The Form of the Effective Interaction in Harmonic-Oscillator-Based Effective Theory (HOBET)

- The form of H^{eff}: short-range expansion/long-range summation
- N^3LO results for two-body interaction: running with Λ, Lepage plot
- State-dependence and κ: implications for Lee-Suzuki, etc.
- Scales and long-wavelength/short-wavelength factorization: implications for numerical strategies with potentials or in ET

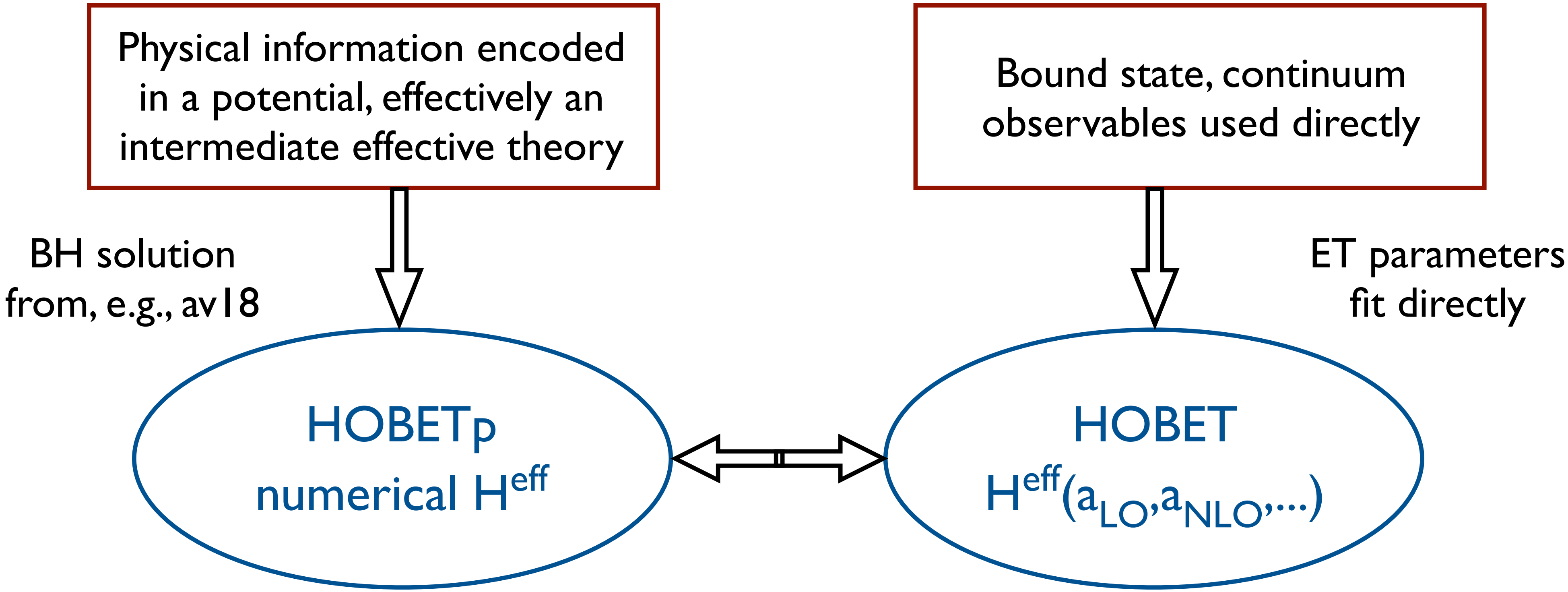
Wick Haxton -- New Approaches in Nuclear Many-Body Theory -- INT, October 2007

- Much of the work on HOBET done in collaboration with

  { Chang-Liang Song
    Tom Luu

- More complete summary of most of this talk available at

  arXiv: 0710:0289



this interplay potentially quite useful:

- HOBETp can be used to determine the features of the systematic expansion that will be needed in HOBET (our main topic)
- But the insight HOBET provides can also be helpful in potential-based approaches -- simple analytic representation for H^{eff}

Overview of Approach

- Low-energy P-space defined by a set of HO states with quanta

$$\leq \Lambda_P \hbar\omega$$

- Hamiltonian a sum of relative KE and potential

$$H = \frac{1}{2} \sum_{i,j=1}^{A} (T_{ij} + V_{ij})$$

- HOBETp's H^{eff} defined by energy-dependent Bloch-Horowitz equation

$$H^{eff} = P \left[H + H \frac{1}{E - QH} QH \right] P$$

$$H^{eff} |\Psi_P\rangle = E |\Psi_P\rangle \qquad |\Psi_P\rangle = P|\Psi\rangle$$

- Solved self consistently. E is the exact eigenvalue and $|\Psi_P\rangle$ the restriction of the exact wave function to P: nontrivial normalization, non-orthogonality

$$P = P(b, \Lambda_P)$$

- P is thus separable: H^{eff}, like H, is translationally invariant
- Solutions independent of b, Λ_P if the ET is executed properly -- though efficiency may be influence by this choice
- Wave function evolves simply with increments $\Lambda_P \rightarrow \Lambda_P+2$: new components added to existing, norm increases, eventually $\rightarrow$ 1
- HOBET's H^{eff} defined by a systematic expansion that encodes in P the physics residing in Q, with parameters fit to bound and continuum data
- Effective operators are in HOBET/HOBETp done in analogy with H^{eff}, taking into account both operator corrections and wf normalization

- 3He
- av18 potential (~hard core)
- numerical BH solution

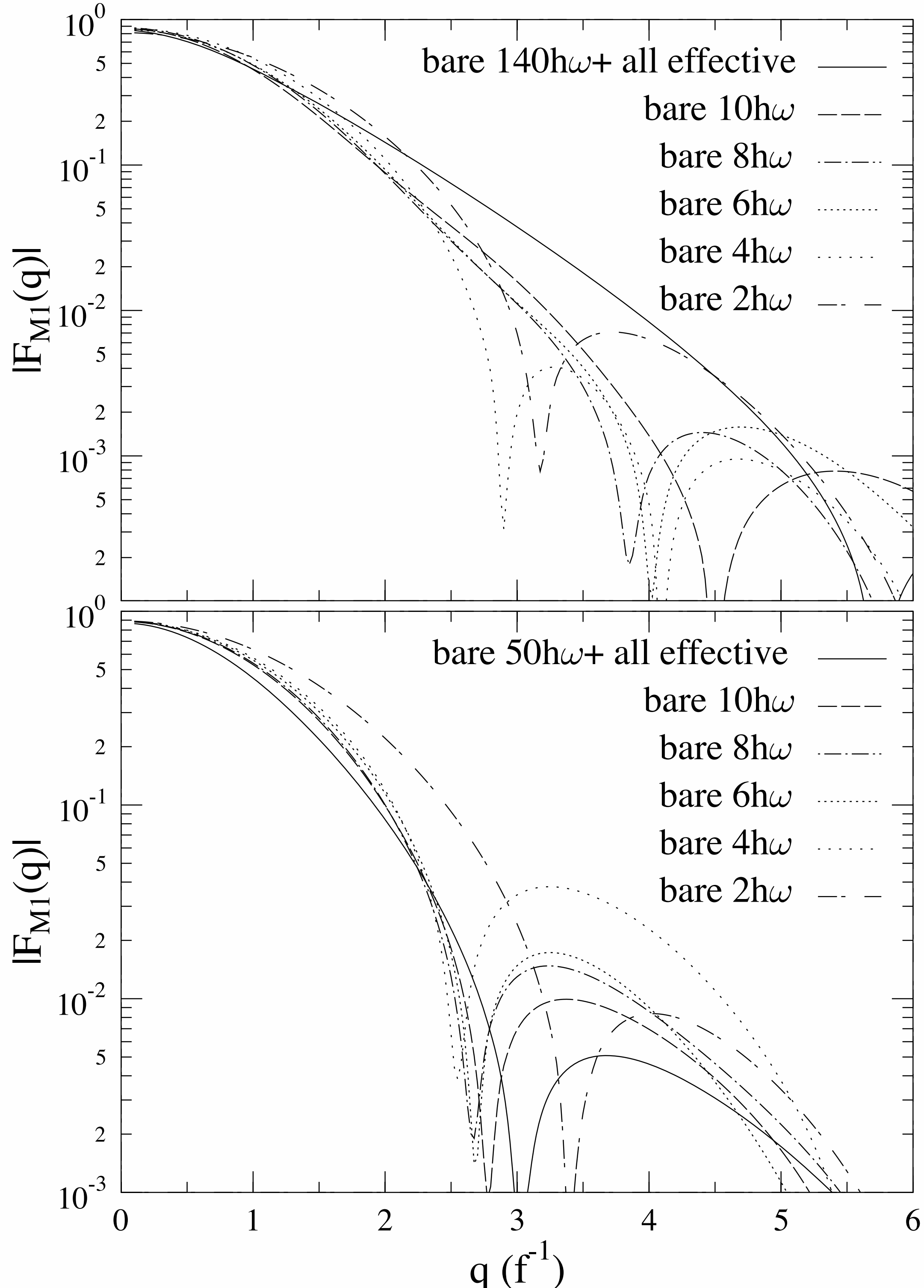
state	amplitude					
	$0\hbar\omega$	$2\hbar\omega$	$4\hbar\omega$	$6\hbar\omega$	$8\ \hbar\omega$	exact
	(31.1%)	(57.4%)	(70.0%)	(79.8%)	(85.5%)	(100%)
$\|0,1\rangle$	0.5579	0.5579	0.5579	0.5579	0.5579	0.5579
$\|2,1\rangle$	0.0000	0.0463	0.0461	0.0462	0.0462	0.0463
$\|2,2\rangle$	0.0000	-0.4825	-0.4824	-0.4824	-0.4824	-0.4826
$\|2,3\rangle$	0.0000	0.0073	0.0073	0.0073	0.0073	0.0073
$\|4,1\rangle$	0.0000	0.0000	-0.0204	-0.0204	-0.0204	-0.0205
$\|4,2\rangle$	0.0000	0.0000	0.1127	0.1127	0.1127	0.1129
$\|4,3\rangle$	0.0000	0.0000	-0.0419	-0.0420	-0.0421	-0.0423

Basic attributes:

- Slow convergence of shell-by-shell expansions -- "SM" missing a great deal of physics
- Attractive evolution of wf
- $<H^{eff}>$ hypersensitive to choice of P

	$2\hbar\omega$	$4\hbar\omega$	$6\hbar\omega$	$8\ \hbar\omega$
$\langle 0,1 \| H^{eff} \| 2,1\rangle$	-4.874	-3.165	-0.449	1.279
$\langle 0,1 \| H^{eff} \| 2,5\rangle$	-0.897	-1.590	-1.893	-2.208
$\langle 2,1 \| H^{eff} \| 2,2\rangle$	6.548	-2.534	-4.144	-5.060

All observables are independent of the parameters one picks to describe the low-energy included space



Task #1: Identifying a Systematic Expansion for HOBET's H^{eff}

- Usual goal in an ET is to describe the low-lying excitations in P
- The HOBET H^{eff} is more ambitious: a *spectral* quantity where
  - ❒ Q contains both missing short- and long-range physics, unlike EFTs: can be viewed as an expansion around $q \sim 1/b$. What is the systematic expansion for such a case?
  - ❒ The relative importance of the missing long- and short-range physics is governed by the binding energy E: one has a finely tuned parameter that can produce an extended state as $\rightarrow 0$ (balance between T, V minimization delicate)
  - ❒ P and Q are strongly coupled by T via nearest-shell interactions -- T is a ladder operator in the HO. Worst possible case for an ET
- Lovely resolution, which we learned about the hard way ...

What is the expansion implicit in the BH "data"?

- Initial attempt mimicked EFT approaches (WH + Luu, NP A690 (2001) 5247)
  - Discrete renormalization group: shell-by-shell integration
  - Started with a LO contact operator at some high scale Λ, integrated progressively to reach Λ_P, the "SM" scale
  - LO was schemed independent; beyond LO introduced scheme dependent counterterms to make shell-by-shell evolution exact
- But results were troubling: the coefficients a_{LO}, a_{NLO}, etc., did not evolve naturally. The short-range expansion used was not systematically correcting the low-energy results
  - Tom's thesis: dissecting this problem, identifying the right expansion
  - Led to other results on making NP perturbative -- will not discuss

TABLE I: Contact-gradient expansion for relative-coordinate two-particle matrix elements. Here $\overrightarrow{D}^2_M = (\overrightarrow{\nabla} \otimes \overrightarrow{\nabla})_{2M}$, $\overrightarrow{D}^0_0 = [(\sigma(1)\otimes\sigma(2))_2 \otimes \overrightarrow{D}^2]_{00}$, $\overrightarrow{F}^3_M = (\overrightarrow{\nabla}\otimes\overrightarrow{D}^2)_{3M}$, $\overrightarrow{F}^1_M = [(\sigma(1)\otimes\sigma(2))_2\otimes \overrightarrow{F}^3]_{1M}$, $\overrightarrow{G}^4_M = (\overrightarrow{D}^2\otimes\overrightarrow{D}^2)_{4M}$, $\overrightarrow{G}^2_M = [(\sigma(1)\otimes\sigma(2))_2\otimes\overrightarrow{G}^4]_{2M}$, and the scalar product of tensor operators is defined as $A^J \cdot B^J = \sum_{M=-J}^{M=J}(-1)^M A^J_M B^J_{-M}$.

