Criticality in the Interacting Boson Model with configuration mixing

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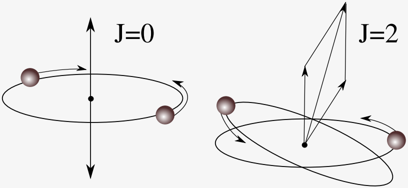
Outline

1. The interacting boson model (IBM)
2. U(5)-QQ mixing
3. QQ-QQ mixing
4. Conclusions and outlook

The interacting boson model

Basic building blocks



Nucleon pairs coupled to $J = 0$ and $J = 2$ are approximately treated as s and d bosons. These are the basic building blocks of the IBM.

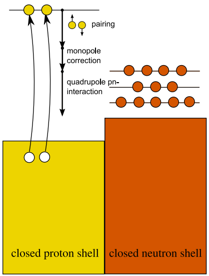
The Hamiltonian is 'generated' by group theory

$$\hat{H}_{\text{cqf}} = \varepsilon \hat{n}_d + \kappa \hat{Q}(\chi) \cdot \hat{Q}(\chi)$$

Three symmetry limits in which the Hamiltonian is analytically solvable

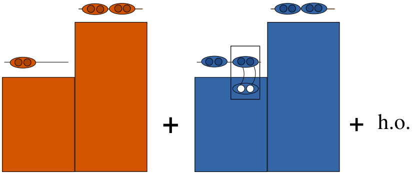
- U(5)-limit : $\kappa = 0$
- SU(3)-limit : $\varepsilon = 0$ and $\chi = \pm\sqrt{7}/2$
- O(6)-limit : $\varepsilon = 0$ and $\chi = 0$

The interacting boson model with configuration mixing



If particle-hole configurations out of a closed shell are sufficiently lowered in energy, configuration mixing occurs.

K. Heyde et al., Nucl. Phys. A466, 189 (1987)



The IBM Hamiltonian can be extended to include particle-hole configurations

$$\hat{H} = \hat{P}_N^\dagger \hat{H}_{\text{cqf}}^N \hat{P}_N + \hat{P}_{N+2}^\dagger \left(\hat{H}_{\text{cqf}}^{N+2} + \Delta^{N+2} \right) \hat{P}_{N+2} + \hat{V}_{\text{mix}}^{N,N+2} + \text{h.o.}$$

Δ^{N+2} takes corrected excitation energy of 2p-2h configuration into account

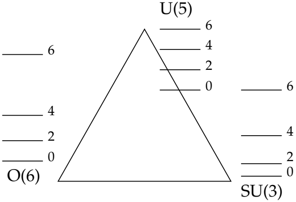
Geometry of the IBM

A single configuration

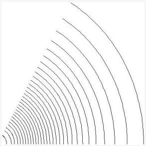
J. N. Ginocchio *et al.*, Phys. Rev. Lett. **44**, 1744 (1980)

IBM $\xrightarrow[\text{in intrinsic state}]{\text{calculate expectation value}}$ energy surface in β and γ

$$E^N(\beta,\gamma;\varepsilon,\kappa,\chi) = \varepsilon N \frac{\beta^2}{1+\beta^2} + \kappa \left[\frac{N\left[5+(1+\chi^2)\beta^2\right]}{1+\beta^2} + \frac{N(N-1)}{(1+\beta^2)^2}\left(\frac{2}{7}\chi^2\beta^4 - 4\sqrt{\frac{2}{7}}\chi\beta^3\cos(3\gamma) + 4\beta^2 \right) \right]$$



U(5): vibrational



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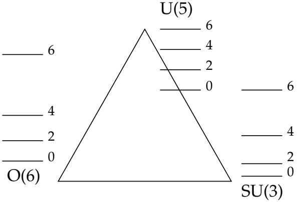
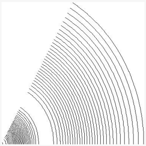


O(6): γ-unstable rotor



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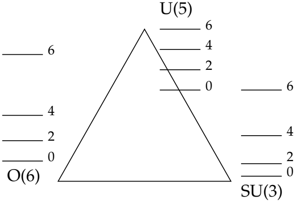
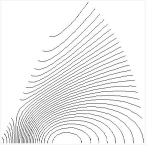


