider QCD process}$  =  $R + (s) - 3\Sigma e^2$ The PQC prediction is of the form of the f ider QUD process:  $\rho = n_{e^+e^-}(s) - 3\angle e_q$ 

 $\mathbb{E} \left[ \mathcal{L} \left( \mathcal{L} \right) \right] = \mathcal{L} \left( \mathcal{L} \right)$  ,  $\mathcal{L} \left( \mathcal{L} \right) = \mathcal{L} \left( \mathcal{L} \right)$ 

 $\mathcal{L} = \mathcal{L} \left( \mathcal{L} \right)$ 

M

1 +

$$
\rho = r_0 \alpha_s(\mu) \left[ 1 + r_1(\mu) \frac{\alpha_s(\mu)}{\pi} + r_2(\mu) \frac{\alpha_s^2(\mu)}{\pi^2} + \cdots \right]
$$

. (1.1)

 $< 10\%$  sensitivity to  $\mu$  $3.007$  $s$ uch as MS, and  $p$  is a particular choice of renormalization scale. Since  $p$  is a physical quantity  $p$  $\mu$ 

Here αs(µ) = g<sup>2</sup>

!

The PQCD prediction is of the form

#### Next-to-leading finite-size corrections:  $\frac{1}{2}$  =  $\frac{1}{2}$  =  $\frac{1}{2}$  =  $\frac{1}{2}$ z + O<br>| Online or fine! z<br>Leize correction <sup>+</sup> <sup>O</sup>(z<sup>−</sup><sup>2</sup> In Appendix II we find g<sup>0</sup> = 1 = −h0. Using eq. 48 we see that there are leading and Next-to-leading fil te-size correction  $\overline{\phantom{a}}$

 $I = \{1, \ldots, n\}$  we find  $q_1 = 1$  we find go that there are leading and there are leading

subleading 1/N corrections that arise from g0, h<sup>0</sup> #= 0, and in addition there is a correction

that arises from the binomial coefficient prefactor, as shown explicitly in eq. 48. Taking into

next-to-leading finite-size corrections to the thermodynamic limit. Relaxing the thermody-

As we are able to evaluate the sums G and H explicitly, we are able to give the leading and

next-to-leading finite-size corrections to the thermodynamic limit. Relaxing the thermody-

subleading 1/N corrections that arise from g0, h<sup>0</sup> #= 0, and in addition there is a correction

that arises from the binomial coefficient prefactor, as shown explicitly in eq. 48. Taking into

) . (67) .

) . (67) .

. (68)

) corrections. It

) corrections. It

 $2$  ,  $2$  ,

$$
\delta\left(\frac{E_0}{\mathbf{N}}\right)_{FS} = \frac{4\alpha'_2\rho}{M\mathbf{N}} \left[1 + \left(\frac{\alpha'_2}{2\pi}\right)\left(\log\alpha'_2 - \log\mathbf{N}\lambda^2\pi^2 + \frac{1}{2}\right)\right]
$$

$$
\frac{\delta \mathcal{E}_{FS}}{\mathcal{E}} = \frac{2}{\mathbf{N}} + \mathcal{O}(\alpha_2')
$$

It is straightforward to include the leading non-universal corrections in the energy density.

One easily checks that this expression is independent of λ up to O(α"

One easily checks that this expression is independent of λ up to O(α"

which constitutes a ten-percent effect in a system with N  $=20$  bosons. The 20 bosons. In a system with N  $=20$ 

follows that

follows that

namic limit in eq. 53 gives:

namic limit in eq. 53 gives:

account both of these contributions gives  $\mathcal{O}_\mathcal{A}$  and  $\mathcal{O}_\mathcal{A}$  and  $\mathcal{O}_\mathcal{A}$  and  $\mathcal{O}_\mathcal{A}$  and  $\mathcal{O}_\mathcal{A}$  and  $\mathcal{O}_\mathcal{A}$ 

$$
\alpha_2' = -\frac{\pi}{\log(\rho \lambda (2\pi)^2 a_2^2)}
$$

which is obtained by comparing eq. 13 and eq. 13 and eq. 13 and eq. 18. This is the traditional way of expression<br>This is the traditional way of expressional way of expression and expressional way of expressional way of ex

scale set by the Landau pole. Nevertheless, the energy density can be expressed in terms of