Transitions	LO	NLO	NNLO	N^{3}LO
$^3S_1 \leftrightarrow {}^3S_1$ or $^1S_0 \leftrightarrow {}^1S_0$	$a^{3S1}_{LO}\delta(\mathbf{r})$	$a^{3S1}_{NLO}(\overleftarrow{\nabla}^2\ \delta(\mathbf{r}) + \delta(\mathbf{r})\ \overrightarrow{\nabla}^2)$	$a^{3S1,22}_{NNLO}\ \overleftarrow{\nabla}^2\ \delta(\mathbf{r})\ \overrightarrow{\nabla}^2$ $a^{3S1,40}_{NNLO}(\overleftarrow{\nabla}^4\ \delta(\mathbf{r}) + \delta(\mathbf{r})\ \overrightarrow{\nabla}^4)$	$a^{3S1,42}_{N^3LO}(\overleftarrow{\nabla}^4\ \delta(\mathbf{r})\ \overrightarrow{\nabla}^2 + \overleftarrow{\nabla}^2\ \delta(\mathbf{r})\ \overrightarrow{\nabla}^4)$ $a^{3S1,60}_{N^3LO}(\overleftarrow{\nabla}^6\ \delta(\mathbf{r}) + \delta(\mathbf{r})\ \overrightarrow{\nabla}^6)$
$^3S_1 \leftrightarrow {}^3D_1$		$a^{SD}_{NLO}(\delta(\mathbf{r})\ \overrightarrow{D}^0 + \overleftarrow{D}^0\ \delta(\mathbf{r}))$	$a^{SD,22}_{NNLO}(\overleftarrow{\nabla}^2\ \delta(\mathbf{r})\ \overrightarrow{D}^0 + \overleftarrow{D}^0\ \delta(\mathbf{r})\ \overrightarrow{\nabla}^2)$ $a^{SD,04}_{NNLO}(\delta(\mathbf{r})\ \overrightarrow{\nabla}^2\overrightarrow{D}^0 + \overleftarrow{D}^0\overleftarrow{\nabla}^2\ \delta(\mathbf{r}))$	$a^{SD,42}_{N^3LO}(\overleftarrow{\nabla}^4\ \delta(\mathbf{r})\ \overrightarrow{D}^0 + \overleftarrow{D}^0\ \delta(\mathbf{r})\ \overrightarrow{\nabla}^4)$ $a^{SD,24}_{N^3LO}(\overleftarrow{\nabla}^2\ \delta(\mathbf{r})\ \overrightarrow{\nabla}^2\overrightarrow{D}^0 + \overleftarrow{D}^0\overleftarrow{\nabla}^2\ \delta(\mathbf{r})\ \overrightarrow{\nabla}^2)$ $a^{SD,06}_{N^3LO}(\delta(\mathbf{r})\ \overrightarrow{\nabla}^4\overrightarrow{D}^0 + \overleftarrow{D}^0\overleftarrow{\nabla}^4\ \delta(\mathbf{r}))$
$^1D_2 \leftrightarrow {}^1D_2$ or $^3D_J \leftrightarrow {}^3D_J$			$a^{1D2}_{NNLO}\ \overleftarrow{D}^2\ \cdot\delta(\mathbf{r})\ \overrightarrow{D}^2$	$a^{1D2}_{N^3LO}(\overleftarrow{D}^2\overleftarrow{\nabla}^2\ \cdot\delta(\mathbf{r})\ \overrightarrow{D}^2 + \overleftarrow{D}^2\ \cdot\delta(\mathbf{r})\ \overrightarrow{\nabla}^2\overrightarrow{D}^2)$
$^3D_3 \leftrightarrow {}^3G_3$				$a^{DG}_{N^3LO}(\overleftarrow{D}^2\ \cdot\delta(\mathbf{r})\ \overrightarrow{G}^2 + \overleftarrow{G}^2\ \cdot\delta(\mathbf{r})\ \overrightarrow{D}^2)$
$^1P_1 \leftrightarrow {}^1P_1$ or $^3P_J \leftrightarrow {}^3P_J$		$a^{1P1}_{NLO}\ \overleftarrow{\nabla}\ \cdot\delta(\mathbf{r})\ \overrightarrow{\nabla}$	$a^{1P1}_{NNLO}(\overleftarrow{\nabla}\overleftarrow{\nabla}^2\ \cdot\delta(\mathbf{r})\ \overrightarrow{\nabla} + \overleftarrow{\nabla}\ \cdot\delta(\mathbf{r})\ \overrightarrow{\nabla}^2\overrightarrow{\nabla})$	$a^{1P1,33}_{N^3LO}\ \overleftarrow{\nabla}\overleftarrow{\nabla}^2\ \cdot\delta(\mathbf{r})\ \overrightarrow{\nabla}^2\overrightarrow{\nabla}$ $a^{1P1,51}_{N^3LO}(\overleftarrow{\nabla}\overleftarrow{\nabla}^4\ \cdot\delta(\mathbf{r})\ \overrightarrow{\nabla} + \overleftarrow{\nabla}\ \cdot\delta(\mathbf{r})\ \overrightarrow{\nabla}^4\overrightarrow{\nabla})$
$^3P_2 \leftrightarrow {}^3F_2$			$a^{PF}_{NNLO}(\overleftarrow{\nabla}\ \cdot\delta(\mathbf{r})\ \overrightarrow{F}^1 + \overleftarrow{F}^1\ \cdot\delta(\mathbf{r})\ \overrightarrow{\nabla})$	$a^{PF,33}_{N^3LO}(\overleftarrow{\nabla}\overleftarrow{\nabla}^2\ \cdot\delta(\mathbf{r})\ \overrightarrow{F}^1 + \overleftarrow{F}^1\ \cdot\delta(\mathbf{r})\ \overrightarrow{\nabla}^2\overrightarrow{\nabla})$ $a^{PF,15}_{N^3LO}(\overleftarrow{\nabla}\ \cdot\delta(\mathbf{r})\ \overrightarrow{\nabla}^2\overrightarrow{F}^1 + \overleftarrow{F}^1\overleftarrow{\nabla}^2\ \cdot\delta(\mathbf{r})\ \overrightarrow{\nabla})$
$^1F_3 \leftrightarrow {}^1F_3$ or $^3F_J \leftrightarrow {}^3F_J$				$a^{1F3}_{N^3LO}\ \overleftarrow{F}^3\ \cdot\delta(\mathbf{r})\ \overrightarrow{F}^3$

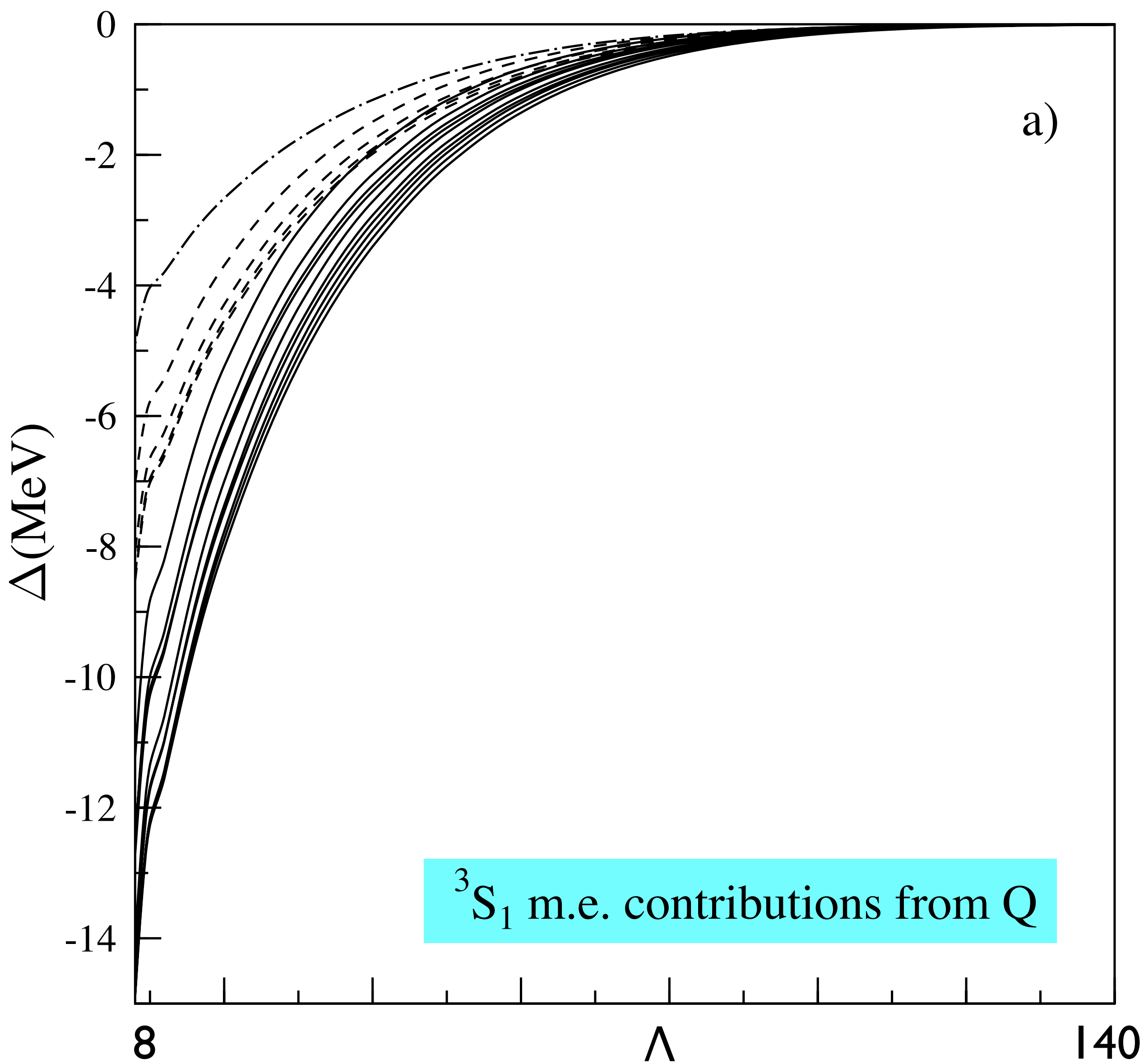
This is the kind of short-range expansion -- the candidate HOBET H^{eff} -- we tried: most general nonlocal contact-gradient potential consistent with P,T, hermiticity, etc.

$a_{LO} = a_{LO}(b,\Lambda_P)$, etc

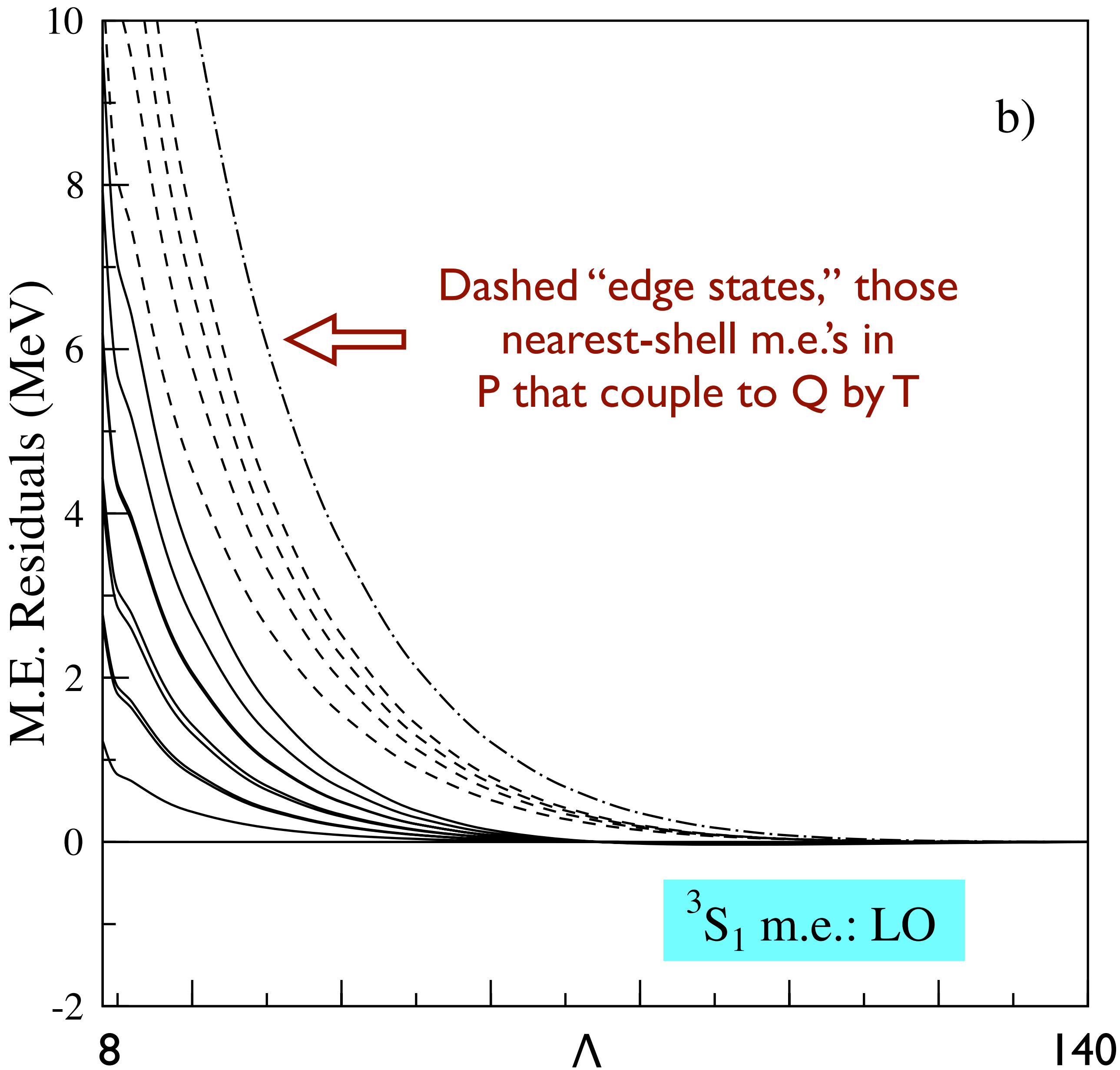
Define $\quad Q_\Lambda \equiv \sum_{\alpha=\Lambda_P+1}^{\Lambda} |\alpha\rangle\langle\alpha| \text{ with } Q_{\Lambda_P} \equiv 0$

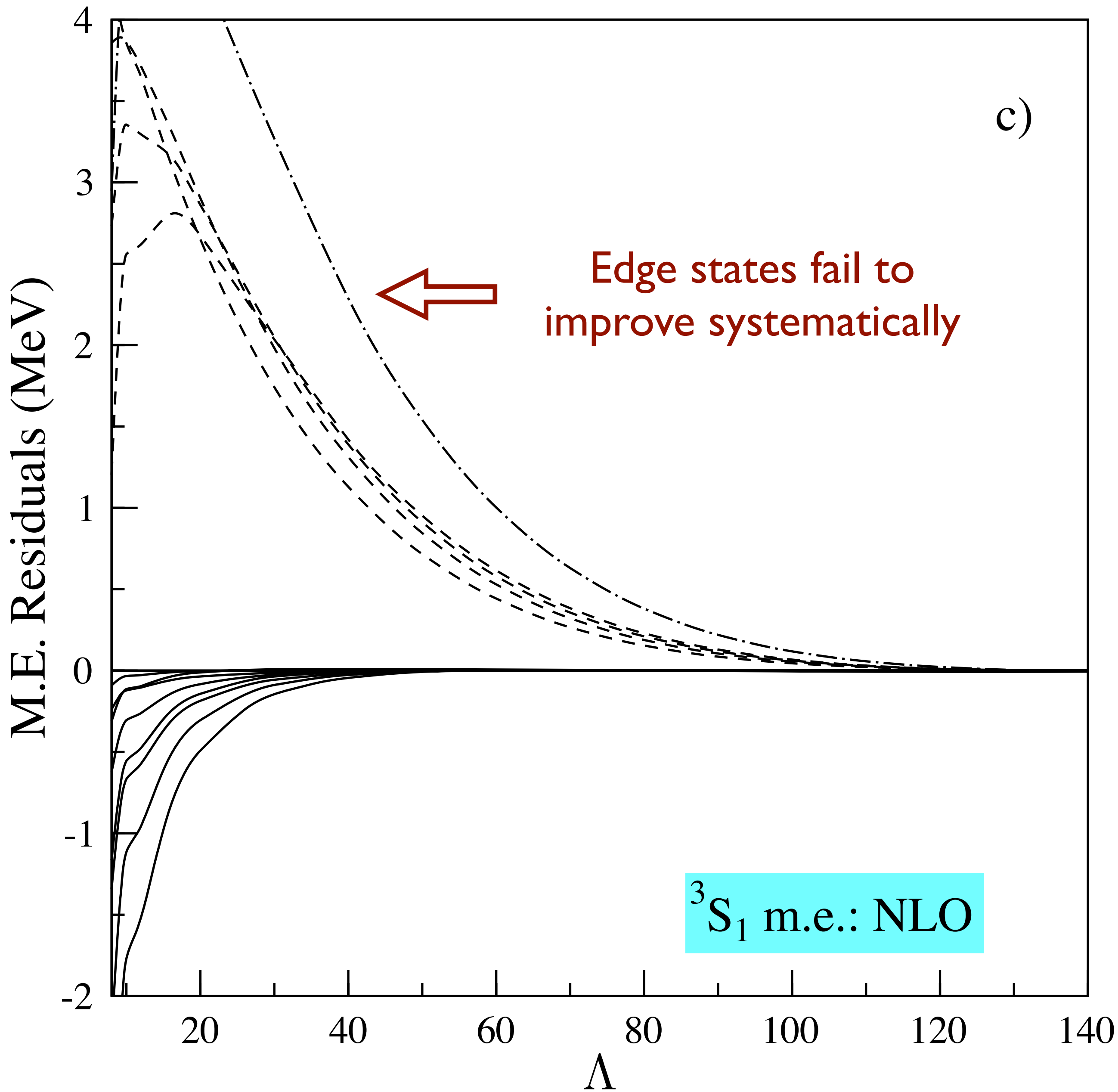
$$\Delta(\Lambda) \equiv H^{eff} - H^{eff}(\Lambda) = H\frac{1}{E-QH}QH - H\frac{1}{E-Q_\Lambda H}Q_\Lambda H \stackrel{\Lambda\to\Lambda_P}{\longrightarrow} H^{eff}$$