SU(3): prolate rotor



Geometry of the IBM

A single configuration

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Mixing between two configurations

A. Frank *et al.*, Phys. Rev. C **69**, 034323 (2004)

IBM with configuration mixing $\longrightarrow$ energy surface in β and γ

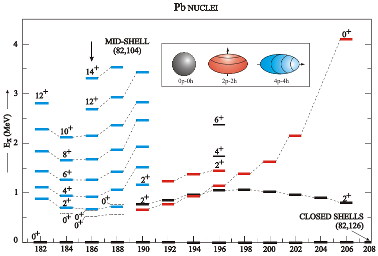
energies are the eigenvalues of

$$\begin{pmatrix} H^N_{\text{cqf}} & V^{N,N+2}_{\text{mix}} \\ \tilde{V}^{N,N+2}_{\text{mix}} & H^{N+2}_{\text{cqf}} + \Delta \end{pmatrix}$$

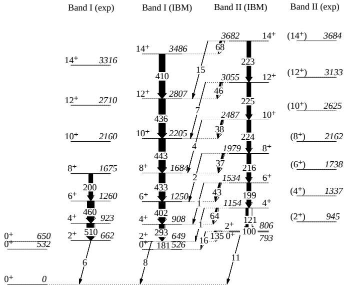
energy surface is lowest eigenvalue of

$$\begin{pmatrix} E^N(\beta,\gamma;\varepsilon_1,\kappa_1,\chi_1) & \omega(\beta) \\ \omega(\beta) & E^{N+2}(\beta,\gamma;\varepsilon_2,\kappa_2,\chi_2) + \Delta \end{pmatrix}$$

Motivation



3-configuration mixing within the IBM

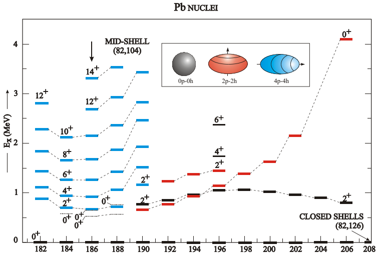


J. Pakarinen *et al.*, Phys. Rev. C 75, 014302 (2007)

T. Grahn *et al.*, Phys. Rev. Lett. 97, 062501

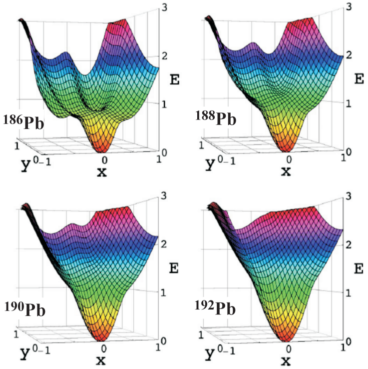
- Z=82 is a magic number
- systematic lowering of two collective bands when proceeding towards neutron midshell
- these collective bands are understood as arising from 2p-2h and 4p4h excitations across the closed Z=82 shell

Motivation



- Z=82 is a magic number
- systematic lowering of two collective bands when proceeding towards neutron midshell
- these collective bands are understood as arising from 2p-2h and 4p4h excitations across the closed Z=82 shell

Geometric interpretation of the IBM



A. Frank *et al.*, Phys. Rev. C **69**, 034323 (2004)

Critical changes of the energy surface

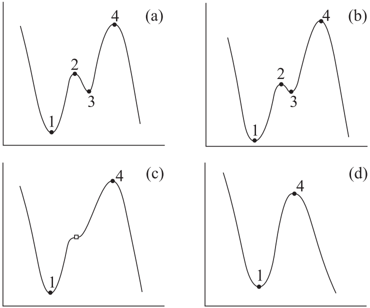
Question

How does the qualitative behaviour of a family of functions $F(x_1,..x_n;a_1,..,a_k)$ change as a function of the parameters $(a_1,..a_k)$?

