Many-body quantum mechanics in flatland

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Outline

- General motivation
- Lattice QCD and many-body physics
- The three-dimensional Bose gas
- The two-dimensional Bose gas
- Conclusion

"...to explore the connections between QCD, cold-atom physics, and few-hadron systems."

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Ultra-cold atoms: At nano-K temperatures, have a non-relativistic few-body system whose inter-particle interaction can be tuned.

"...to explore the connections between QCD, cold-atom physics, and few-hadron systems."

Ultra-cold atoms: At nano-K temperatures, have a non-relativistic few-body system whose inter-particle interaction can be tuned.

It gets better.... consider atoms tightly confined in the z direction:

$$V_H(z) = \frac{1}{4} m\omega_0^2 z^2$$

$$\ell_0 = \sqrt{\frac{\hbar}{m\omega_0}}$$

Can continuously move from 3 to 2 spatial dimensions!

Controversy: ground state energy of 2-d Bose gas

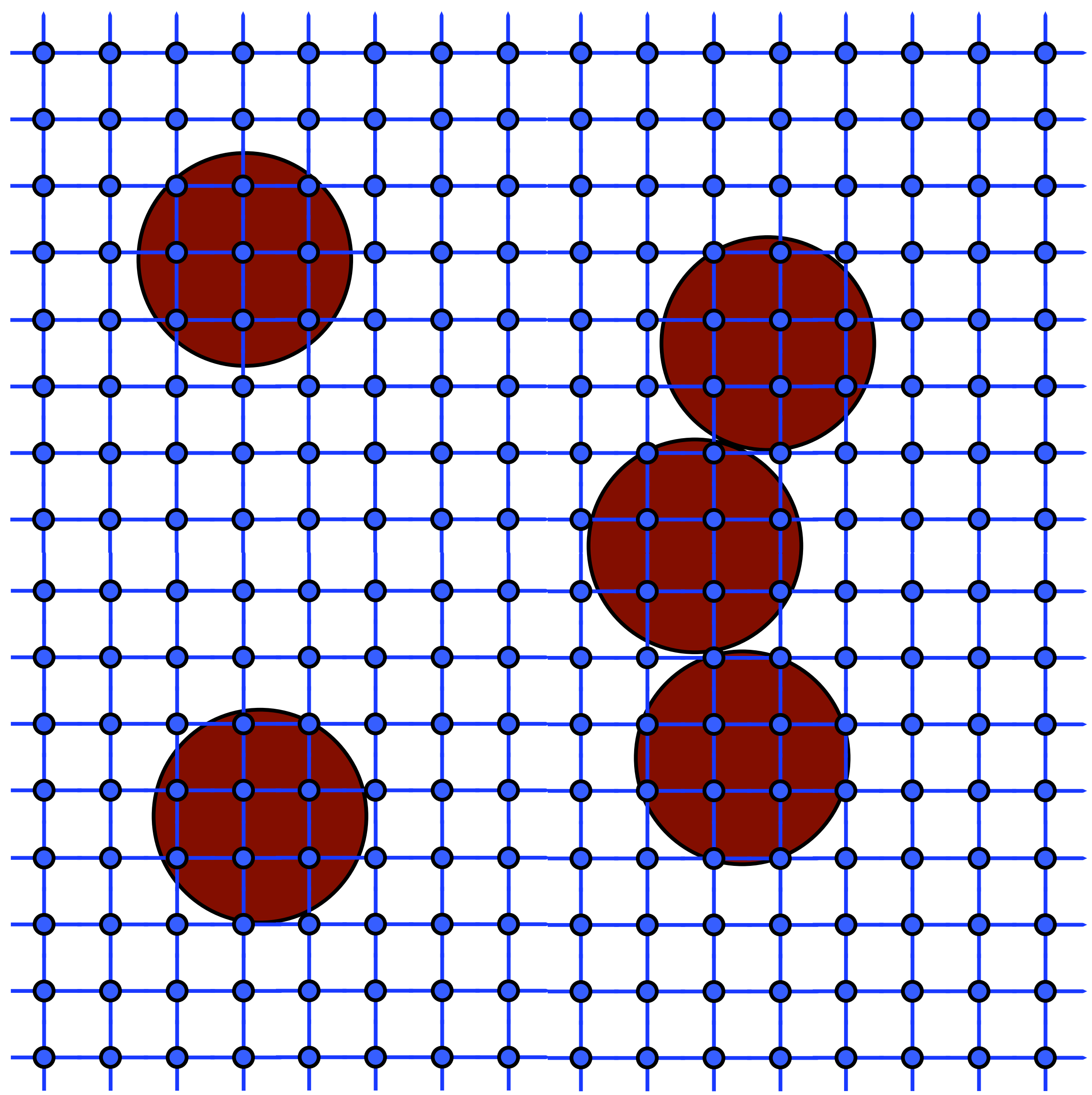
G.E. Astrakharchik *et al.*, Phys. Rev. A **79**, 051602(R) (2009)

C. Mora and Y. Castin, Phys. Rev. Lett. **102**, 180404 (2009)

claim discrepancy in sub-leading corrections and cite error in:

J.O. Andersen Eur. Phys. J. **B28**, 389 (2002)