- Q contribution above scale Λ
- b = 1.7 f, av18, deuteron, Λ_P=8
- 10 MeV is roughly the scale of Q's contribution at Λ_P=8

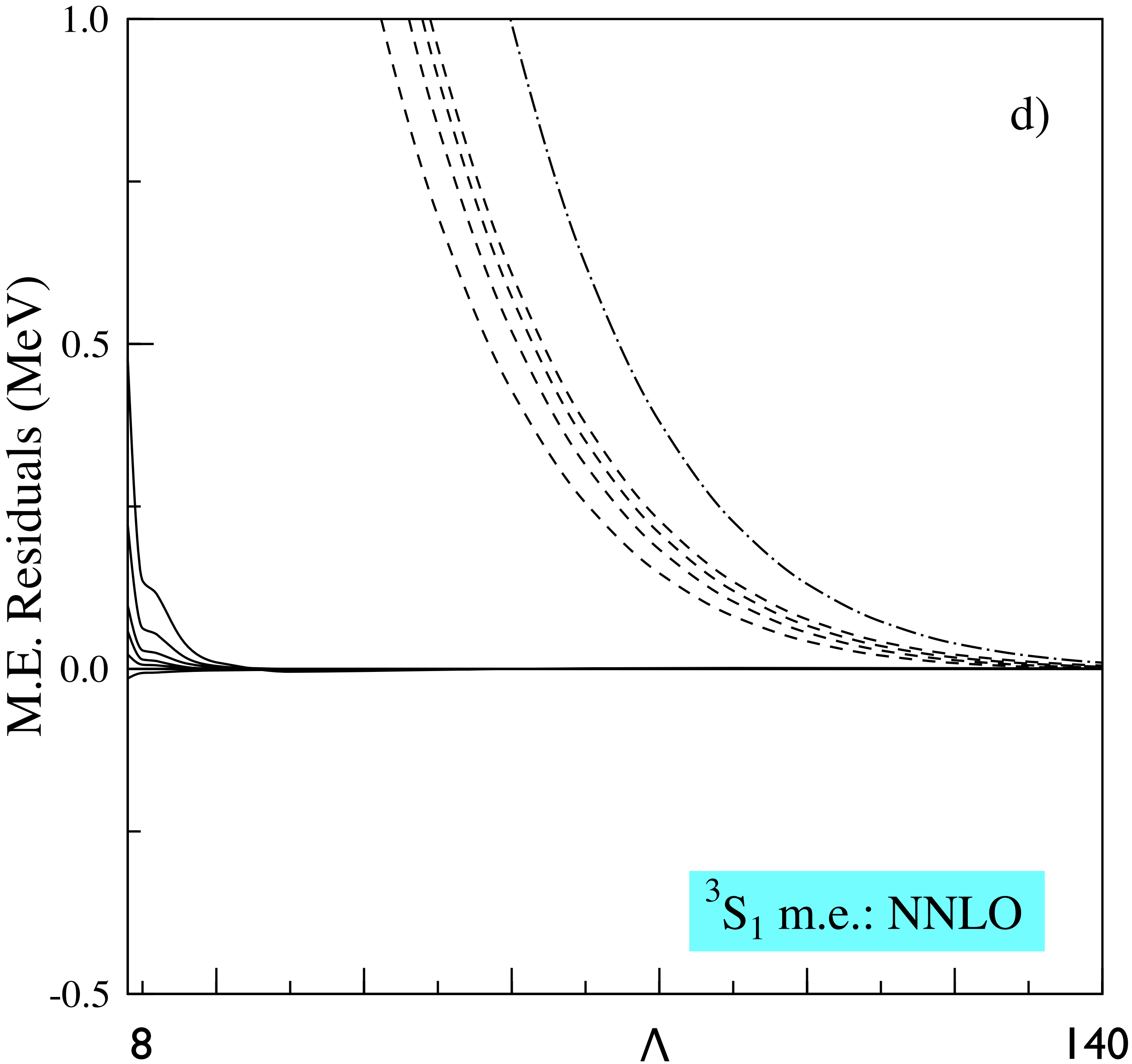


Repeated our troubled HOBET effort, dissecting the results at the individual matrix element level in HOBETp, to see why the wheels feel off:

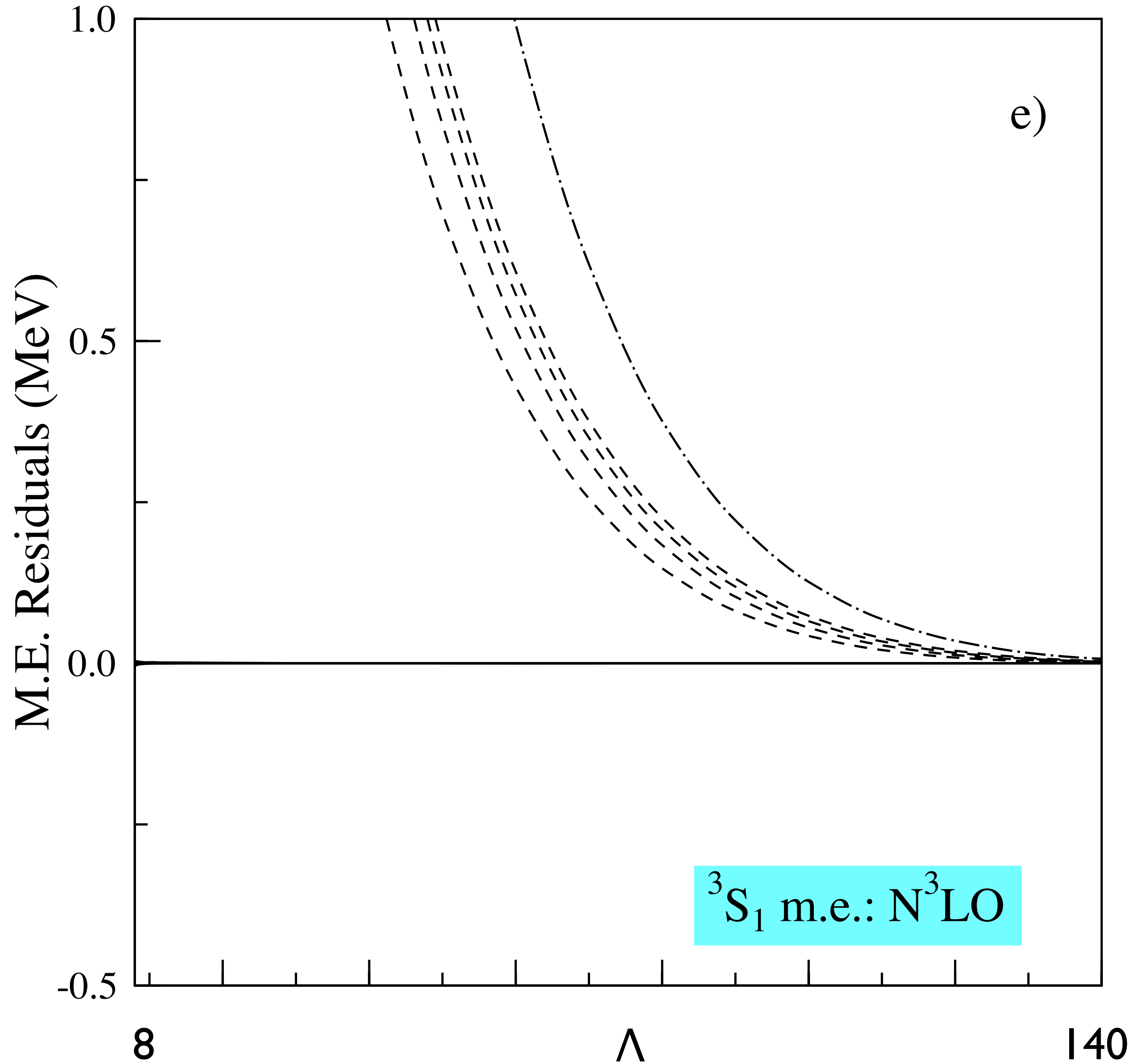




c)
Edge states fail to improve systematically
$^3S_1$ m.e.: NLO
M.E. Residuals (MeV)
Λ
4
3
2
1
0
-1
-2
20
40
60
80
100
120
140

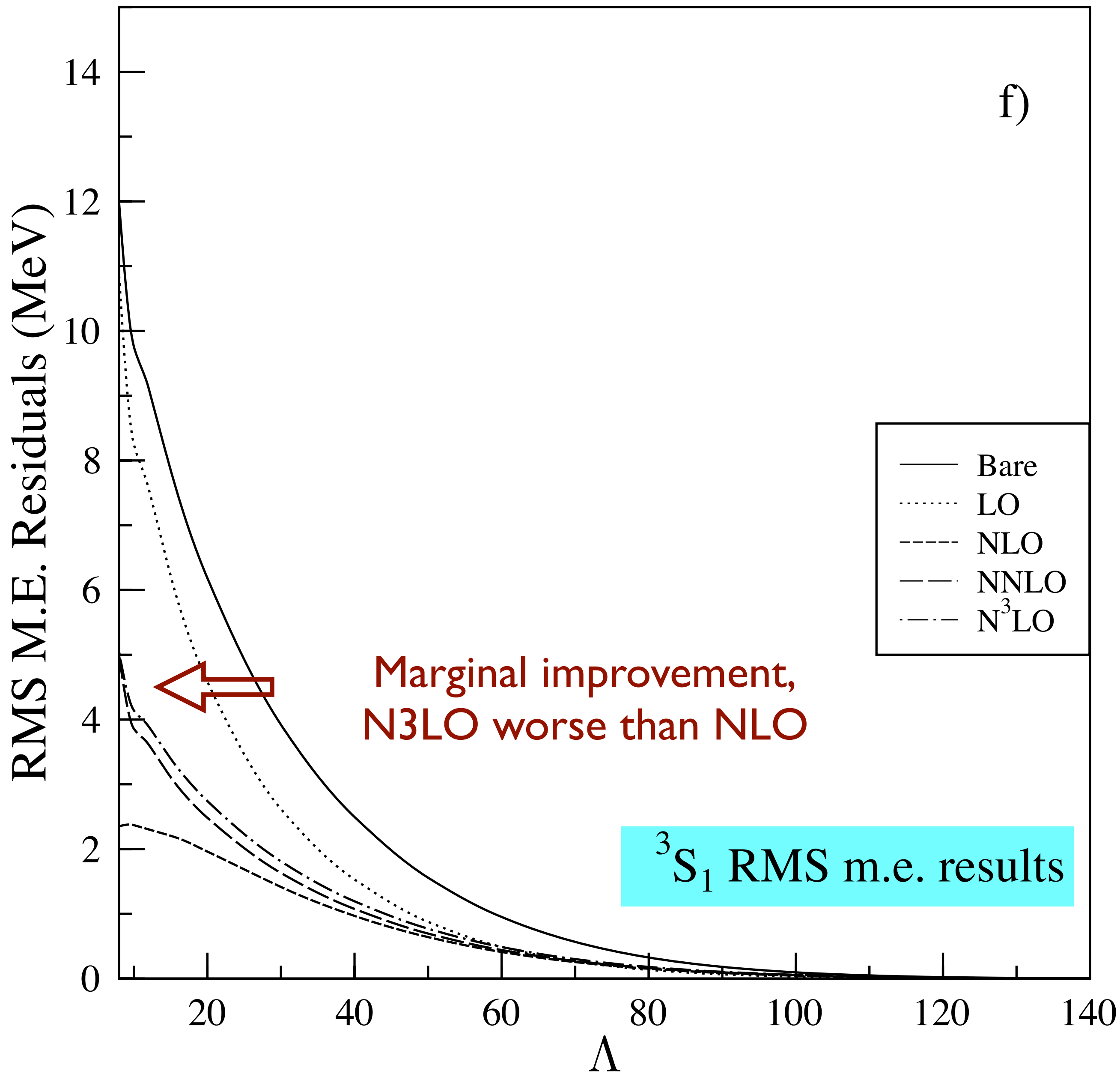


1.0
0.5
0.0
-0.5
M.E. Residuals (MeV)
d)
$^3S_1$ m.e.: NNLO
8
Λ
140



1.0
0.5
0.0
-0.5
M.E. Residuals (MeV)
e)
$^3S_1$ m.e.: N^3LO
8
Λ
140

One measure of m.e. quality:



Formulating HOBET

- HOBET -- and any confined basis -- excludes low- and high-momentum excitations: tension between T(E) and V_{hard}: sum QT to all orders

$$H^{eff} = \frac{E}{E-TQ}\left[T - T\frac{Q}{E}T + V + V\frac{1}{E-QH}QV\right]\frac{E}{E-QT}$$

- This redefines bare V, bare T, and rescattering contributions as:

  - $\text{bare T} : \langle\alpha|T\frac{E}{E-QT}|\beta\rangle = \langle\alpha|\frac{E}{E-TQ}T|\beta\rangle \overset{\text{nonedge}}{\longrightarrow} \langle\alpha|T|\beta\rangle$
  - $\text{bare V} : \langle\alpha|\frac{E}{E-QT}V\frac{E}{E-TQ}|\beta\rangle \overset{\text{nonedge}}{\longrightarrow} \langle\alpha|V|\beta\rangle$
  - $\langle\alpha|\frac{E}{E-TQ}V\frac{1}{E-QH}QV\frac{E}{E-QT}|\beta\rangle \overset{\text{nonedge}}{\longrightarrow} \langle\alpha|V\frac{1}{E-QH}QV|\beta\rangle$