A one-dimensional example

For the function $f(x;a_1,..,a_k)$, the degenerate critical points are determined by

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial^2 f}{\partial x^2} = 0$$



Critical changes of the energy surface

Answer

Degenerate critical points mark out the regions where qualitative behaviour of $F(x_1, ..x_n; a_1, .., a_k)$ remains unaltered

A one-dimensional example

For the function $f(x; a_1, .., a_k)$, the degenerate critical points are determined by

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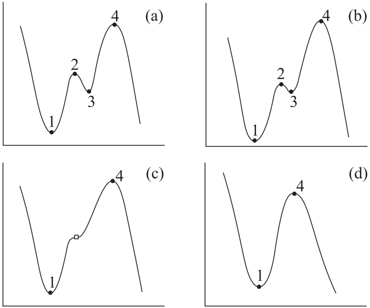
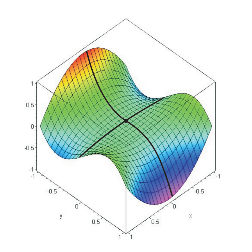


An energy surface associated with the IBM-CM

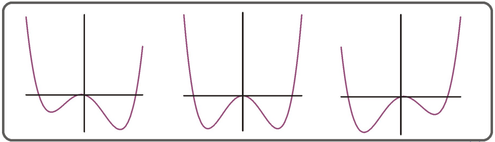
For $E_-(\beta,\gamma;\varepsilon_1,\varepsilon_2,\kappa_1,\kappa_2,\chi_1,\chi_2,\omega,\Delta,N)$, the degenerate critical points are determined by

$$\frac{\partial E_-}{\partial \beta} = 0\,,\ \frac{\partial E_-}{\partial \gamma} = 0\,,$$

$$\det(\mathcal{S}) = \begin{pmatrix} \frac{\partial^2 E_-}{\partial \beta^2} & \frac{\partial^2 E_-}{\partial \beta \partial \gamma} \\ \frac{\partial^2 E_-}{\partial \gamma \partial \beta} & \frac{\partial^2 E_-}{\partial \gamma^2} \end{pmatrix} = 0$$

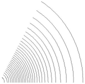
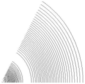


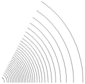
In regions where the energy surface has several minima, it is of interest to know the Maxwell points



Criticality in the IBM with two-configuration mixing

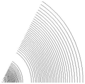
A study of the full phase space $(\varepsilon_1, \varepsilon_2, \kappa_1, \kappa_2, \chi_1, \chi_2, \omega, \Delta, N)$ of the energy surface $E_-(\beta, \gamma; \varepsilon_1, \varepsilon_2, \kappa_1, \kappa_2, \chi_1, \chi_2, \omega, \Delta, N)$ is a tremendous, if not impossible, task. Therefore, we focus on mixing cases between the dynamical symmetry limits which are the benchmarks of the model.

1. $U(5) - \hat{Q}(\chi) \cdot \hat{Q}(\chi)$ mixing $\longrightarrow$ $U(5) - O(6)$:  $\leftrightarrow$ 

   $\longrightarrow$ $U(5) - SU(3)$:  $\leftrightarrow$ 

2. $\hat{Q}(\chi_1) \cdot \hat{Q}(\chi_1) - \hat{Q}(\chi_2) \cdot \hat{Q}(\chi_2)$ mixing

   $\longrightarrow$ encompasses mixing between the $SU(3)$, the $\overline{SU}(3)$, and the $O(6)$ limit

$U(5) - \hat{Q}(\chi) \cdot \hat{Q}(\chi)$ mixing

The energy surface is given as

$$E_- = \frac{|\kappa|}{2(1+\beta^2)^2}\left(\left[\varepsilon' N - (N+2)(1+\chi^2) - \frac{2}{7}(N+2)(N+1)\chi^2 + \Delta'\right]\beta^4 \right.$$
$$+\left[\varepsilon' N - (N+2)(6+\chi^2) - 4(N+2)(N+1) + 2\Delta'\right]\beta^2$$
$$+\frac{4}{7}(N+2)(N+1)\sqrt{14}\chi\beta^3\cos(3\gamma) - 5(N+2) + \Delta'$$
$$-\left[\left(\left[\varepsilon' N + (N+2)(1+\chi^2) + \frac{2}{7}(N+2)(N+1)\chi^2 - \Delta'\right]\beta^4\right.\right.$$
$$+\left[\varepsilon' N + (N+2)(6+\chi^2) + 4(N+2)(N+1) - 2\Delta'\right]\beta^2$$
$$\left.-\frac{4}{7}(N+2)(N+1)\sqrt{14}\chi\beta^3\cos(3\gamma) + 5(N+2) - \Delta'\right)^2$$
$$\left.\left. + \omega'^2(1+\beta^2)^4\right]^{\frac{1}{2}}\right)$$