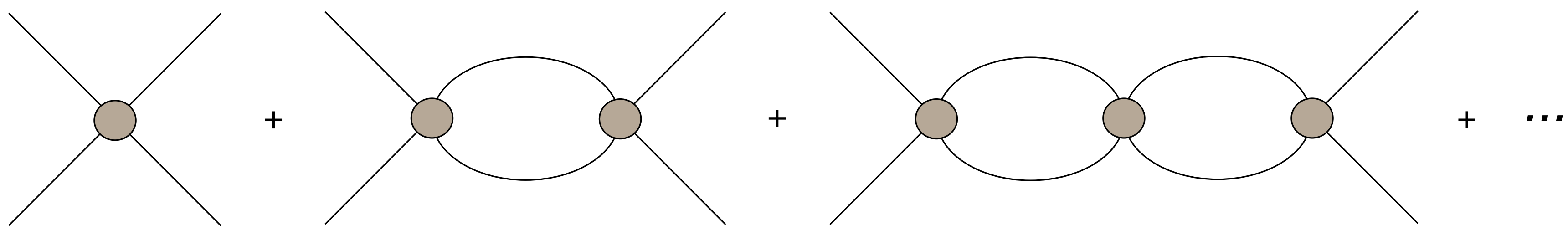
My interest in many-body QM began with Lattice QCD calculations of multi-pion interactions



Consider two boson scattering in QM:

Assume: finite range interaction in d dimensions

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{4} (\psi^\dagger \psi)^2 - \frac{C_2}{8} \nabla(\psi^\dagger \psi) \nabla(\psi^\dagger \psi) + \ldots$$



$$\mathcal{A}_2(p) \;=\; -\frac{\sum C_{2n}\; p^{2n}}{1 - I_0(p) \sum C_{2n}\; p^{2n}} \qquad = \frac{-1}{\mathrm{Im}(I_0(p)) \left[\cot \delta(p) - i \right]}$$

d=4 Effective range theory:

$$p \cot\delta(p) \;=\; -\frac{1}{a_3} \;+\; r_3\, p^2 \;+\; \mathcal{O}(p^4)$$

$$a_3 \;=\; \frac{MC_0}{8\pi} \qquad\qquad r_3 \;=\; \frac{16\pi C_2}{MC_0^2}$$

No running couplings in $\overline{MS}$

$$\mathcal{A}_2(p) = \frac{8\pi}{M}\frac{1}{p\cot\delta(p) - ip}$$

$a_3 \to \infty \qquad r_3 \to 0$ unitarity limit

What about two bosons in a confined geometry?
(e.g. a lattice)

$(d-1)$-dimensional torus

$$q^{d-3}\cot\delta(p) = \Gamma\left(\frac{d-1}{2}\right)\pi^{-\frac{d+1}{2}}\sum_{\mathbf{n}\in\mathbb{Z}^{d-1}}^{\Lambda_n}\frac{1}{\mathbf{n}^2-q^2} + \frac{2\Lambda_n^{d-1}}{\pi(d-1)q^2}\mathrm{Re}\left[{}_2\mathcal{F}_1\left(1,\frac{d-1}{2},\frac{d+1}{2};\frac{\Lambda_n^2}{q^2}\right)\right]$$

gives energy levels: $q \equiv pL/2\pi$

Finite Volume

$$p\cot\delta(p) = \frac{1}{\pi L}\,\mathcal{S}_3\left(\frac{pL}{2\pi}\right) \qquad \mathcal{S}_3\left(\,\eta\,\right) \equiv \sum_{\mathbf{n}}^{\Lambda_n} \frac{1}{\mathbf{n}^2 - \eta^2} - 4\pi\Lambda_n$$

Weak coupling expansion:

$$E_0(2,L) = \frac{4\pi\, a_3}{M\, L^3}\left\{1 - \left(\frac{a_3}{\pi\, L}\right)\mathcal{Q}_2 + \left(\frac{a_3}{\pi\, L}\right)^2\left[\mathcal{Q}_2^2 - \mathcal{Q}_4\right] + \left(\frac{a_3}{\pi\, L}\right)^3\left[-\mathcal{Q}_2^3 + 3\mathcal{Q}_2\mathcal{Q}_4 - \mathcal{Q}_6\right]\right\} + \frac{8\pi^2 a_3^3}{M\, L^6} r_3 + \mathcal{O}\left(L^{-7}\right)$$

$$\mathcal{Q}_2 = \lim_{\Lambda_n\to\infty} \sum_{\mathbf{n}\neq\mathbf{0}}^{|\mathbf{n}|\leq\Lambda_n} \frac{1}{\mathbf{n}^2} - 4\pi\Lambda_n = -8.91363291781 \qquad \mathcal{Q}_4 = \sum_{\mathbf{n}\neq\mathbf{0}} \frac{1}{\mathbf{n}^4} = 16.532315959$$

$$\mathcal{Q}_6 = \sum_{\mathbf{n}\neq\mathbf{0}} \frac{1}{\mathbf{n}^6} = 8.401923974433$$

What about N bosons in a confined geometry?