- Effectively absorbs into a new P' the "soft" physics residing in Q that governs the asymptotic behavior of w.f. -- new orthogonal space

- Identify contact-gradient expansion with the short-range term

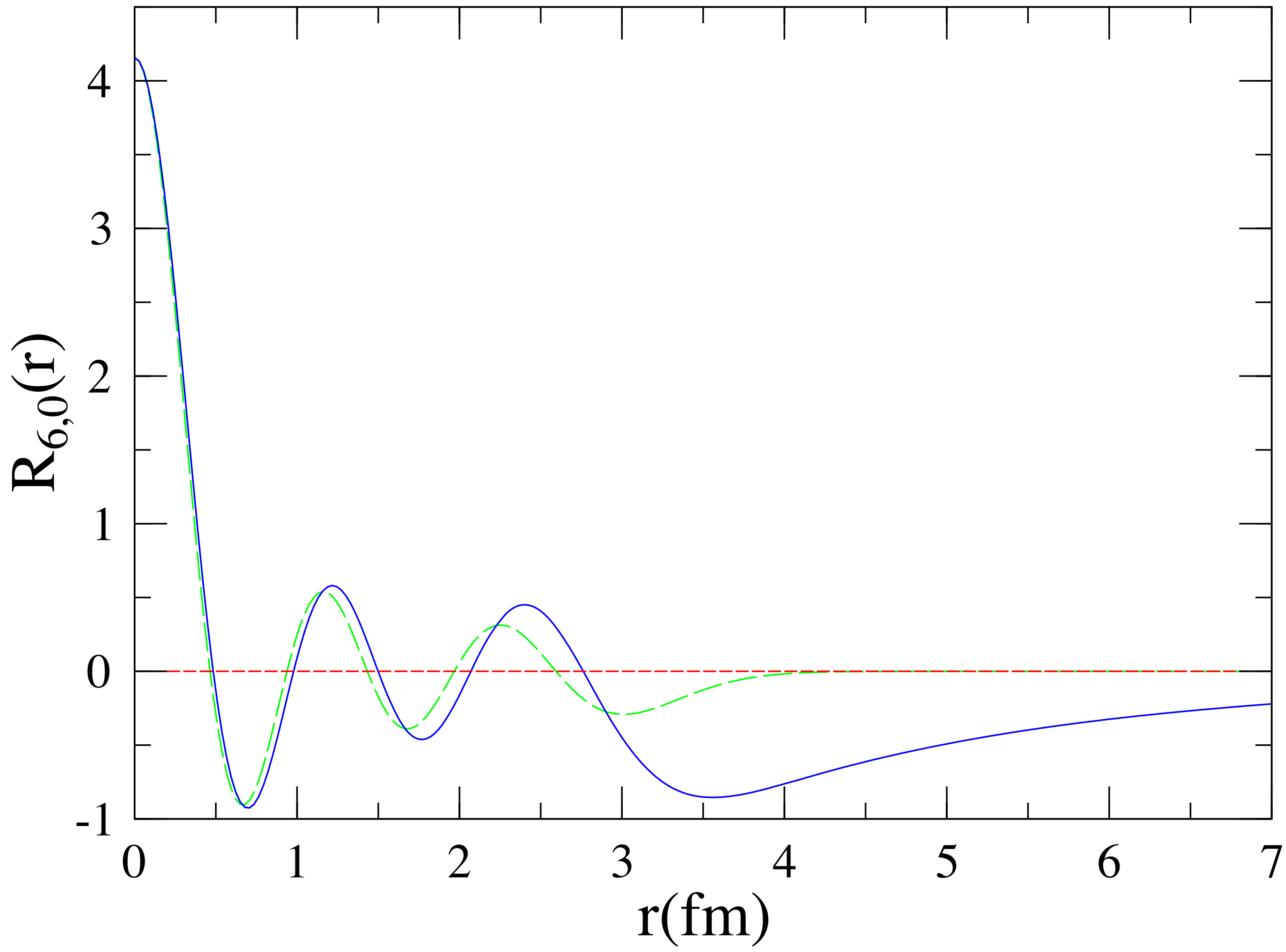
$$\frac{E}{E-TQ}V\frac{1}{E-QH}QV\frac{E}{E-QT}\xrightarrow{\text{HOBET}}\frac{E}{E-TQ}\bar{O}\frac{E}{E-QT}$$

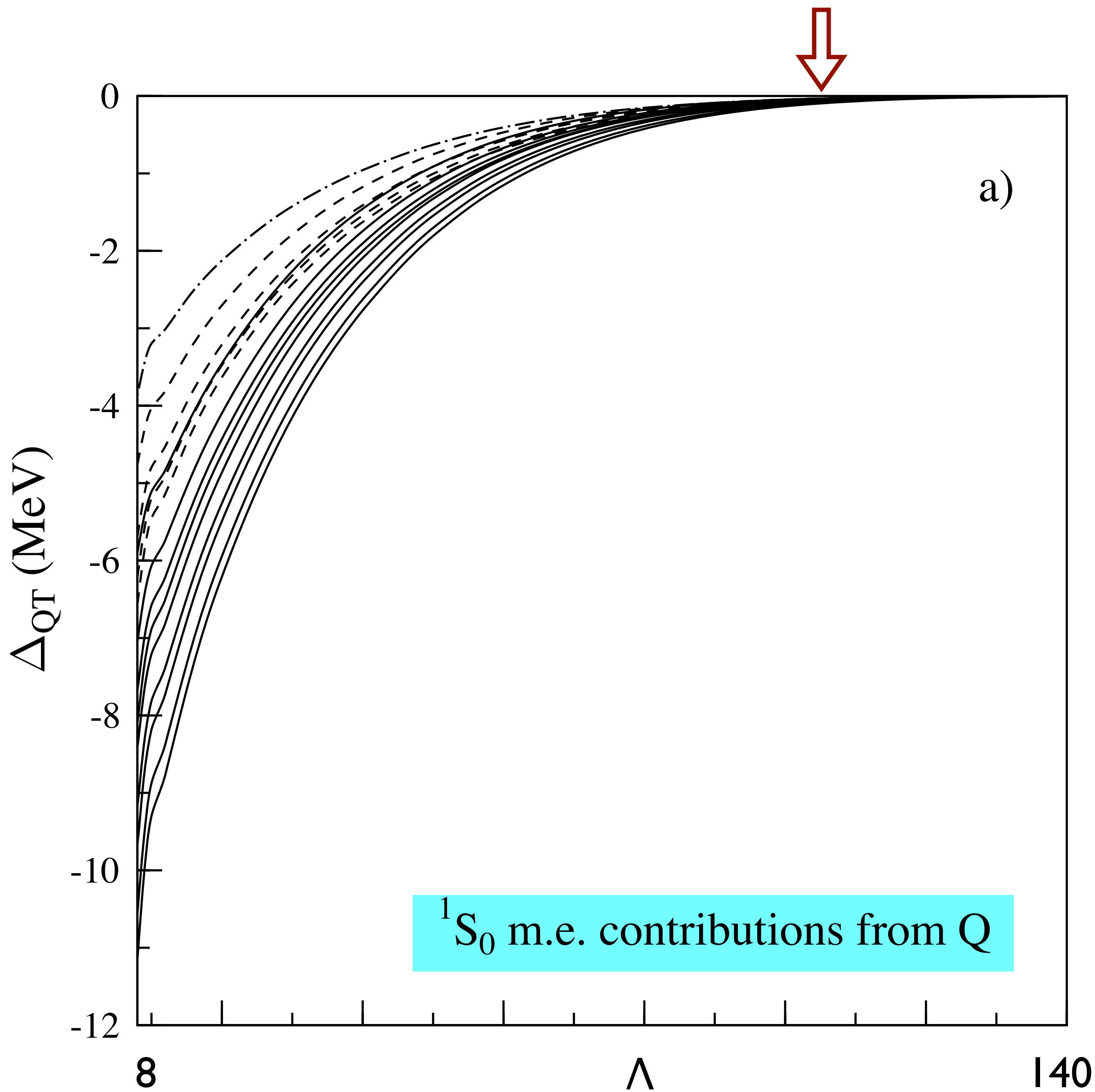
- Define gradients as an expansion around $r_0 \sim 1/b$

  - $\text{EFT}\ \vec{\nabla}^2 e^{i\vec{k}\cdot\vec{r}}\big|_{\vec{k}=0}=0 \quad \Rightarrow \quad \text{HOBET}\ \vec{\nabla}^2\psi_{1s}(b)=0$
  - $\text{contact}-\text{gradient operators}\ O\rightarrow\bar{O}\equiv e^{r^2/2}Oe^{r^2/2}$
  - $\text{expansion nodal q.n.s}: \vec{\nabla}^2\sim-4(n-1), \quad \vec{\nabla}^4\sim 16(n-1)(n-2)$
  - $\text{no op. mixing : e.g.,}\ a_{LO}\ \leftrightarrow\ 1s-1s,\ \text{remains fixed, higher order}$
  - $a's\sim\int_0^\infty\int_0^\infty e^{-r_1^2}\left[r_1^{n'}V(r_1,r_2)r_2^n\right]e^{-r_2^2}r_1^2r_2^2dr_1dr_2$

- Summation over QT involves single parameter, $\kappa = \sqrt{2|E|/\hbar\omega}$
  - Long-wavelength corrections severe as κ→0: limit of small binding (halo nucleus) or small b. But significant in all cases.
  - Remarkable that these effects are encoded in a single parameter κ
  - Summation non-perturbative in both QT and V -- strong potential consequences contained in |E| (correct asymptotic correlations)
- $|\tilde{\alpha}\rangle = \frac{E}{E - QT}|\alpha\rangle$ from free Green's function or via HO expansion
  - $(E - T)|\tilde{\alpha}\rangle = \left[P\frac{1}{E-T}P\right]^{-1}|\alpha\rangle$ (Jacobi basis, HO Fourier)
  - $|\tilde{n}\tilde{l}\rangle = \sum_{i=0}^{\infty} \tilde{g}_i(-\kappa^2; n, l)|n + i\ l\rangle$ (continued fractions exploiting HO ladder properties ⇒ hyperspherical basis)

QT-summed transformation of an s-wave edge state (ΛP=10): renormalized at r=0 to show short-range behavior unchanged