The parameters

- N number of bosons
- χ parameter in quadrupole operator (prolate, oblate, or γ-unstable rotor)
- κ strength of quadrupole interaction
- $\varepsilon' = \varepsilon/|\kappa|$ scaled strength of vibrational contribution
- $\omega' = 2\omega/|\kappa|$ scaled mixing strength
- $\Delta' = \Delta/|\kappa|$ scaled excitation energy of intruders

An analytical solution to the criticality conditions is obtained from a Taylor expansion in $(\beta, \gamma) = (0, n\pi/3)$

$$E_- = t_{00} + \frac{1}{2!} t_{20} \beta^2 + \frac{1}{3!} t_{30} \beta^3 + \frac{1}{4!} t_{40} \beta^4 + \frac{1}{5!} t_{50} \beta^5 + \cdots$$

- $t_{20}=0 \quad \Rightarrow \quad \varepsilon'_c = -\frac{(N+2)(4N+\chi^2)}{N} \frac{5(N+2)-\Delta'+\sqrt{(5(N+2)-\Delta')^2+\omega_c'^2}}{5(N+2)-\Delta'-\sqrt{(5(N+2)-\Delta')^2+\omega_c'^2}}$

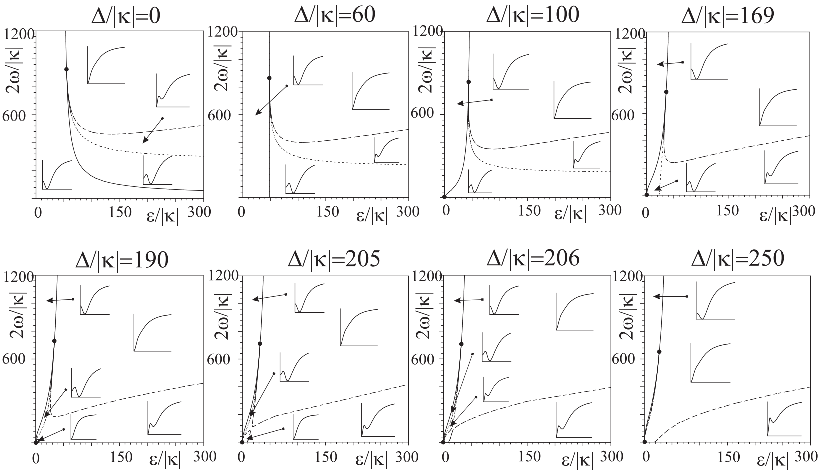
  χ is part of a scaling factor

- triple point?

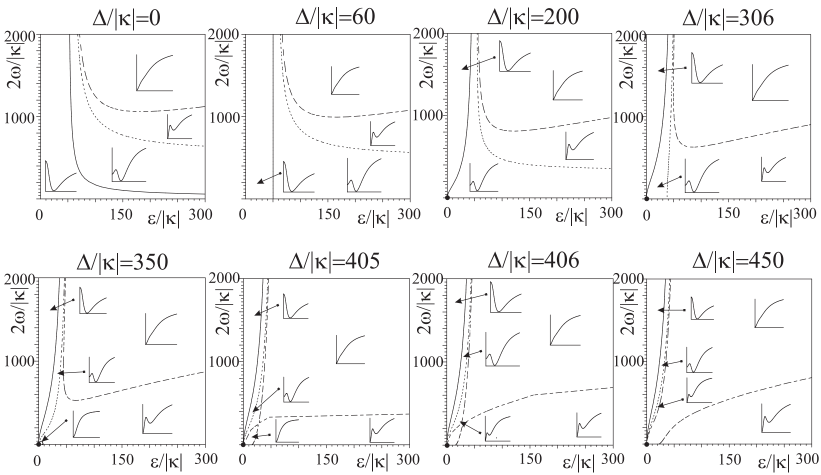
  $$t_{30} = \frac{24}{7}(N+2)(N+1)\sqrt{14}\chi \cos(n\pi) \frac{5(N+2)-\Delta'+\sqrt{(5(N+2)-\Delta')^2+\omega'^2}}{\sqrt{(5(N+2)-\Delta')^2+\omega'^2}}$$