Rayleigh-Schrodinger PT:

$$\hat{H} = \hat{H}_0 + \lambda \hat{V} \qquad\qquad E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + ...$$

$$\begin{aligned}
E_n^{(1)} &= \langle n^{(0)}|\hat{V}|n^{(0)}\rangle = V_{nn} \\
E_n^{(2)} &= \sum_{k\neq n} \frac{|\langle n^{(0)}|\hat{V}|k^{(0)}\rangle|^2}{E_n^{(0)} - E_k^{(0)}} = \sum_{k\neq n} \frac{|V_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} \\
E_n^{(3)} &= \sum_{k\neq n}\sum_{p\neq n} \frac{V_{np}V_{pk}V_{kn}}{(E_n^{(0)} - E_k^{(0)})(E_n^{(0)} - E_p^{(0)})} - V_{nn}\sum_{k\neq n} \frac{|V_{nk}|^2}{(E_n^{(0)} - E_k^{(0)})^2} \\
E_n^{(4)} &= \sum_{k\neq n}\sum_{p\neq n}\sum_{s\neq n} \frac{V_{np}V_{ps}V_{sk}V_{kn}}{(E_n^{(0)} - E_k^{(0)})(E_n^{(0)} - E_p^{(0)})(E_n^{(0)} - E_s^{(0)})} \\
&- V_{nn}\sum_{k\neq n}\sum_{p\neq n} \frac{V_{np}V_{pk}V_{kn}}{(E_n^{(0)} - E_k^{(0)})^2(E_n^{(0)} - E_p^{(0)})} \\
&- E_n^{(1)}\left(\sum_{k\neq n}\sum_{p\neq n} \frac{V_{np}V_{pk}V_{kn}}{(E_n^{(0)} - E_k^{(0)})(E_n^{(0)} - E_p^{(0)})^2} - V_{nn}\sum_{k\neq n} \frac{|V_{nk}|^2}{(E_n^{(0)} - E_k^{(0)})^3}\right) \\
&- E_n^{(2)}\sum_{k\neq n} \frac{|V_{nk}|^2}{(E_n^{(0)} - E_k^{(0)})^2} \quad ,
\end{aligned}$$

N bosons in a finite volume

Detmold et al, (2007) Tan, (2007)

Two low-density regimes:

$$L \gg \mathbf{N} a_3$$

$$E_0(\mathbf{N},L) = \frac{4\pi\, a_3}{M\, L^3}\left\{\binom{\mathbf{N}}{2} - \left(\frac{a_3}{\pi\, L}\right)\binom{\mathbf{N}}{2}\mathcal{Q}_2 + \left(\frac{a_3}{\pi\, L}\right)^2\left\{\binom{\mathbf{N}}{2}\mathcal{Q}_2^2 - \left[\binom{\mathbf{N}}{2}^2 - 12\binom{\mathbf{N}}{3} - 6\binom{\mathbf{N}}{4}\right]\mathcal{Q}_4\right\}\right.$$
$$+\left(\frac{a_3}{\pi\, L}\right)^3\left[-\binom{\mathbf{N}}{2}\mathcal{Q}_2^3 + 3\binom{\mathbf{N}}{2}^2\mathcal{Q}_2\mathcal{Q}_4 - \binom{\mathbf{N}}{2}^3\mathcal{Q}_6 - 24\binom{\mathbf{N}}{3}\left(\mathcal{Q}_2\mathcal{Q}_4 + 2\mathcal{Q} + \mathcal{R} - \mathcal{Q}_6\binom{\mathbf{N}}{2}\right)\right.$$
$$\left.\left. -6\binom{\mathbf{N}}{4}\left(3\mathcal{Q}_2\mathcal{Q}_4 + 51\mathcal{Q}_6 - 2\binom{\mathbf{N}}{2}\mathcal{Q}_6\right) - 300\binom{\mathbf{N}}{5}\mathcal{Q}_6 - 90\binom{\mathbf{N}}{6}\mathcal{Q}_6\right]\right\}$$
$$+\binom{\mathbf{N}}{3}\frac{64\pi a_3^4}{M\, L^6}(3\sqrt{3} - 4\pi)\log(\mu\ L) + \binom{\mathbf{N}}{2}\frac{8\pi^2 a_3^3}{M\, L^6}r_3 + \binom{\mathbf{N}}{3}\frac{\eta(\mu)}{L^6} + \mathcal{O}\left(L^{-7}\right)$$

$$\mathbf{N}^{\frac{1}{3}} a_3 \ll L \ll \mathbf{N} a_3$$

BEC: thermodynamic limit

$$\frac{E_0}{\mathbf{N}} = \frac{2\pi\rho a_3}{M}\left(1 + \frac{128}{15\sqrt{\pi}}(\rho a_3^3)^{1/2} + \frac{8}{3}(4\pi - 3\sqrt{3})\rho a_3^3 \ln(\rho a_3^3) + \rho a_3^3\eta' + \ldots\right) + \ldots$$

How does one calculate in BEC regime?