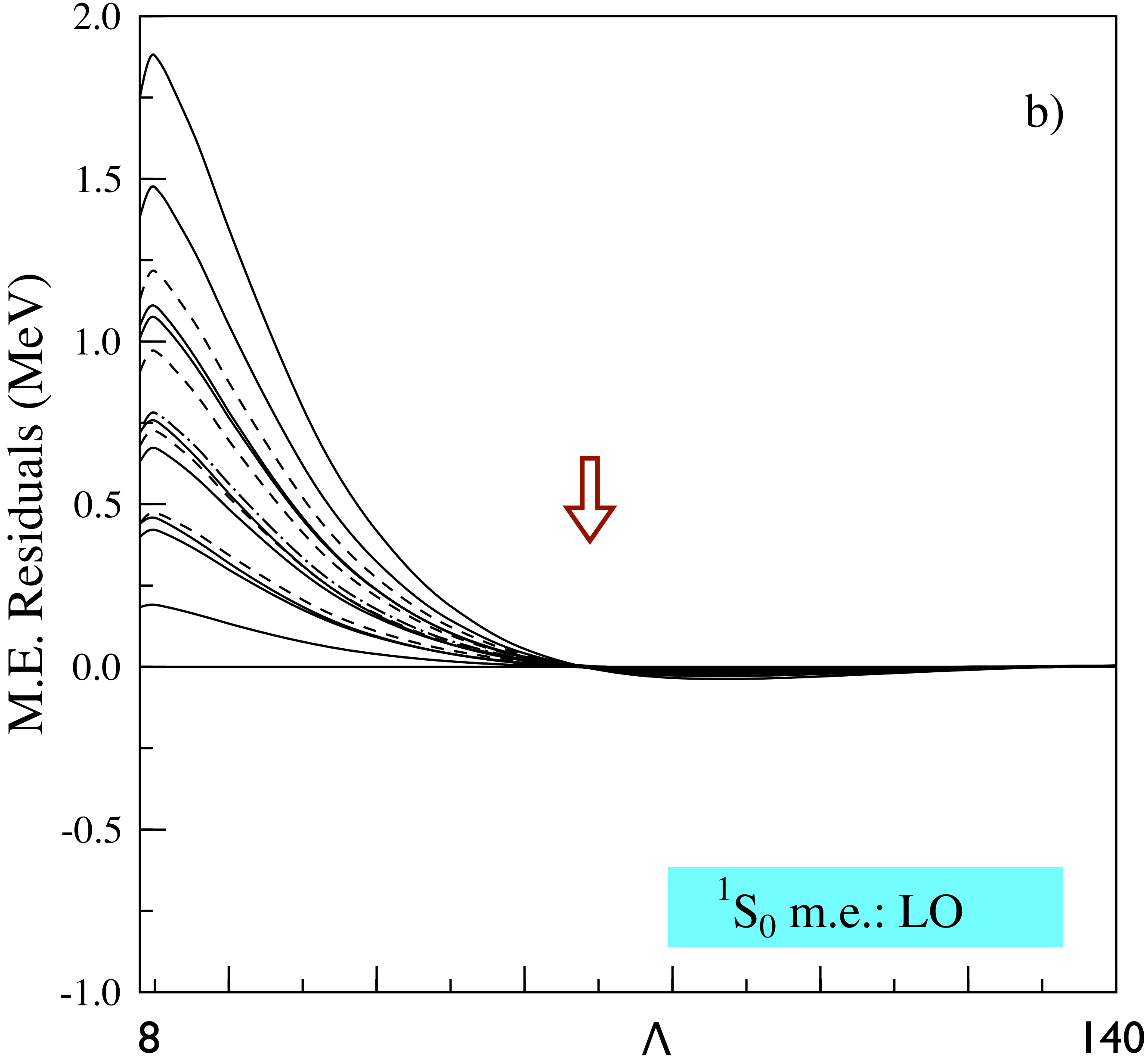


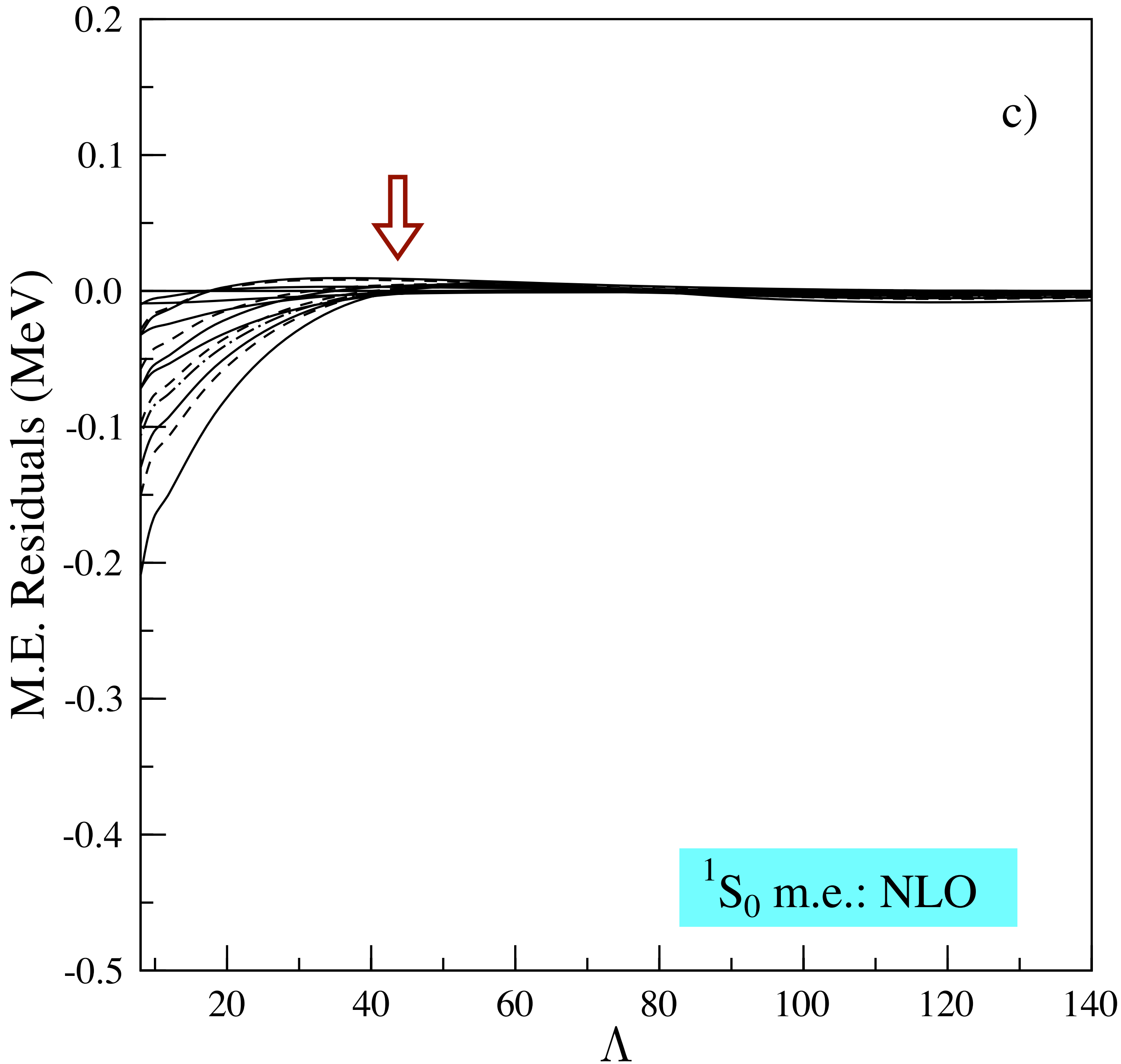


0
-2
-4
-6
-8
-10
-12
a)
Δ_{QT} (MeV)
$^1S_0$ m.e. contributions from Q
8
Λ
140

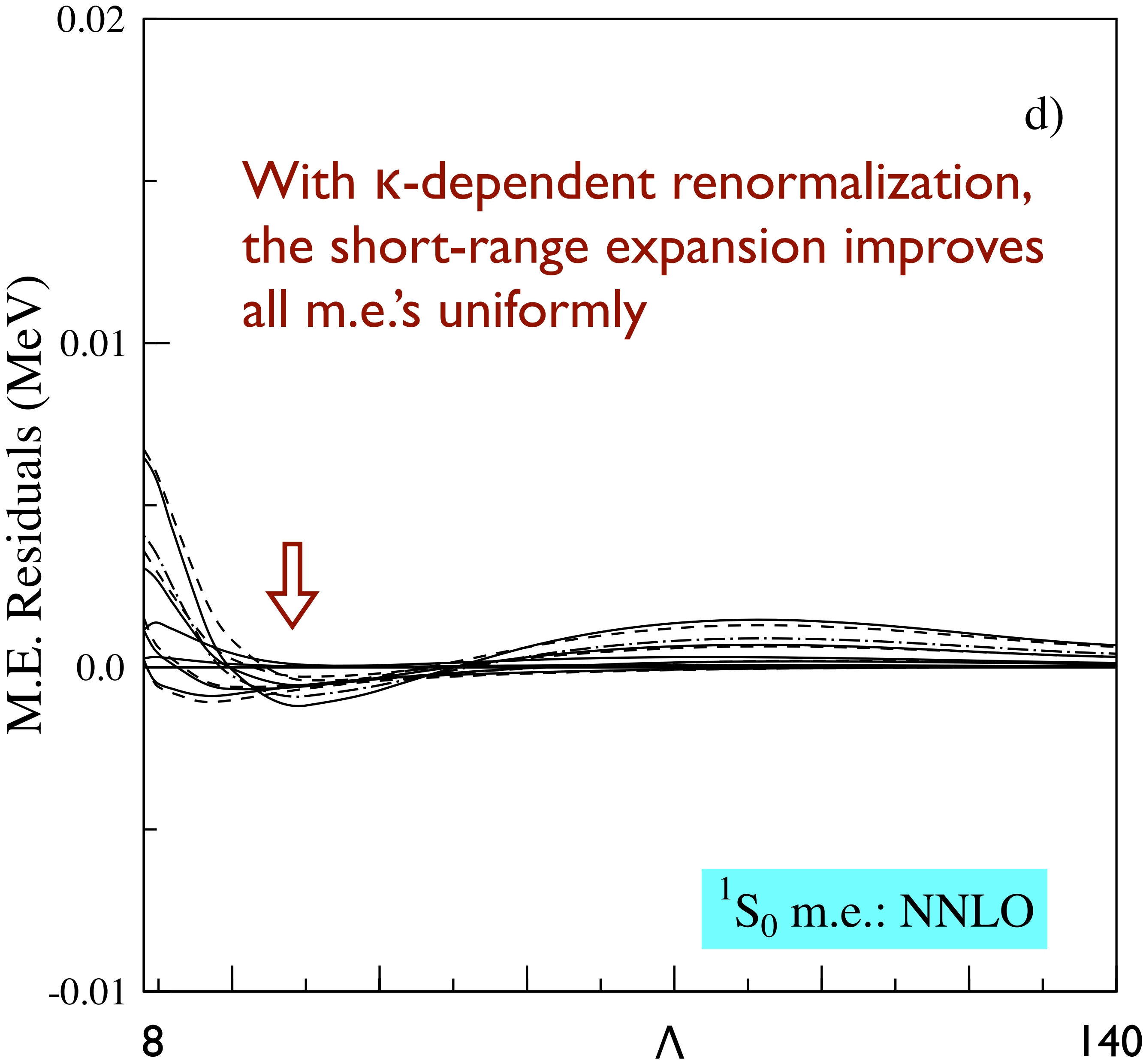
$$\Delta_{QT}(\Lambda) = \frac{E}{E - TQ}\left[V\frac{1}{E - QH}QV - V\frac{1}{E - Q_\Lambda H}Q_\Lambda V\right]\frac{E}{E - QT}$$

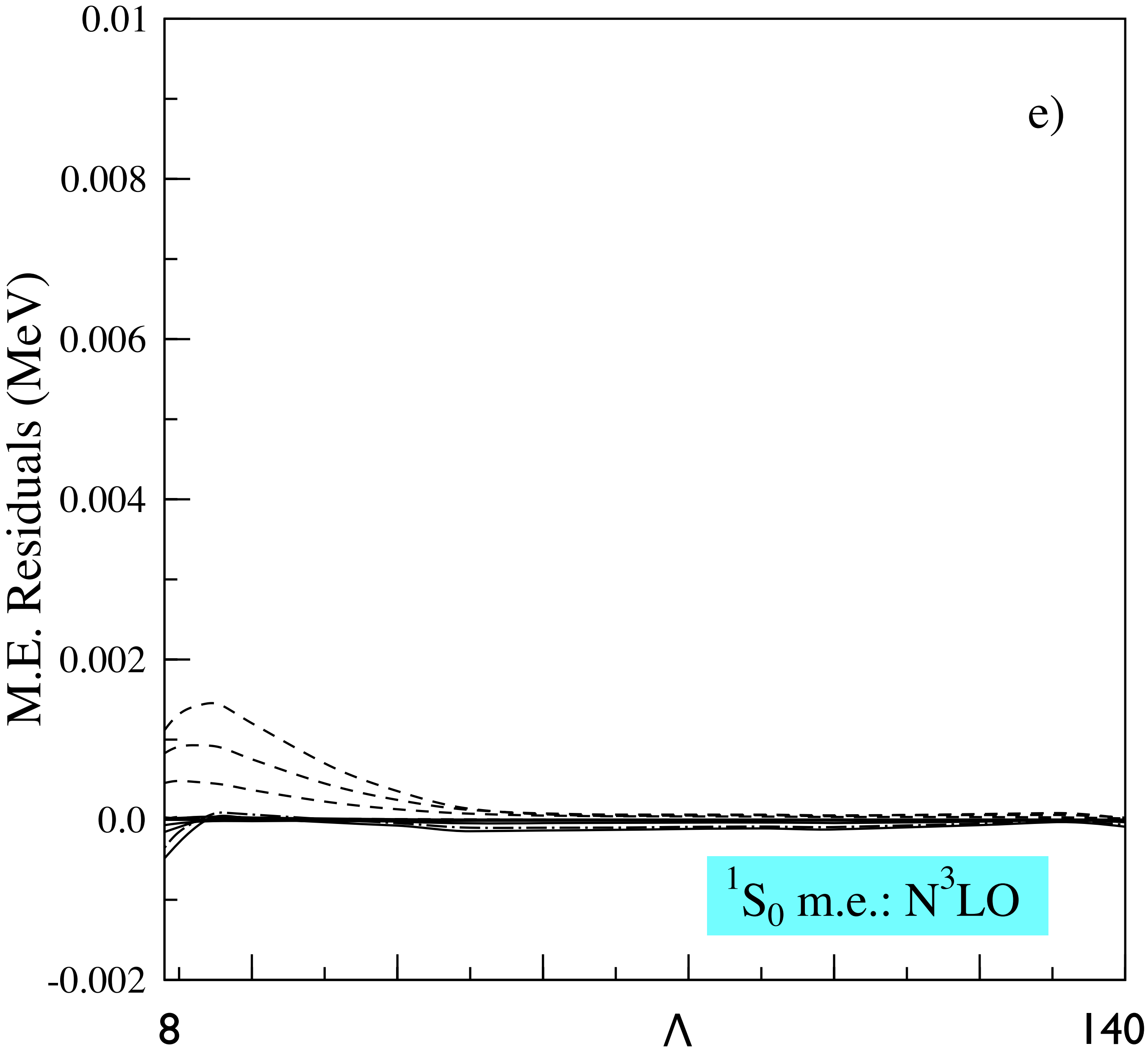
QT
summed





0.2
0.1
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
M.E. Residuals (MeV)
20
40
60
80
100
120
140
Λ
c)
$^1S_0$ m.e.: NLO


0.02
0.01
0.0
-0.01
M.E. Residuals (MeV)
d)
With κ-dependent renormalization,
the short-range expansion improves
all m.e.'s uniformly
1S0 m.e.: NNLO
8
Λ
140