  $\Rightarrow$ only in case of $U(5)$-$O(6)$ mixing
  $\Rightarrow$ triple point is obtained from $t_{20} = t_{40} = 0$

Phase diagram for U(5)-O(6) mixing



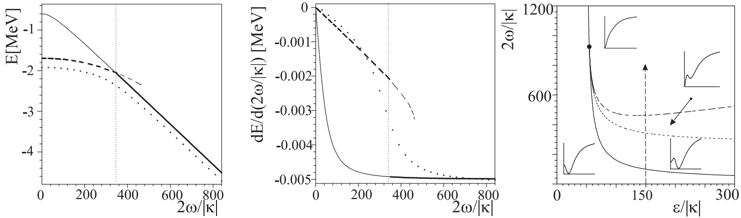
Phase diagram for U(5)-SU(3) mixing



Phase transitions for U(5)-O(6) mixing

In analogy to the Ehrenfest classification for thermodynamical phase transition, an analogous classification for quantum phase transitions can be proposed

- 1st order : discontinuity in first derivative of the energy of the global minimum



- 2nd order : discontinuity in second derivative of the energy of the global minimum

In the transition from a spherical to a deformed minimum in case of $U(5) - O(6)$ mixing, the deformation β_0 exhibits powerlaw behaviour.

From the condition that $\partial E_- / \partial \beta = 0$, we derive

$$\omega'_{\pm} = \pm \frac{4\sqrt{-(1+\beta^2)\varepsilon' N(N+2)[(N+2)\beta^2 - N]}}{(1+\beta^2)^2\left([\varepsilon' N + 4(N+2)^2]\beta^2 + N[\varepsilon' - 4(N+2)]\right)} \times \left([\varepsilon' N - 4(N+2) + \zeta]\beta^4 + [\varepsilon' N + 4N(N+2) + 2\zeta]\beta^2 + \zeta\right)$$

where $\zeta = -\Delta' + 5(N+2)$.

Powerlaw at the degenerate critical points

the deformation of the global minimum in the vicinity of the degenerate critical points

$$\beta_0 = \sqrt{\frac{\zeta N}{2\varepsilon'_c\left[4N^2(N+2) + \zeta(N+1) - \varepsilon'_c N^2\right]}} \left(\varepsilon'_c - \varepsilon'\right)^{1/2}$$

for $\omega'_c > \omega'_t$ and $\varepsilon' < \varepsilon'_c$

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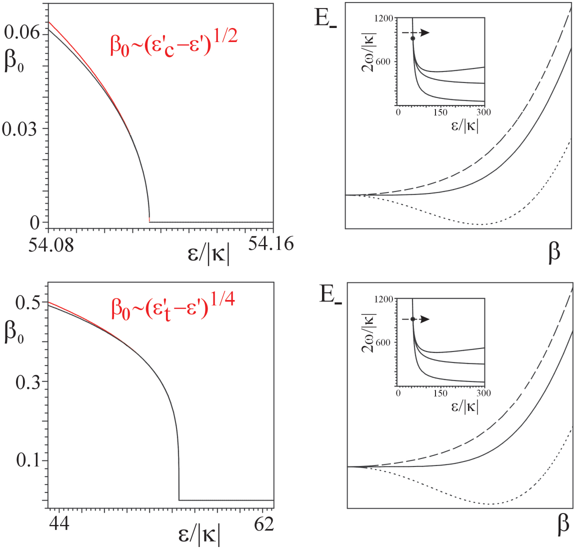
where $\zeta = -\Delta' + 5(N+2)$.