Braaten and Nieto, (1999)

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2M} + \mu \right) \psi - \frac{C_0}{4}(\psi^\dagger\psi)^2 - \frac{C_2}{8}\nabla(\psi^\dagger\psi)\nabla(\psi^\dagger\psi) - \frac{D_0}{36}(\psi^\dagger\psi)^3 \; + \; \ldots$$

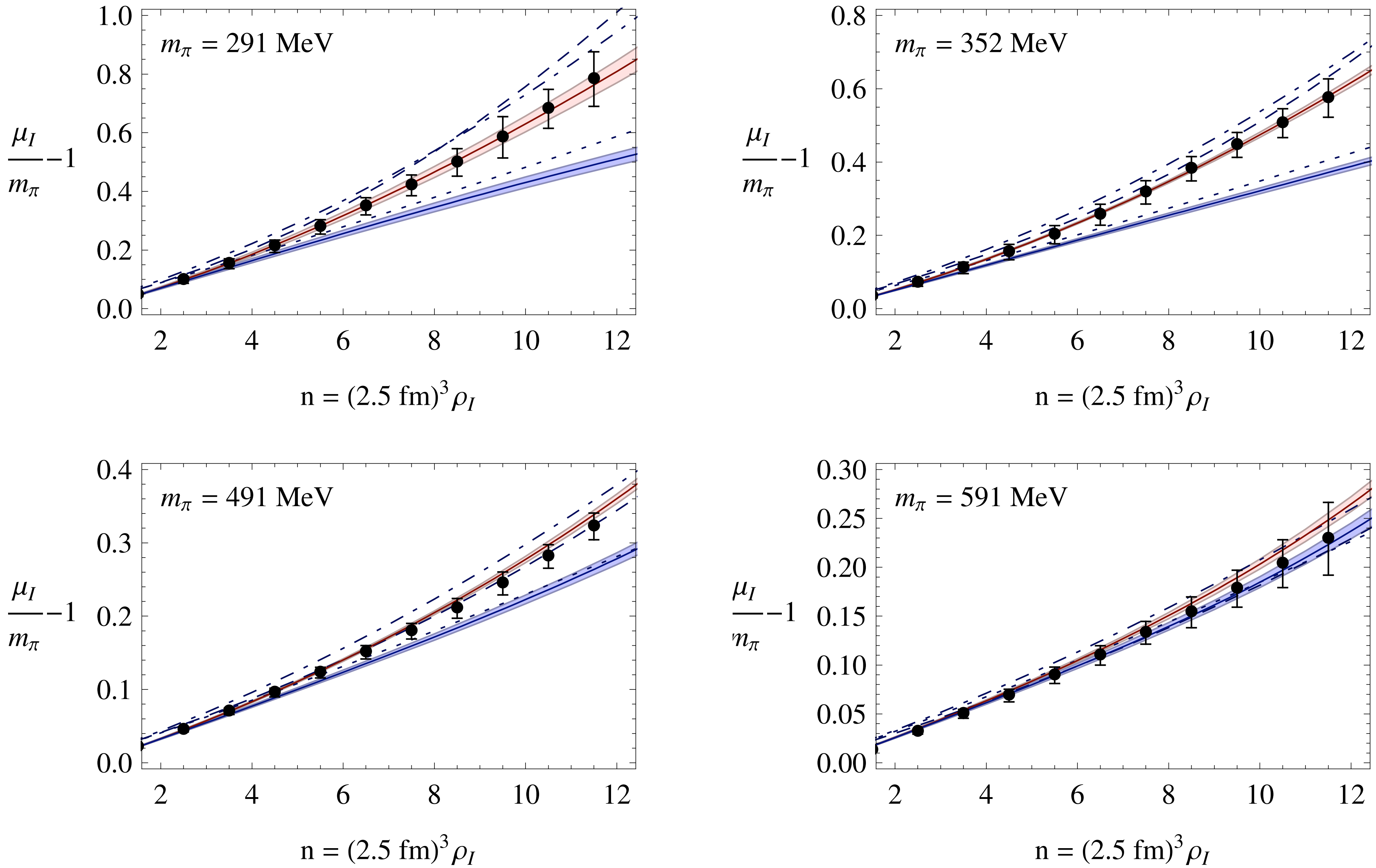
$$\rho(\mu) = < \psi^\dagger\psi >_\mu$$

quantum fluctuations around a mean field

$$\psi(\mathbf{r},t) = v + \frac{\xi(\mathbf{r},t) + \mathrm{i}\eta(\mathbf{r},t)}{\sqrt{2}}$$

mean field theory: $\rho_0 = v^2$

Example: multi-pion interactions from Lattice QCD:



$$\chi\text{–PT}: \qquad \rho_I = \frac{1}{2} f_\pi^2 \mu_I \left(1 - \frac{m_\pi^4}{\mu_I^4}\right)$$

Can one take the thermodynamic limit directly?

(How does the Bose gas lose knowledge of its container?)

$$\mathbf{N} \to \infty \qquad V \to \infty \qquad \rho \equiv \frac{\mathbf{N}}{V} \ \textit{fixed}$$

leading order is trivial

corrections look mysterious...