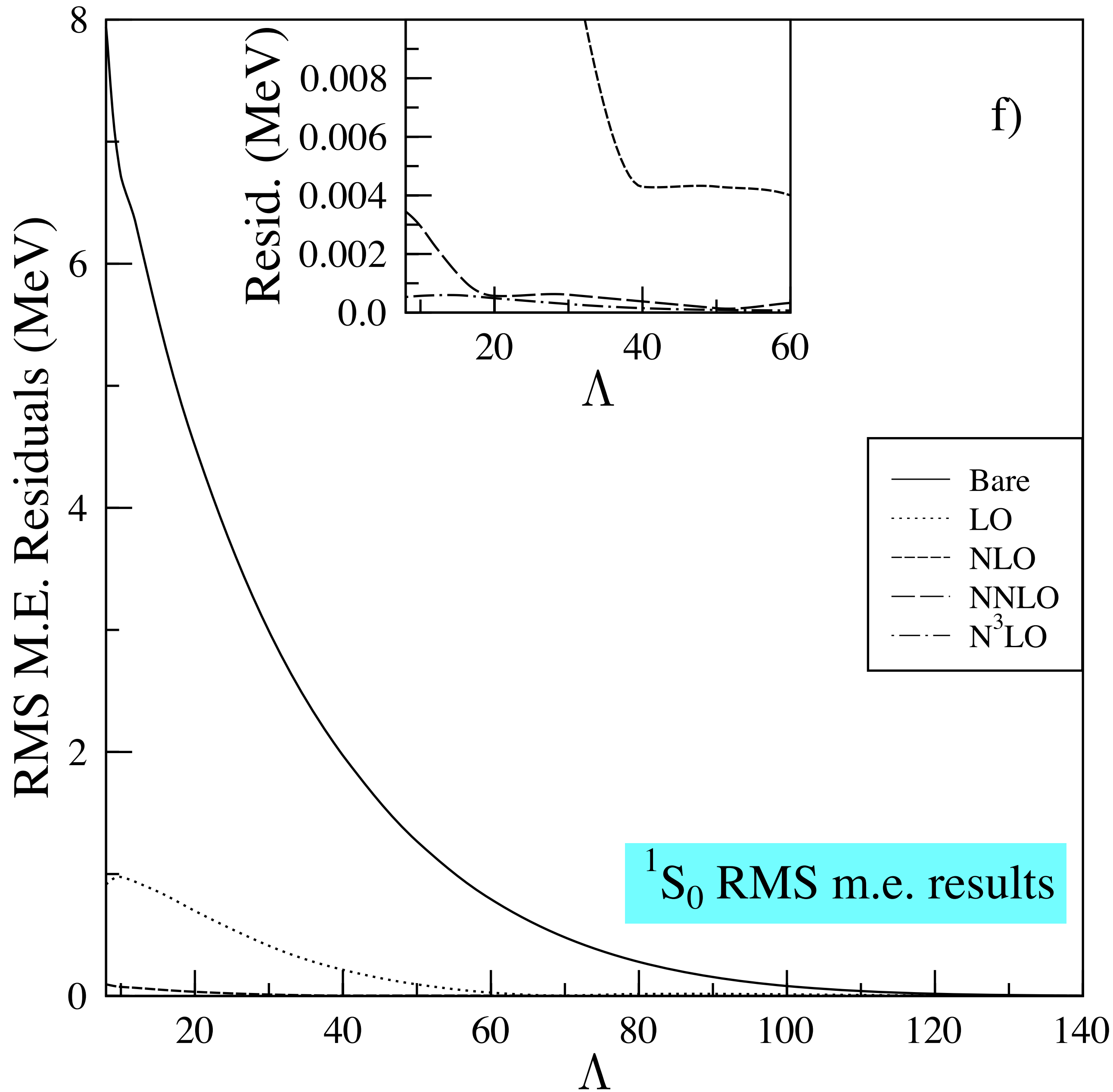


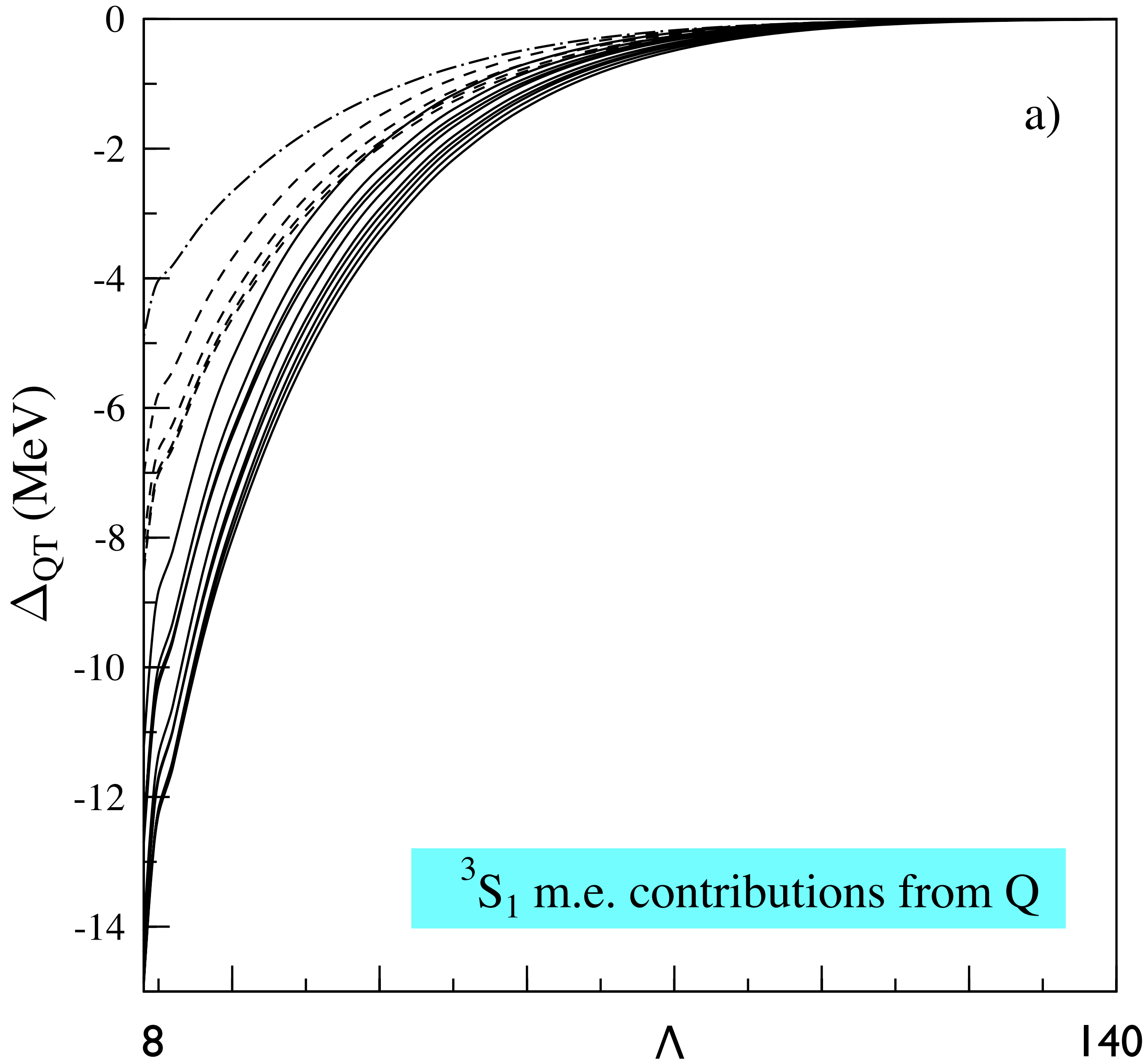
0.01
0.008
0.006
0.004
0.002
0.0
-0.002
M.E. Residuals (MeV)
e)
$^1S_0$ m.e.: N^3LO
8
Λ
140

Fitting procedure uses only m.e.'s with lowest quanta: in particular, has no knowledge of the edge states

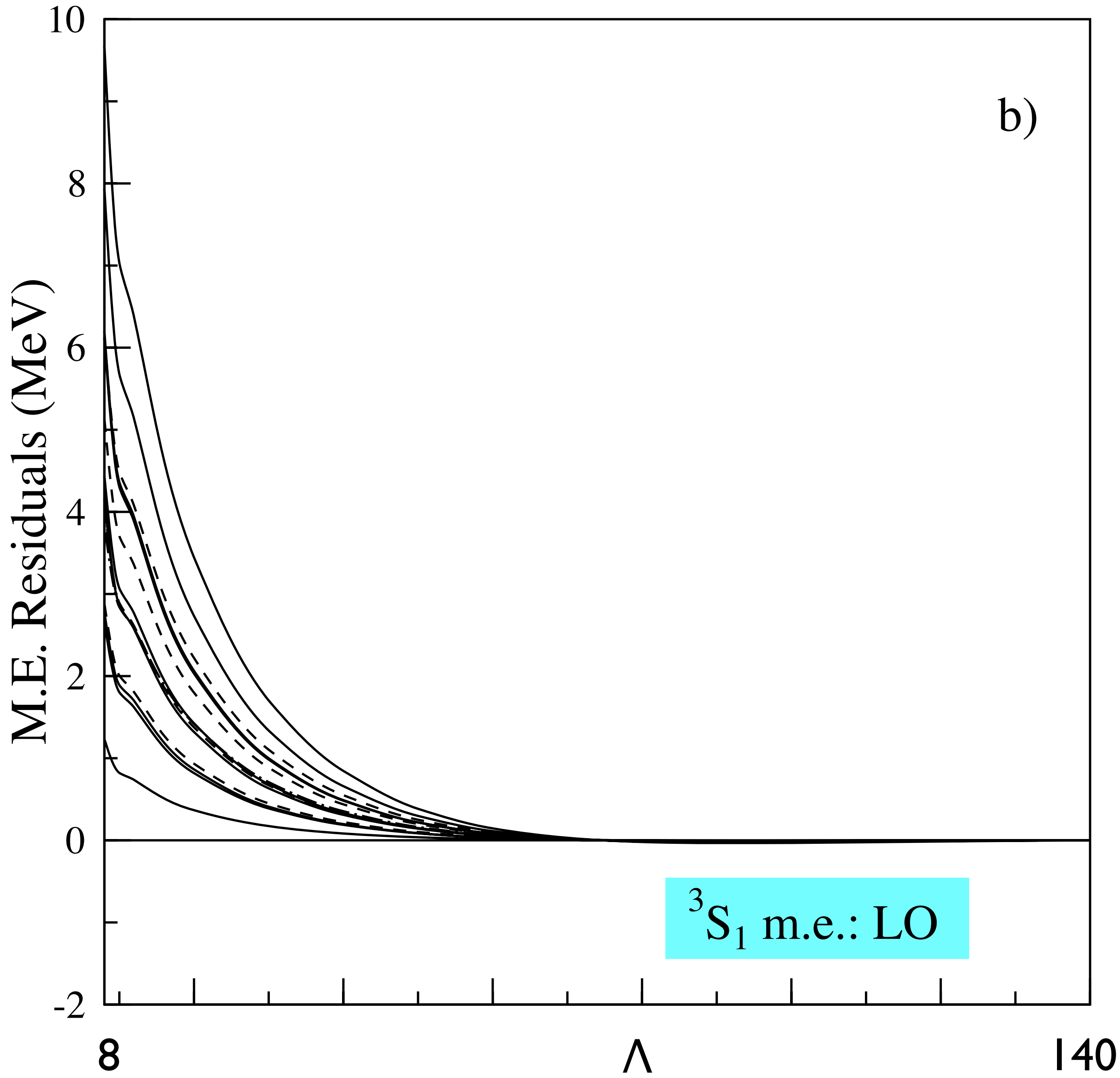
rms error based on all unconstrained H^{eff} m.e.'s (9-14)

result ~ 0.5 keV

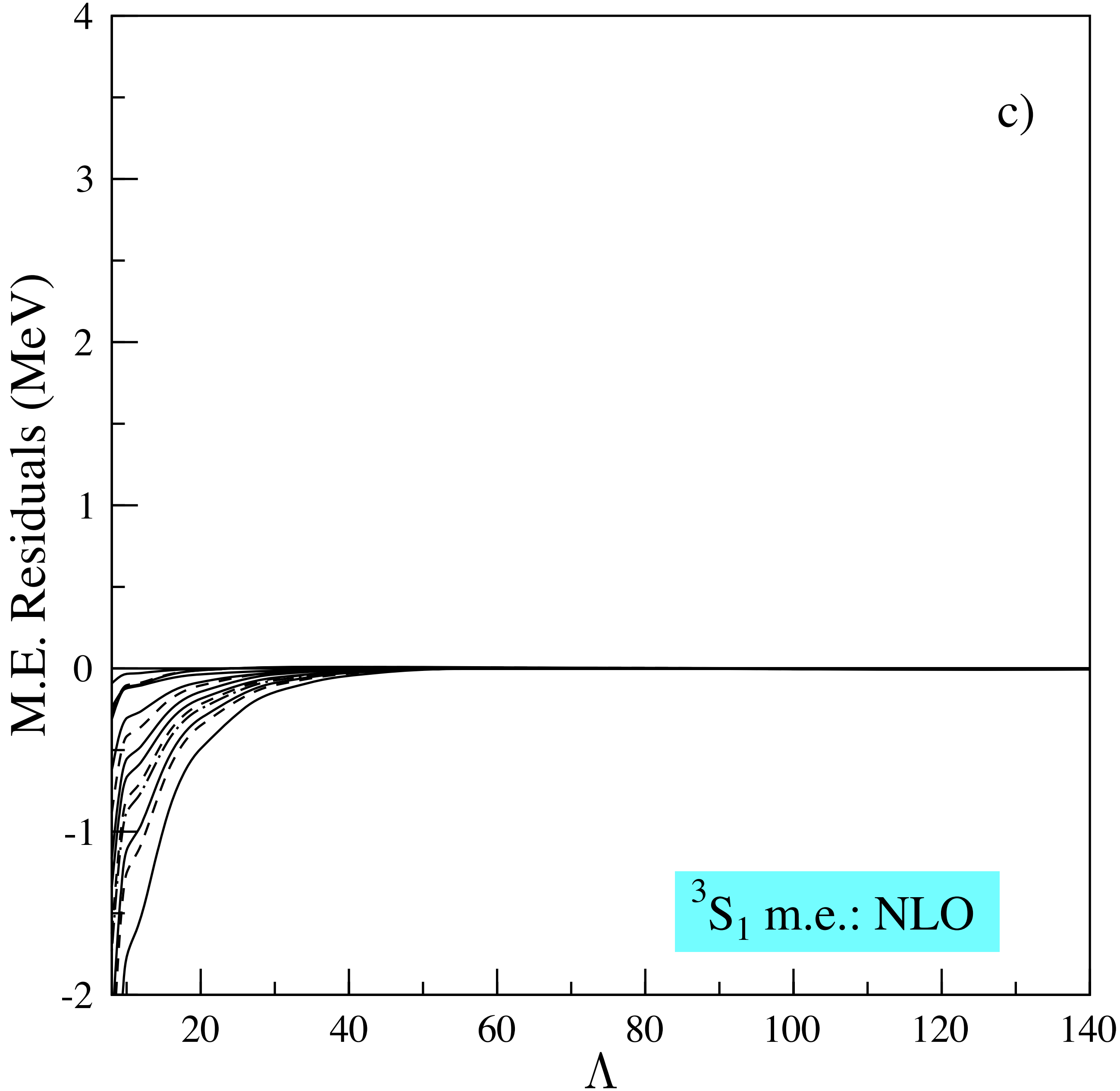




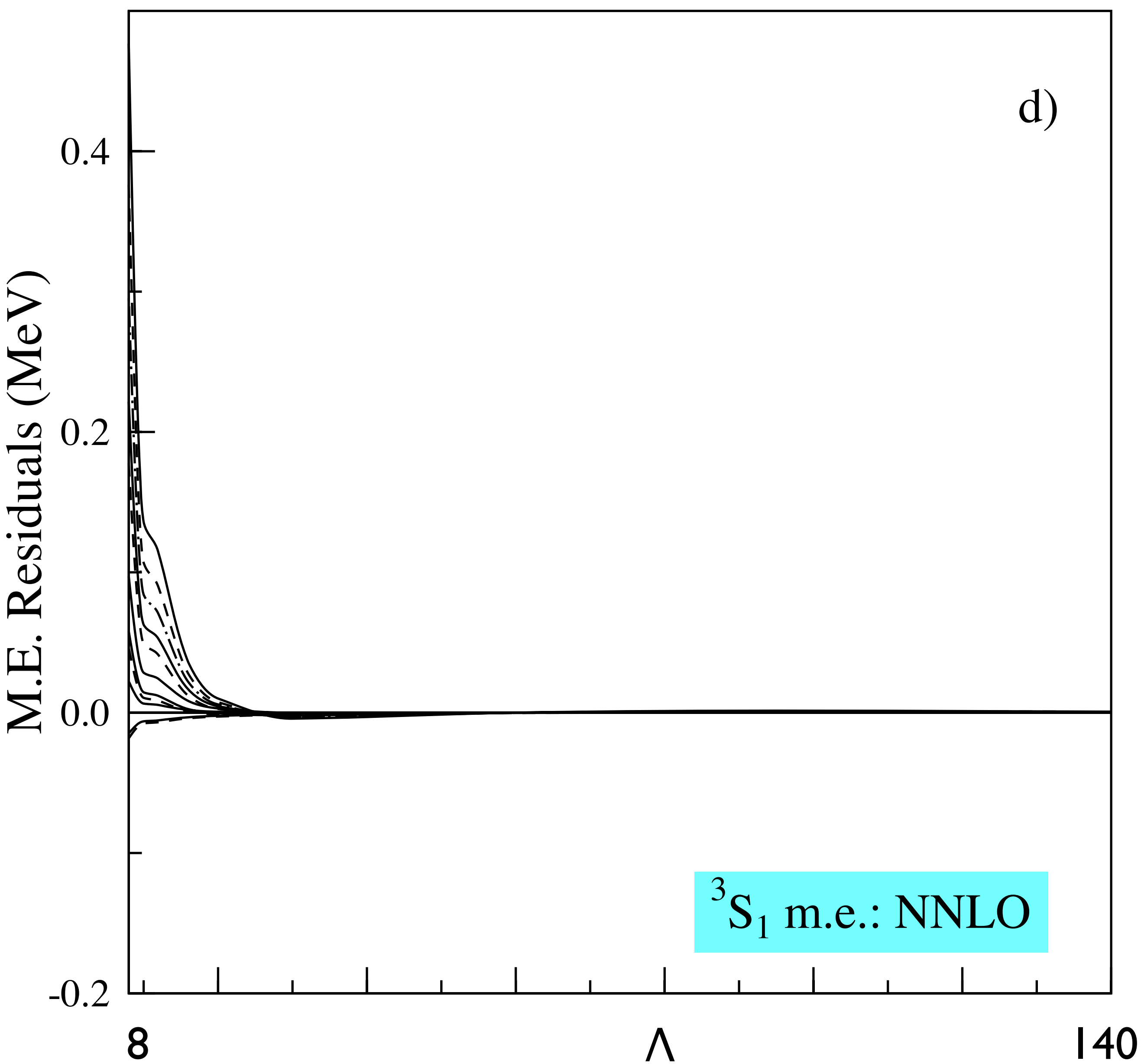
0
-2
-4
-6
-8
-10
-12
-14
Δ_{QT} (MeV)
a)
$^3S_1$ m.e. contributions from Q
8
Λ
140