Powerlaw at the triple point

the deformation of the global minimum in the vicinity of the triple point

$$\beta_0 = \left(\frac{N^4}{3(N+1)^2[4N^2(N+2) + \zeta(N+1)]}\right)^{1/4} (\varepsilon'_t - \varepsilon')^{1/4}$$



- powerlaw behaviour fingerprint for 2nd order phase transitions

$\hat{Q}(\chi_1) \cdot \hat{Q}(\chi_1) - \hat{Q}(\chi_2) \cdot \hat{Q}(\chi_2)$ mixing

The energy surface is given as

$$E_- = \frac{|\kappa|}{2(1+\beta^2)^2} \Big[a_1\beta^4 + a_2\beta^2 + a_3\beta^3\cos(3\gamma) - 5(N + |\sigma'|(N+2)) + \Delta' - \Big[\Big(b_1\beta^4 + b_2\beta^2 + b_3\beta^3\cos(3\gamma) - 5(N - |\sigma'|(N+2)) - \Delta' \Big)^2 + \omega'^2(1+\beta^2)^4 \Big]^{1/2} \Big]$$

with

$$a_1 = -N\Big(1 + \frac{1}{7}(2N+5)\chi_1^2\Big) - |\sigma'|(N+2)\Big(1 + \frac{1}{7}(2N+9)\chi_2^2\Big) + \Delta' \,,$$
$$a_2 = -N(\chi_1^2 + 2(2N+1)) - |\sigma'|(N+2)(\chi_2^2 + 2(2N+5)) + 2\Delta' \,,$$
$$a_3 = \frac{4}{7}\sqrt{14}(N(N-1)\chi_1 + |\sigma'|(N+1)(N+2)\chi_2) \,,$$
$$b_1 = -N\Big(1 + \frac{1}{7}(2N+5)\chi_1^2\Big) + |\sigma'|(N+2)\Big(1 + \frac{1}{7}(2N+9)\chi_2^2\Big) - \Delta' \,,$$
$$b_2 = -N(\chi_1^2 + 2(2N+1)) + |\sigma'|(N+2)(\chi_2^2 + 2(2N+5)) - 2\Delta' \,,$$
$$b_3 = \frac{4}{7}\sqrt{14}(N(N-1)\chi_1 - |\sigma'|(N+1)(N+2)\chi_2)$$

The parameters

- N number of bosons
- χ_1 and χ_2 parameter in quadrupole operator (prolate, oblate, or γ-unstable rotor)
- κ strength of quadrupole interaction for the regular configuration
- $|\sigma'| = |\sigma|/|\kappa|$ scaled strength of quadrupole interaction for the intruder configuration
- $\omega' = 2\omega/|\kappa|$ scaled mixing strength
- $\Delta' = \Delta/|\kappa|$ scaled excitation energy of intruders

Two classes of solutions

From the condition $\frac{\partial E_-}{\partial \gamma} = 0$, it follows that γ can be frozen $n\pi/3$ such that the other criticality conditions reduce to

$$\left.\frac{\partial E_-}{\partial \beta}\right|_{\gamma=n\pi/3} = 0\,, \qquad \det\begin{pmatrix} \left.\frac{\partial^2 E_-}{\partial \beta^2}\right|_{\gamma=n\pi/3} & 0 \\ 0 & \left.\frac{\partial^2 E_-}{\partial \gamma^2}\right|_{\gamma=n\pi/3} \end{pmatrix} = 0$$

Hence, there are two classes of solutions to the criticality conditions :

1. degenerate critical points indicating changes in the number of extrema in the β-direction
2. degenerate critical points indicating changes in the number of extrema in the γ-direction

Prolate-oblate coexistence?

From the condition $\frac{\partial^2 E_-}{\partial \gamma^2}\Big|_{\gamma=n\pi/3} = 0$, we derive