Seems to require understanding of:

$$\mathcal{Q}_{2s} \equiv \sum_{\mathbf{n}\in\mathbb{Z}^3 \neq 0}^{\infty} \frac{1}{(\mathbf{n}^2)^s}$$

Carl Gustav Jacob Jacobi

Relations among elliptic integrals gives:

$$\sum_{\mathbf{n}\in\mathbb{Z}^2\neq 0}^{\infty} \frac{1}{(\mathbf{n}^2)^s} = 4\zeta(s)\beta(s)$$

$$\zeta(s) \equiv \sum_{m=0}^{\infty} \frac{1}{(m+1)^s}$$

Riemann zeta function

$$\beta(s) \equiv \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^s}$$

Dirichlet beta function

$$\sum_{\mathbf{n}\in\mathbb{Z}^4\neq 0}^{\infty} \frac{1}{(\mathbf{n}^2)^s} = 8\left(1-2^{2-2s}\right)\zeta(s)\zeta(s-1)$$

As usual, three spatial dimensions is a pain in the..

Let’s consider flatland

d=3 Effective range theory:

$$\cot\delta(p) \;=\; \frac{1}{\pi}\log\left(\frac{p^2}{\mu^2}\right) \;-\; \frac{1}{\alpha_2(\mu)} \;+\; \sigma_2\, p^2 \;+\; \mathcal{O}(p^4)$$

$$\alpha_2(\mu) \;=\; \frac{MC_0(\mu)}{8} \qquad\qquad \sigma_2 \;=\; \frac{8C_2(\mu)}{MC_0^2(\mu)}$$

$\overline{MS}$

$$\alpha_2(\mu) \;=\; \frac{\alpha_2(\nu)}{1 - \frac{2}{\pi}\alpha_2(\nu)\log\left(\frac{\mu}{\nu}\right)}$$

Asymptotically free for attractive case!

Landau pole for repulsive case

Weirdness of two spatial dimensions:

Bound state for attractive *and* repulsive coupling:

$$\gamma \;=\; \mu \exp(\pi/2\alpha_2(\mu))$$

In repulsive case corresponds to Landau pole!

cutoff of EFT

Here will focus on repulsive case

Many-boson state with attraction

Hammer and Son, (2004)

$$B_N = c_1 B_2 c^{N-2}$$

$$c \approx 8.567$$

$$B_{N+1}/B_N \approx 8.567$$

Note: in CM literature

$$\cot\delta(p) \;=\; \frac{1}{\pi}\log\left(p^2 a_2^2\right) \;+\; \sigma_2\, p^2 \;+\; \mathcal{O}(p^4)$$

a_2 is the scattering length or.. $a_2 = a e^{\gamma}/2$

Horrible object!

$$a_2^{-1} = \mu \exp(\pi/2\alpha_2(\mu))$$

is the position of the Landau pole!

Finite Area

zero mode removed!

$$\cot\delta'(p) = \frac{1}{\pi^2}\mathcal{S}_2\left(\frac{pL}{2\pi}\right)$$

$$\cot\delta'(p) \equiv -\frac{1}{\alpha_2} + \sigma_2\, p^2 + \mathcal{O}(p^4) \qquad \mathcal{S}_2(\eta) \equiv \sum_{\mathbf{n}}^{\Lambda_n} \frac{1}{\mathbf{n}^2 - \eta^2} - 2\pi\log\Lambda_n$$

$$\alpha_2 \equiv \alpha_2(2\pi/L)$$

repulsive coupling is weak in the infrared!

N bosons in a finite area

$$|\alpha_2|\mathbf{N} \ll 1$$

$$E_0 = \frac{4\,\alpha_2}{ML^2}\Bigg[\binom{\mathbf{N}}{2} - \left(\frac{\alpha_2}{\pi^2}\right)\binom{\mathbf{N}}{2}\mathcal{P}_2 + \left(\frac{\alpha_2}{\pi^2}\right)^2\left(\binom{\mathbf{N}}{2}\mathcal{P}_2^2 - \left[\binom{\mathbf{N}}{2}^2 - 12\binom{\mathbf{N}}{3} - 6\binom{\mathbf{N}}{4}\right]\mathcal{P}_4\right)$$
$$+ \left(\frac{\alpha_2}{\pi^2}\right)^3\Bigg(-\binom{\mathbf{N}}{2}\mathcal{P}_2^3 + 3\binom{\mathbf{N}}{2}^2\mathcal{P}_2\mathcal{P}_4 - \binom{\mathbf{N}}{2}^3\mathcal{P}_6 - 24\binom{\mathbf{N}}{3}\left(\mathcal{P}_2\mathcal{P}_4 + 2\mathcal{Q}_0 + \mathcal{R}_0 - \mathcal{P}_6\binom{\mathbf{N}}{2}\right)$$
$$-6\binom{\mathbf{N}}{4}\left(3\mathcal{P}_2\mathcal{P}_4 + 51\mathcal{P}_6 - 2\binom{\mathbf{N}}{2}\mathcal{P}_6\right) - 300\binom{\mathbf{N}}{5}\mathcal{P}_6 - 90\binom{\mathbf{N}}{6}\mathcal{P}_6\Bigg) + \mathcal{O}(\alpha_2^4)\Bigg]$$
$$+ \frac{16\,\alpha_2^3\,\sigma_2}{ML^4}\binom{\mathbf{N}}{2}$$

$$|\alpha_2|\ln|\alpha_2| \ll 1$$

BEC: thermodynamic limit

Schick, (1971)

$$\frac{E_0}{\mathbf{N}} = \frac{2\alpha_2'\rho}{M}\left[1 + \mathcal{O}(\alpha_2'\ln\alpha_2') \qquad ?? \qquad \right]$$