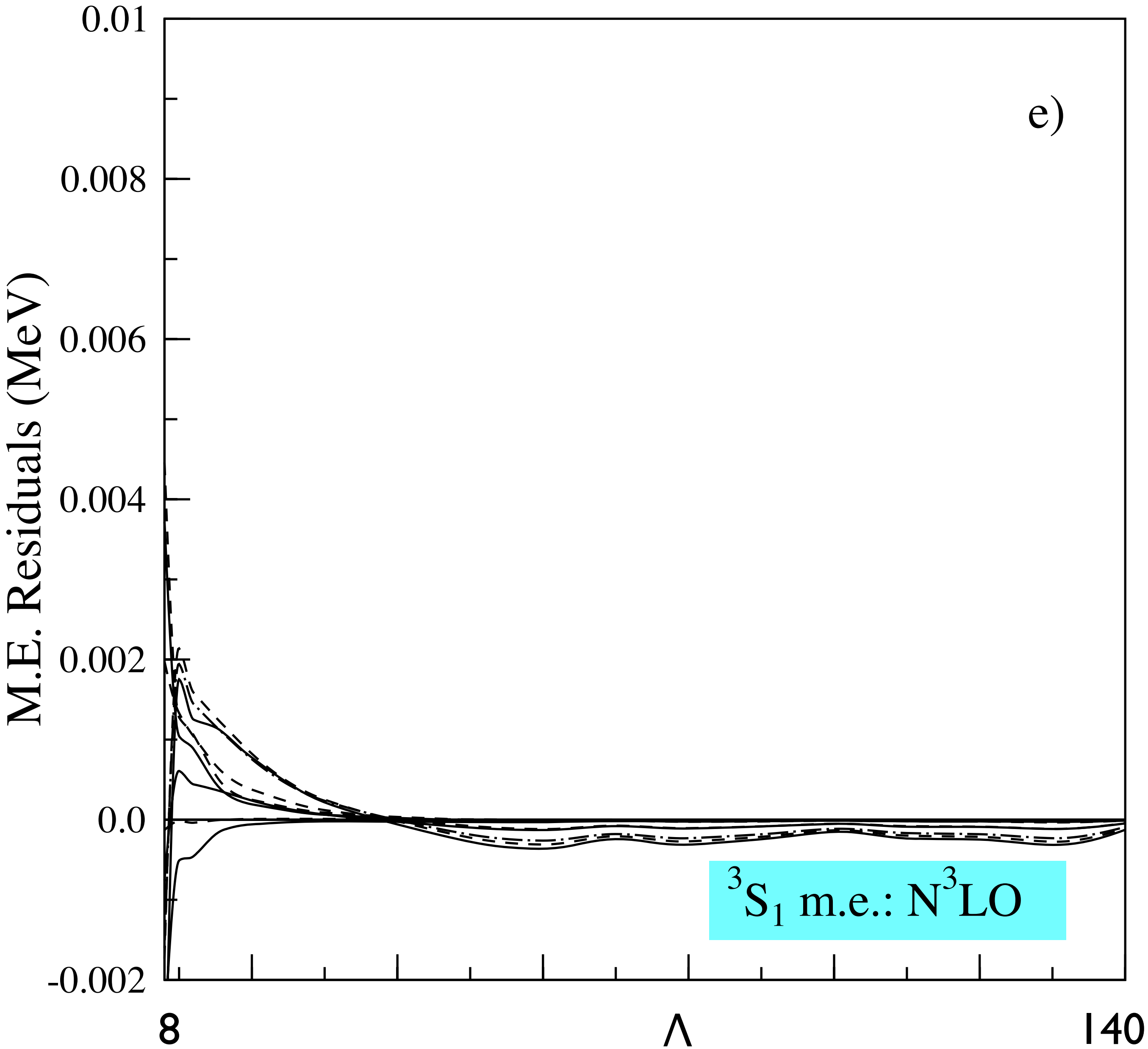


10
8
6
4
2
0
-2
M.E. Residuals (MeV)
b)
3S1 m.e.: LO
8
Λ
140

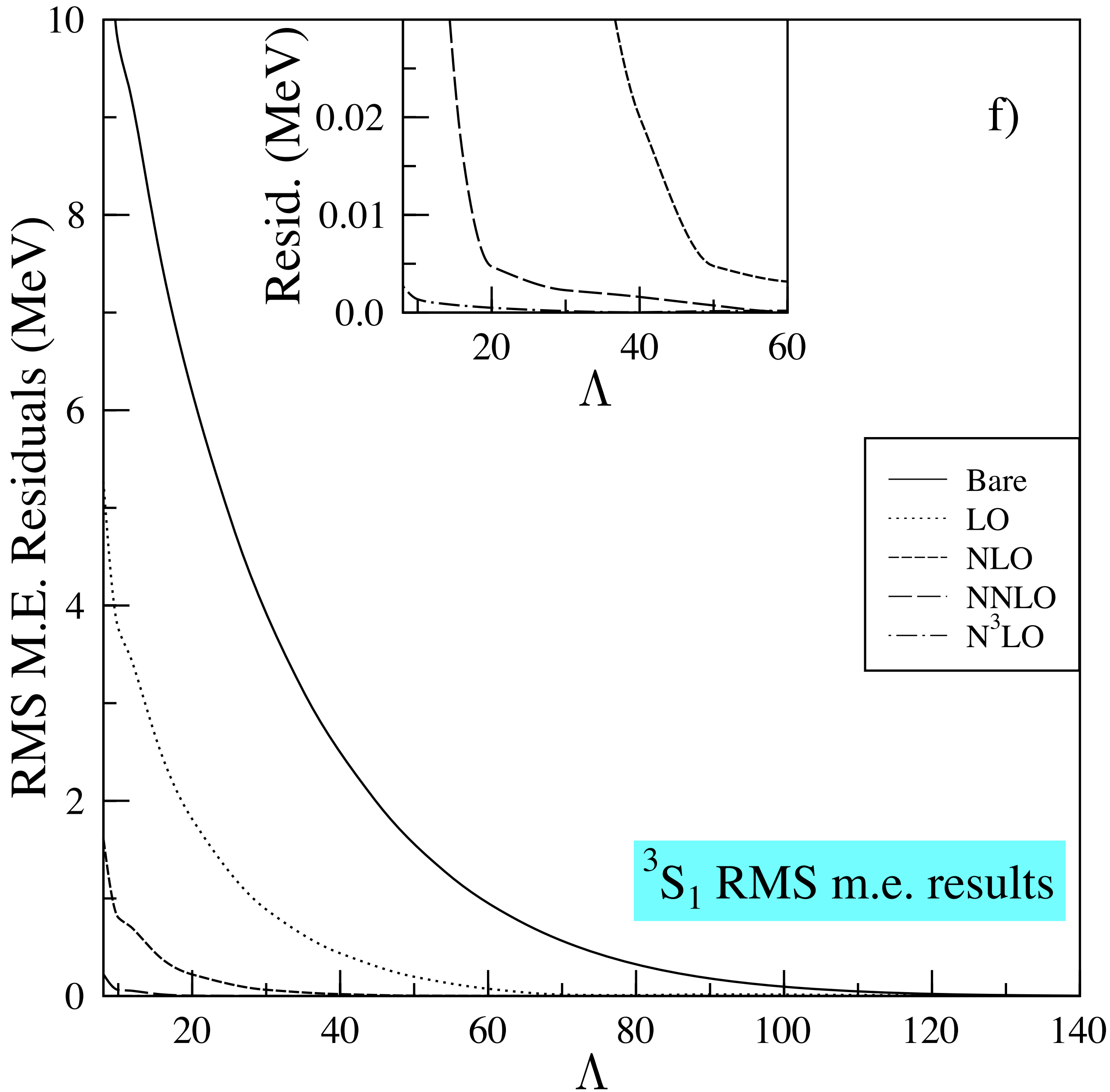


c)
$^3S_1$ m.e.: NLO
M.E. Residuals (MeV)
Λ
4
3
2
1
0
-1
-2
20
40
60
80
100
120
140


d)
M.E. Residuals (MeV)
0.4
0.2
0.0
-0.2
$^3S_1$ m.e.: NNLO
8
Λ
140



0.01
0.008
0.006
0.004
0.002
0.0
-0.002
M.E. Residuals (MeV)
e)
$^3S_1$ m.e.: N^3LO
8
Λ
140



RMS M.E. Residuals (MeV)
Λ
f)
Resid. (MeV)
Bare
LO
NLO
NNLO
N^{3}LO
$^3S_1$ RMS m.e. results

TABLE II: The effective interaction for LO through N^3LO, with $\Lambda_P = 8$ and b=1.7f.†

Channel	Couplings (MeV)							$\langle$M.E.$\rangle_{RMS}$ (MeV)	$\langle$Resid.$\rangle_{RMS}$ (keV)
	a^{S}_{LO}	a^{S}_{NLO}	$a^{S,22}_{NNLO}$	$a^{S,40}_{NNLO}$	$a^{S,42}_{N^3LO}$	$a^{S,60}_{N^3LO}$			
$^1S_0 - {}^1S_0$	-32.851	-2.081E-1	-2.111E-3	-1.276E-3	-7.045E-6	-1.8891E-6		7.94	0.53
$^3S_1 - {}^3S_1$	-62.517	-1.399	-5.509E-2	-1.160E-2	-5.789E-4	-1.444E-4		11.97	2.71
		a^{SD}_{NLO}	$a^{SD,22}_{NNLO}$	$a^{SD,04}_{NNLO}$	$a^{SD,42}_{N^3LO}$	$a^{SD,24}_{N^3LO}$	$a^{SD,06}_{N^3LO}$		
$^3S_1 - {}^3D_1$		2.200E-1	1.632E-2	2.656E-2	2.136E-4	3.041E-4	-1.504E-4	0.160	2.45
			a^{D}_{NNLO}		$a^{D}_{N^3LO}$				
$^1D_2 - {}^1D_2$			-6.062E-3		-1.189E-4			0.027	1.21
$^3D_1 - {}^3D_1$			-1.034E-2		-1.532E-4			0.051	2.27
$^3D_2 - {}^3D_2$			-3.048E-2		-5.238E-4			0.141	1.20
$^3D_3 - {}^3D_3$			-9.632E-2		-4.355E-3			0.303	122‡
					$a^{SD}_{N^3LO}$				
$^3D_3 - {}^3G_3$					3.529E-4			0.012	12.2‡
		a^{P}_{NLO}	a^{P}_{NNLO}		$a^{P,33}_{N^3LO}$	$a^{P,51}_{N^3LO}$			
$^1P_1 - {}^1P_1$		-8.594E-1	-7.112E-3		-6.822E-5	1.004E-5		0.694	0.11
$^3P_0 - {}^3P_0$		-1.641	-1.833E-2		-2.920E-4	-1.952E-4		1.283	2.26
$^3P_1 - {}^3P_1$		-1.892	-1.588E-2		-1.561E-4	-6.737E-6		1.526	0.08
$^3P_2 - {}^3P_2$		-4.513E-1	-1.257E-2		-5.803E-4	-1.421E-4		0.285	5.61
			a^{PF}_{NNLO}		$a^{PF,33}_{N^3LO}$	$a^{PF,15}_{N^3LO}$			
$^3P_2 - {}^3F_2$			-4.983E-3		1.729E-5	-5.166E-5		0.034	1.43
					$a^{F}_{N^3LO}$				
$^1F_3 - {}^1F_3$					-3.135E-4			0.007	1.03
$^3F_2 - {}^3F_2$					-8.537E-4			0.020	2.34
$^3F_3 - {}^3F_3$					-2.647E-4			0.006	0.61
$^3F_4 - {}^3F_4$					-5.169E-4			0.008	6.23

† The appropriate LO, NLO, and NNLO interactions are obtained by truncating the table at the desired order.
‡ An N^4LO calculation in the $^3D_3 - {}^3D_3$ channel yields $a^{3D3,44}_{N^4LO}$=-2.510E-4 MeV and $a^{3D3,62}_{N^4LO}$ = -7.550E-5 MeV, and reduces $\langle$Resid.$\rangle_{RMS}$ to 22.3 keV; and in the $^3D_3 - {}^3G_3$ channel yields $a^{DG,44}_{N^4LO}$ = -2.141E-5 MeV and $a^{DG,26}_{N^4LO}$ = 1.180E-5 MeV and reduces $\langle$Resid.$\rangle_{RMS}$ to 3.26 keV.