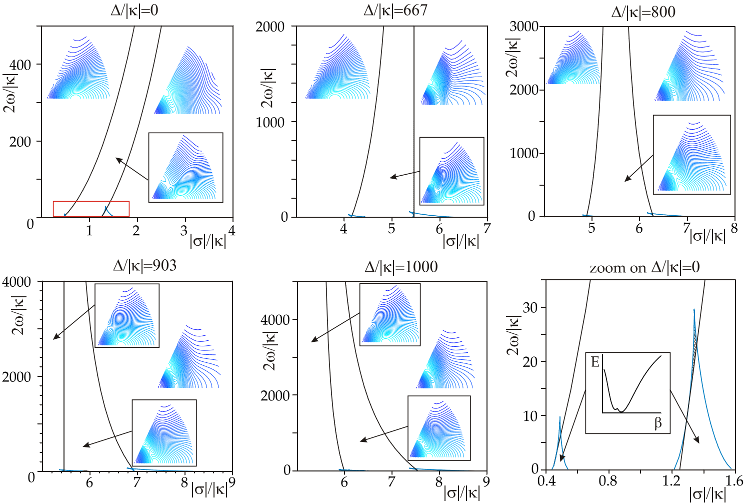
$$\omega' = \pm\sqrt{(b_3 - a_3)(b_3 + a_3)}\,\frac{b_1\beta^4 + b_2\beta^2 + b_3\beta^3 - 5(N - |\sigma'|(N+2)) - \Delta'}{a_3(1+\beta^2)^2}$$

with

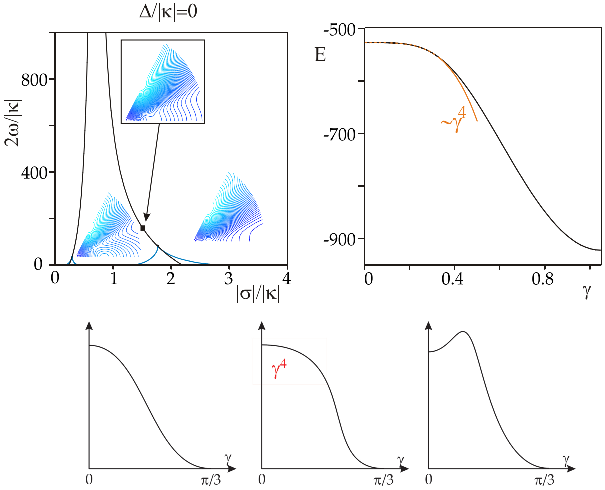
$$\sqrt{(b_3 - a_3)(b_3 + a_3)} = \frac{8}{7}\sqrt{-14|\sigma'|(N-1)N(N+1)(N+2)\chi_1\chi_2}$$

$\Rightarrow$ Prolate-oblate shape coexistence only occurs if χ_1 and χ_2 have opposite sign.

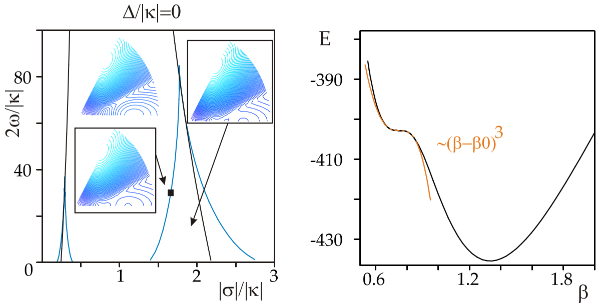
Phase diagram for $SU_{-}(3)$–$\hat{Q}(\chi_2)\hat{Q}(\chi_2)$ mixing ($\chi_2 = \sqrt{7}/16$)

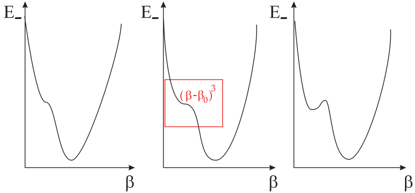


By means of a Taylor expansion, the behaviour in the degenerate critical point can be derived



By means of a Taylor expansion, the behaviour in the degenerate critical point can be derived





Conclusions and outlook

- An extensive region of shape coexistence is found for various excitation energies of the intruder configuration in the phase diagrams for the IBM with configuration mixing. This makes shape coexistence a important aspect of the IBM with configuration mixing.
- In case of mixing between two deformed configurations, shape coexistence between a prolate and an oblate minimum can only arise if χ_1 and χ_2 have opposite sign
- Future perspectives :
    - calculation of the Maxwell points in case of mixing between two deformed configurations and possibility of first-order shape phase transitions?
    - study of the energy surface associated with configuration mixing in less schematic cases
    - ...