CM I and CM II

Can one take the thermodynamic limit directly?

$$\alpha_2(2\pi/L)$$

ill defined in thermo limit

So what? Choose new scale using RG evolution

$$\nu = 2\pi\sqrt{\rho\lambda}$$

λ is scale ambiguity!

$$\alpha'_2 \equiv \alpha_2(\nu)$$

Rewrite finite-area energy:

$$\frac{E_0}{\mathbf{N}} = \frac{2\alpha_2'}{M}\left(\rho + \frac{1}{L^2}\right)\Bigg[1 + \frac{1}{\mathbf{N}}\,\mathcal{G} + \frac{1}{\mathbf{N}^2}\Big(\pi\log(\mathbf{N}\lambda)\,\mathcal{H} + \mathcal{I}\Big)$$

$$- \left(\frac{\alpha_2'}{\pi^2}\right)\Big(\mathcal{P}_2 + \pi\log(\mathbf{N}\lambda)\Big)$$

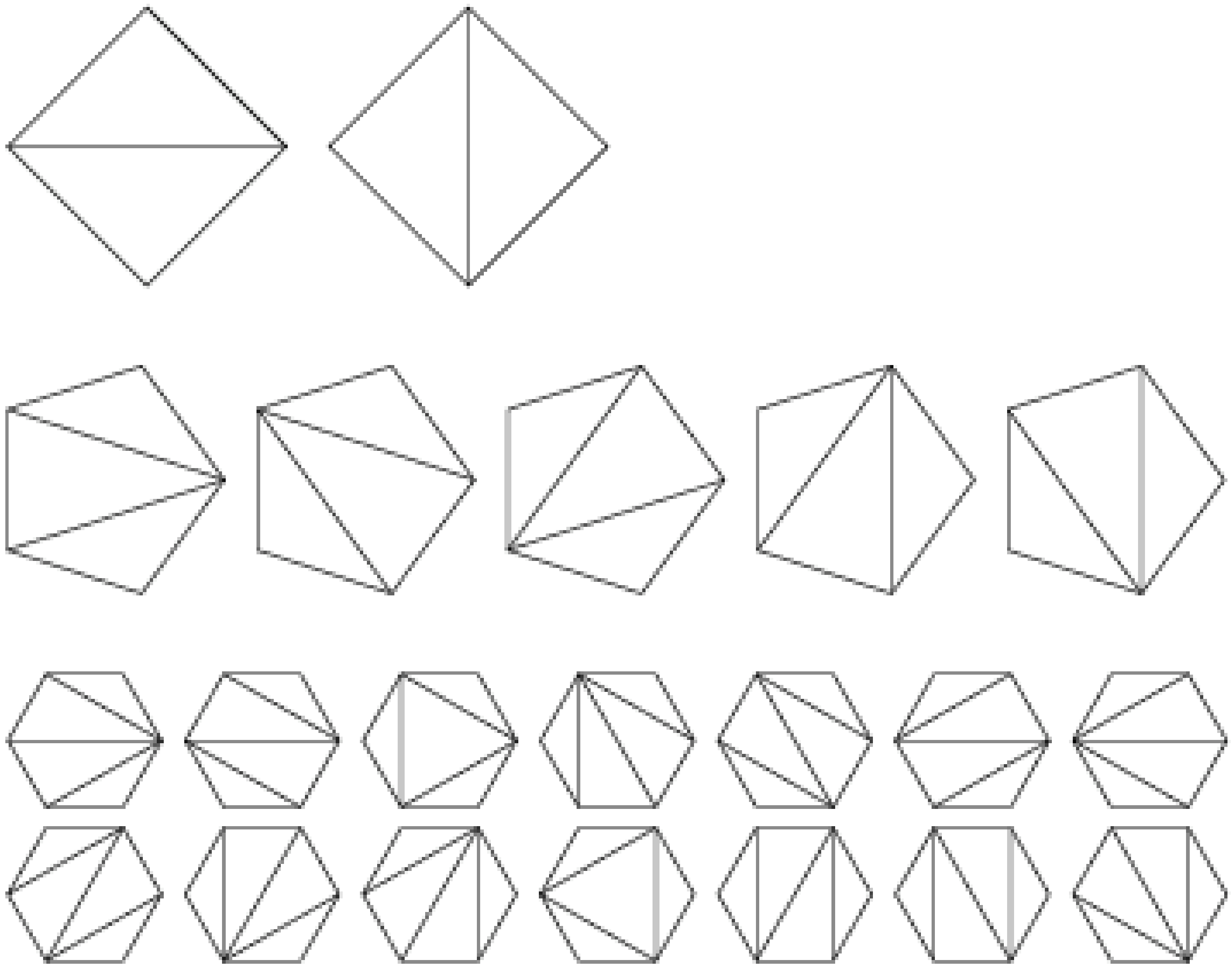
$$+ \left(\frac{\alpha_2'}{\pi^2}\right)^2\Big(\mathcal{P}_2^2 - 5\mathcal{P}_4 + \pi\log(\mathbf{N}\lambda)\Big(2\mathcal{P}_2 + \pi\log(\mathbf{N}\lambda)\Big)\Big) + \mathcal{O}({\alpha_2'}^3)\Bigg]$$