### $\lambda = 1/(2\pi)^2$  rocovars CN1  $\alpha$  and is the physical quantity; and  $\alpha$  physical quantity; any attempt to assign definite meaning to its meaning  $\lambda = 1/(2\pi)^2$  recovers CM I

In this paper we have computed the ground-state energy of N identical bosons which interact

the two-dimensional scattering length via the formula

$$
\frac{E_0}{\mathbf{N}} = \frac{2\pi\rho}{M|\log\rho a_2^2|} \left[ 1 - \frac{1}{|\log\rho a_2^2|} \left( \log|\log\rho a_2^2| - \log 4\pi - \frac{1}{2} \right) \right.\n+ \frac{1}{|\log\rho a_2^2|^2} \left( \log^2|\log\rho a_2^2| - 2(1 + \log 4\pi) \log|\log\rho a_2^2| \right.\n+ \log^2 4 + \log 16(1 + \log \pi) + \log \pi (2 + \log \pi) - 1 - 2C \right) \right]
$$

<sup>2</sup><sup>|</sup> <sup>−</sup> log <sup>4</sup><sup>π</sup> <sup>−</sup> <sup>1</sup>

In this paper we have computed the ground-state energy of N identical bosons which interact

via the most general finite-range potential in a finite area. This energy is expressed as a

 $2$ 

$$
a_2 = ae^{\gamma}/2 \qquad \lambda = e^{-2\gamma}/\pi^2 \qquad \text{recovers CM II}
$$

incorrect. As pointed out above, we find no discrepancy between these two calculations. We believe that

VI. CONCLUSION There is no discrepancy! two scattering length conventions. double perturbative expansion in the two-body interaction strength, which is logarithmically interaction strength, which is logarithmically interaction strength, which is logarithmically interaction strength, which is loga There is no discrepancy! two scattering length conventions!

break scale invariance at the classical level. Effective range corrections and the leading effective range corrections and the level. Effective range corrections and the leading effective range  $\alpha$ 

is futile.

M| log ρa<sup>2</sup>

<sup>E</sup> <sup>=</sup> <sup>2</sup>πρ<sup>2</sup>

λ = e<sup>−</sup>2<sup>γ</sup>/π<sup>2</sup> in eq. 76.

#### $RFCAL$  $\sum_{i=1}^n \sum_{j=1}^n \sum_{j$  $\blacksquare$  $R_{\text{F}} \cap \Delta \mid \cdot$  $\begin{array}{ccc} \text{?} & \text{?} & \text{?} \\ \text{?} & \text{?} & \text{?} \\ \text{?} & \text{?} & \text{?} \end{array}$ RECALL:

[25] D. B. Kaplan, arXiv:nucl-th/0510023; Unpublished Notes.

Controvatory around state energy of 2 d Rees Sonicoversy. Ground State energy of 2 a. [28] M. Luscher and U. Wolff, Nucl. Phys. B 339, 222 (1990). Controversy. Ground state energy of  $\alpha$ [4] I. Bloch, J. Dalibard and W. Zwerger, Rev. Mod. Phys. 80, 885 (2008). Controversy: ground state energy of 2-d Bose gas

G.E. Astrakharchik et al., Phys. Rev. A  $79$ ,  $051602(R)$   $(2009)$ 

C. Mora and Y. Castin, Phys. Rev. Lett.  $102$ ,  $180404$   $(2009)$ .

 $\frac{1}{100}$  erg none of the computation, https://mathworld.wolfram.com/  $\sim$  Soft $\sim$  Superince  $\sim$  $\alpha$ pandy in oad roading conceatorio and oile circums. claim discrepancy in sub-leading corrections and cite error in:

[12] S. R. Beane, P. F. Bedaque, A. Parren˜o and M. J. Savage, Phys. Lett. B 585, 106 (2004).

J.O. Andersen Eur. Phys. J. **B28**, 389 (2002).

### Andersen vindicated! [26] R. Jackiw, in \*Jackiw, R.: Diverse topics in theoretical and mathematical physics\*, 35-53.

## **Conclusion**

- Experiments with ultra-cold atoms provide playground for those interested in non-relativistic quantum mechanics of few-body systems in various dimensions.
- In two spatial dimensions, starting from N weakly interacting particles in a finite area, one can explicitly take the thermodynamic limit and obtain the low-density BEC energy.
- The BEC energy has a scale ambiguity reminiscent of perturbative QCD.
- Finite-size corrections are calculable and can be checked against quantum Monte-Carlo simulations.