$$z \equiv \mathbf{N}\alpha_2'/\pi^2$$

$$\begin{aligned}
\mathcal{G}(z) &= 2\,z^2\,\mathcal{P}_4 - 5\,z^3\,\mathcal{P}_6 + 14\,z^4\,\mathcal{P}_8 + \mathcal{O}(z^5)\\
\mathcal{H}(z) &= -6\,z^3\,\mathcal{P}_4 + 20\,z^4\,\mathcal{P}_6 - 70\,z^5\,\mathcal{P}_8 + \mathcal{O}(z^6)\\
\mathcal{I}(z) &= -z^3\Big(2\mathcal{P}_2\mathcal{P}_4 - 41\mathcal{P}_6 + 8(2\mathcal{Q}_0 + \mathcal{R}_0)\Big)\\
&\quad + z^4\left(4\mathcal{P}_4\mathcal{P}_6 - \mathcal{P}_4^2 + 227\mathcal{P}_8 + \ldots\right) + \mathcal{O}(z^5)
\end{aligned}$$

$$\mathcal{G}(z) = \sum_{n=2}^{\infty}(-1)^n C(n)\, z^n \mathcal{P}_{2n}$$

The Catalan Numbers



"In how many ways can a regular n-gon be divided into n-2 triangles if different orientations are counted separately?"

$$C(n-2)$$

Scale invariance constrains the thermo limit..

$$\frac{E_0}{\mathbf{N}} = \frac{2\alpha_2'}{M}\left(\rho + \frac{1}{L^2}\right)\Bigg[1 + \frac{1}{\mathbf{N}}\,\mathcal{G} + \frac{1}{\mathbf{N}^2}\left(\pi\log(\mathbf{N}\lambda)\,\mathcal{H} + \mathcal{I}\right)$$
$$- \left(\frac{\alpha_2'}{\pi^2}\right)\left(\mathcal{P}_2 + \pi\log(\mathbf{N}\lambda)\right)$$
$$+ \left(\frac{\alpha_2'}{\pi^2}\right)^2\left(\mathcal{P}_2^2 - 5\mathcal{P}_4 + \pi\log(\mathbf{N}\lambda)\left(2\mathcal{P}_2 + \pi\log(\mathbf{N}\lambda)\right)\right) + \mathcal{O}({\alpha_2'}^3)\Bigg]$$

$$z \equiv \mathbf{N}\alpha_2'/\pi^2$$

$$\lim_{z\to\infty}\frac{1}{z}\,\mathcal{G}(z) \equiv \mathfrak{g}(z)\;;\quad \lim_{z\to\infty}\frac{1}{z^2}\,\mathcal{H}(z) \equiv \mathfrak{h}(z)\;;\quad \lim_{z\to\infty}\frac{1}{z^2}\,\mathcal{I}(z) \equiv \mathfrak{i}(z)$$

$$\begin{aligned}\mathfrak{g}(z) &= \pi\log z + \bar{g}\;;\\ \mathfrak{h}(z) &= -2\pi\log z + \bar{h}\;;\\ \mathfrak{i}(z) &= \pi^2\log^2 z - \pi\left(\bar{h} + 2\mathcal{P}_2\right)\log z + \bar{i}\end{aligned}$$

$$
\begin{aligned}
\frac{E_0}{\mathbf{N}} \;=\; \frac{2\alpha_2'\rho}{M}\Bigg[\, 1 \;+\; \left(\frac{\alpha_2'}{\pi}\right)\left(\log\alpha_2' \;-\; \log\lambda\pi^2 \;-\; \tfrac{1}{\pi}\left(\mathcal{P}_2 \;-\; \bar{g}\right)\right) \\
+ \left(\frac{\alpha_2'}{\pi}\right)^2\Big(\log^2\alpha_2' \;-\; \left(2\log\lambda\pi^2 \;+\; \tfrac{1}{\pi}\left(2\mathcal{P}_2 \;+\; \bar{h}\right)\right)\log\alpha_2' \\
+ \log\lambda\pi^2\,\tfrac{1}{\pi}\left(2\mathcal{P}_2 \;+\; \bar{h}\right) \;+\; \log^2\lambda\pi^2 \;+\; \tfrac{1}{\pi^2}\left(\mathcal{P}_2^2 \;-\; 5\mathcal{P}_4 \;+\; \bar{i}\right)\Big) \\
+ \mathcal{O}\left({\alpha_2'}^3\right)\Bigg]
\end{aligned}
$$

$$
2\bar{g} + \bar{h} + \pi = 0
$$

To go further must evaluate the sums

$$\mathcal{G}(z) = \sum_{n=2}^{\infty} (-1)^n C(n) z^n \mathcal{P}_{2n}$$

$$\mathcal{G}(z) \;=\; \frac{16}{\pi} \int_0^{\infty} \frac{d\omega\, \omega^2}{(1+\omega^2)^2} \sum_{n=2}^{\infty} \bar{z}^n \xi(n) \beta(n)$$

Can evaluate at large argument!

$$\mathcal{G}(z) \;=\; \pi z \left(\log z + \frac{1}{2} + \frac{\mathcal{P}_2}{\pi} \right) \;+\; 1 \;+\; \mathcal{O}(z^{-1})$$

Gives thermo limit and finite-size corrections!

$$\frac{E_0}{\mathbf{N}} = \frac{2\alpha_2'\rho}{M}\left[1 + \left(\frac{\alpha_2'}{\pi}\right)\left(\log\alpha_2' - \log\lambda\pi^2 + \tfrac{1}{2}\right)\right.$$

$$+ \left(\frac{\alpha_2'}{\pi}\right)^2\left(\log^2\alpha_2' + 2(1 - \log\lambda\pi^2)\log\alpha_2' + \log\lambda\pi^2\left(\log\lambda\pi^2 - 2\right) - 1 - 2C\right)$$

$$\left. + \mathcal{O}\left({\alpha_2'}^3\right)\right]$$

$$C \equiv -\frac{1}{2}\left(1 + \frac{1}{\pi^2}\left(\mathcal{P}_2^2 - 5\mathcal{P}_4 + \bar{i}\right)\right)$$

Energy density has scale ambiguity

$$\frac{E_0}{\mathbf{N}} = \frac{2\alpha_2'\rho}{M}\Bigg[1 + \left(\frac{\alpha_2'}{\pi}\right)\left(\log\alpha_2' - \log\lambda\pi^2 + \tfrac{1}{2}\right)$$

$$+ \left(\frac{\alpha_2'}{\pi}\right)^2 \left(\log^2\alpha_2' + 2(1 - \log\lambda\pi^2)\log\alpha_2' + \log\lambda\pi^2\left(\log\lambda\pi^2 - 2\right) - 1 - 2C\right)$$

$$+ \mathcal{O}\left({\alpha_2'}^3\right)\Bigg]$$

$$C \equiv -\frac{1}{2}\left(1 + \frac{1}{\pi^2}\left(\mathcal{P}_2^2 - 5\mathcal{P}_4 + \bar{i}\right)\right)$$

Energy density has scale ambiguity

Consider QCD process: $\rho = R_{e^+e^-}(s) - 3\Sigma e_q^2$

$$\rho = r_0\alpha_s(\mu)\left[1 + r_1(\mu)\,\frac{\alpha_s(\mu)}{\pi} + r_2(\mu)\,\frac{\alpha_s^2(\mu)}{\pi^2} + \cdots\right]$$

$< 10\%$ sensitivity to μ

Next-to-leading finite-size corrections:

$$\delta\left(\frac{E_0}{\mathbf{N}}\right)_{FS} = \frac{4\alpha_2'\rho}{M\mathbf{N}}\left[1 + \left(\frac{\alpha_2'}{2\pi}\right)\left(\log\alpha_2' - \log\mathbf{N}\lambda^2\pi^2 + \tfrac{1}{2}\right)\right]$$

$$\frac{\delta\mathcal{E}_{FS}}{\mathcal{E}} = \frac{2}{\mathbf{N}} + \mathcal{O}(\alpha_2')$$

$$\alpha_2' = -\frac{\pi}{\log\left(\rho\lambda(2\pi)^2 a_2^2\right)}$$

$\lambda = 1/(2\pi)^2$ recovers CM I

$$\frac{E_0}{\mathbf{N}} = \frac{2\pi\rho}{M|\log\rho a_2^2|}\Bigg[1 - \frac{1}{|\log\rho a_2^2|}\Big(\log|\log\rho a_2^2| - \log 4\pi - \tfrac{1}{2}\Big) + \frac{1}{|\log\rho a_2^2|^2}\Big(\log^2|\log\rho a_2^2| - 2(1 + \log 4\pi)\log|\log\rho a_2^2| + \log^2 4 + \log 16(1+\log\pi) + \log\pi(2+\log\pi) - 1 - 2C\Big)\Bigg]$$

$a_2 = ae^{\gamma}/2$ $\lambda = e^{-2\gamma}/\pi^2$ recovers CM II

There is no discrepancy! two scattering length conventions!

RECALL:

Controversy: ground state energy of 2-d Bose gas

G.E. Astrakharchik *et al.*, Phys. Rev. A **79**, 051602(R) (2009)

C. Mora and Y. Castin, Phys. Rev. Lett. **102**, 180404 (2009)

claim discrepancy in sub-leading corrections and cite error in:

J.O. Andersen Eur. Phys. J. **B28**, 389 (2002)

Andersen vindicated!

Conclusion

- Experiments with ultra-cold atoms provide playground for those interested in non-relativistic quantum mechanics of few-body systems in various dimensions.

- In two spatial dimensions, starting from N weakly interacting particles in a finite area, one can explicitly take the thermodynamic limit and obtain the low-density BEC energy.

- The BEC energy has a scale ambiguity reminiscent of perturbative QCD.

- Finite-size corrections are calculable and can be checked against quantum Monte-Carlo simulations.