Various Properties of H^{eff}

- Convergence patterns similar to EFT
  - spin-aligned channels -- $^3S_1$,$^3P_2$,$^3D_3$ -- show slowest convergence
  - convergence within each channel highly regular: assume scattering in Q generates an effective local potential $V_0 e^{-r_{12}^2/a^2}$

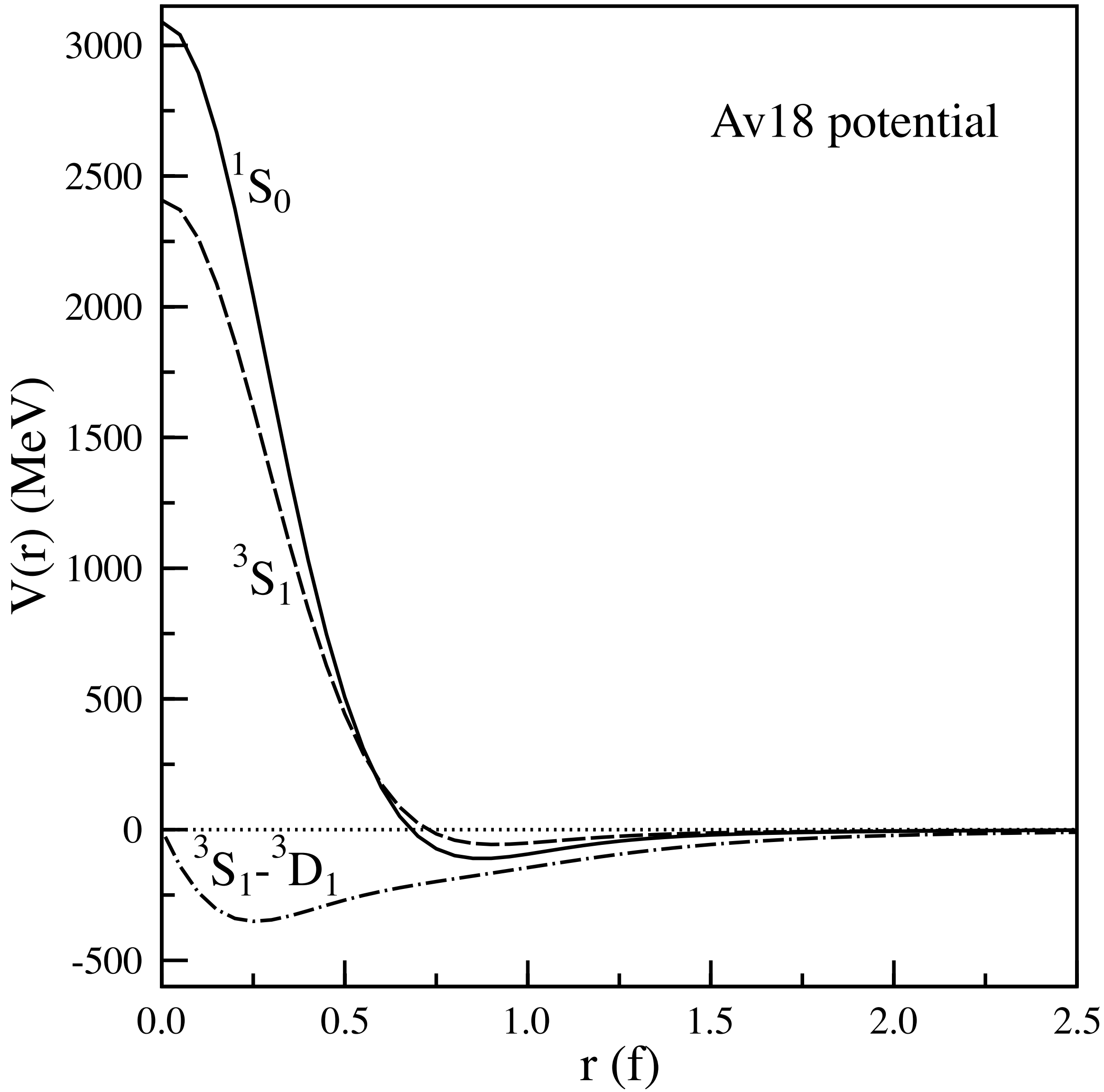
		a_{LO}	a_{NLO}	a^{22}_{NNLO}	a^{40}_{NNLO}	$a^{42}_{N^3LO}$	$a^{60}_{N^3LO}$
$^1S_0$	predicted	1	$6.3E-3$	$6.7E-5$	$2.0E-5$	$3.0E-7$	$4.2E-8$
	found	1	$6.3E-3$	$6.4E-5$	$3.9E-5$	$2.1E-7$	$5.7E-8$
$^3S_1$	predicted	1	$2.2E-2$	$8.3E-4$	$2.5E-4$	$13.1E-6$	$1.9E-6$
	found	1	$2.2E-2$	$8.8E-4$	$1.9E-4$	$9.3E-6$	$2.3E-6$

  - the predicted parameter governing expansion is $\left[\frac{a^2}{a^2+2b^2}\right]$

$^1S_0$ a ~ 0.39f V0 ~ -1.50 GeV

$^3S_1$ a ~ 0.75f V0 ~ -0.42 GeV

contrasting ranges



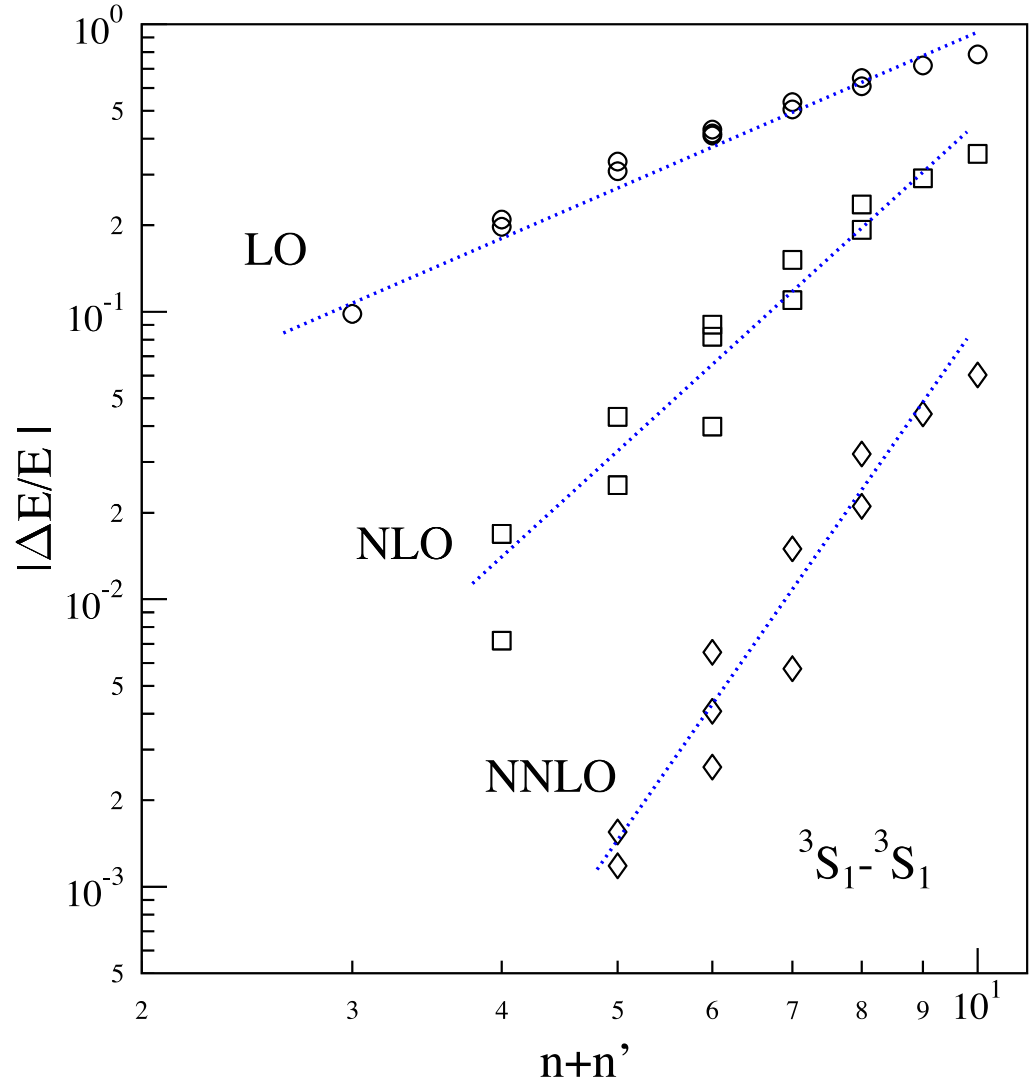
$$\langle V_{SD} \rangle \frac{1}{\langle E \rangle} \langle V_{DS} \rangle$$

Consistent with $^3S_1$ coupling to $^3D_1$ to generate a more extended interaction, with corresponding enhancements due to favorable <E>

- Lepage plot: test whether contact-gradient expansion is systematic -- that improvement is not a matter of additional parameters

  - errors at LO predicted to be linear in (n'+n); errors in NLO and NNLO predicted to be quadratic, cubic in (n',n)

Unconstrained m.e.s displayed;
good convergence even for most exotic high (n',n) matrix elements of Heff;
expected steepening with order



- Various spectral measures of representation of H^{eff} in $^3S_1$-$^3D_1$
  - ground-state A=2 error -- 40 eV
  - spectral first moment accurate to 1.81 keV
  - rms average deviation in eigenvalue spacing 3.52 keV
  - eigenvalue overlaps > 99.99

- Physics: completely removed nearest-shell strong coupling of P,Q via T
  - this instructs us to introduce large renormalizations of bare T, V, and VGV -- the cross-talk of propagating QV and QT

- Rapidly converging, systematic short-range expansion
  - reproduces all nonedge matrix elements to high accuracy
  - in our example, 78 otherwise poorly reproduced edge m.e.s are shown to be reproduced by the same set of strong parameters, but only if the analytic dependence on long-range physics encoded in κ is included
  - this dependence on $\kappa = \sqrt{2|E|/\hbar\omega}$ exists for an isolated state: has nothing to do with BH or other state-dependence

- For HOBET this demonstrates that the needed expansion exist
    - a necessary condition if one hopes to fix the strong coefficients directly from data (an issue also connected to $|\tilde{\alpha}\rangle$), to avoid introducing a potential to take one from QCD to the HO scale
- But the results also important for potential-based approaches like the SM, HOBETp
    - the state-dependence handled through numerical techniques like Lee-Suzuki or in the BH equation
    - the possibility of porting SM techniques into HOBETp, to make those techniques more powerful
- Plane-wave limit (e.g., V_{low-k}): $b \rightarrow \infty$, $\Lambda_P \rightarrow \infty$, Λ_P/b fixed (thus $\kappa \rightarrow \infty$)

State-dependence and κ

- The BH equation is traditionally solved numerically: self-consistency generates state-dependence in $H^{eff}(E)$ that is essential for a proper ET.

- Alternatively, SM approaches often employ a transformation due to Lee and Suzuki to removed energy-dependence

  - Hermitian, energy-independent: this violates the basic rule of an ET that the P-space wave functions are restrictions

  - NonHermitian, energy-dependent: this can be done

- Will argue that these techniques -- nontrivial numerically -- are obscuring the fact that the state dependence is the long-range problem addressed analytically here

Reorganized BH equation identifies four sources of state dependence

- The rescattering of QT to all orders (quite sensitive to |E|)

$$\langle\alpha|T\frac{1}{E-QT}QT|\beta\rangle \xrightarrow{\text{new ``bare''}} \langle\alpha|T|\tilde{\beta}(\kappa)\rangle$$

- The effects of QT to all orders on matrix elements linear in V

$$\langle\alpha|\frac{E}{E-TQ}V\frac{E}{E-QT}|\beta\rangle \xrightarrow{\text{new ``bare''}} \langle\tilde{\alpha}(\kappa)|T|\tilde{\beta}(\kappa)\rangle$$

- The matrix elements of the short-range operators

$$\langle\alpha|\frac{E}{E-TQ}\bar{O}\frac{E}{E-QT}|\beta\rangle \longrightarrow \langle\tilde{\alpha}(\kappa)|\bar{O}|\tilde{\beta}(\kappa)\rangle$$

- The implicit energy dependence embedded in the strong operators

$$\bar{O} \equiv V\frac{E}{E-QH}QV \rightarrow \bar{O}(E)$$

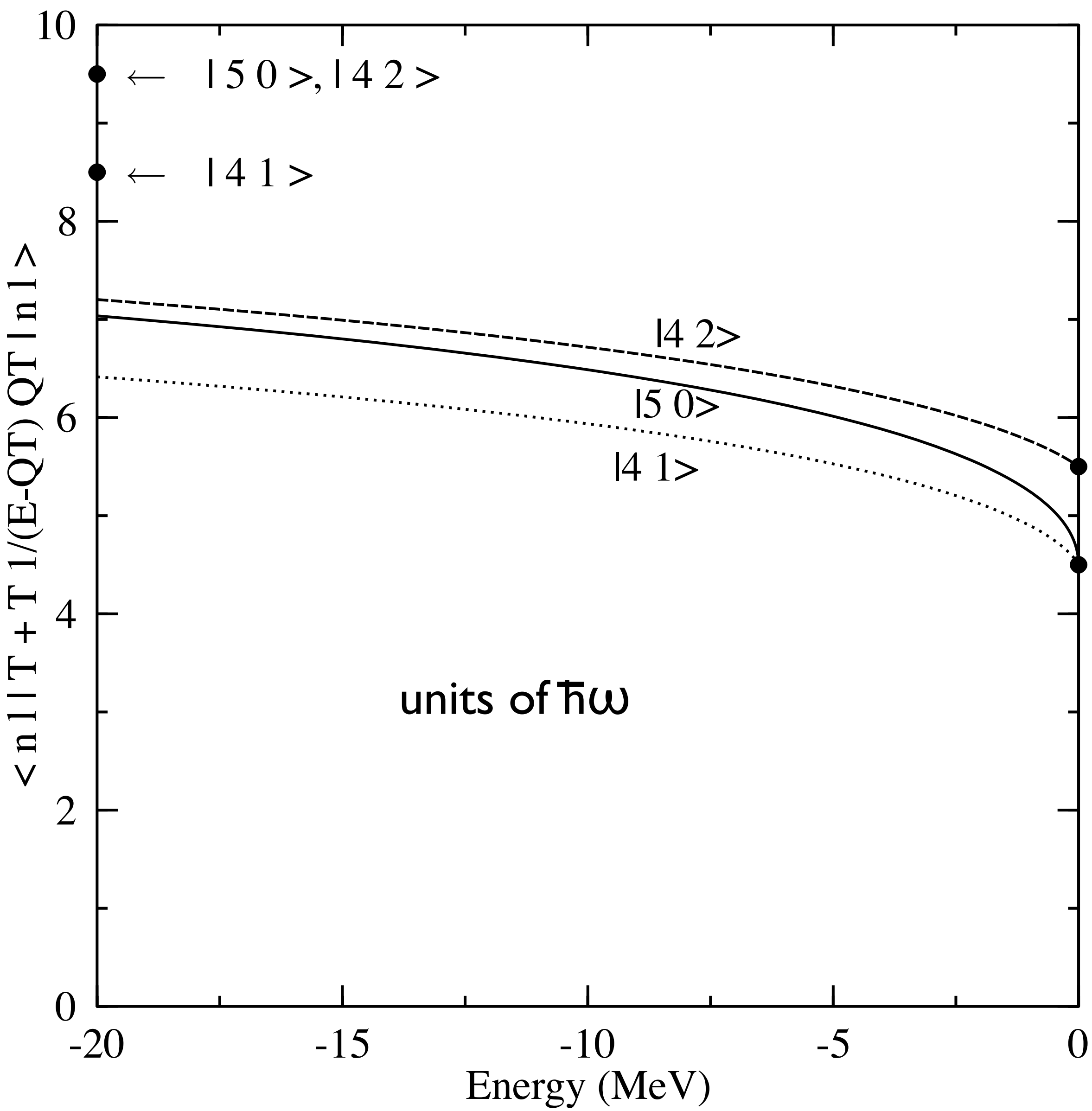
All but the last have been isolated analytically: sizes?

Rescattering of T:

Large shifts ~ 2 ħω over 20 MeV

Binding energy of 20 MeV still far from asymptotic

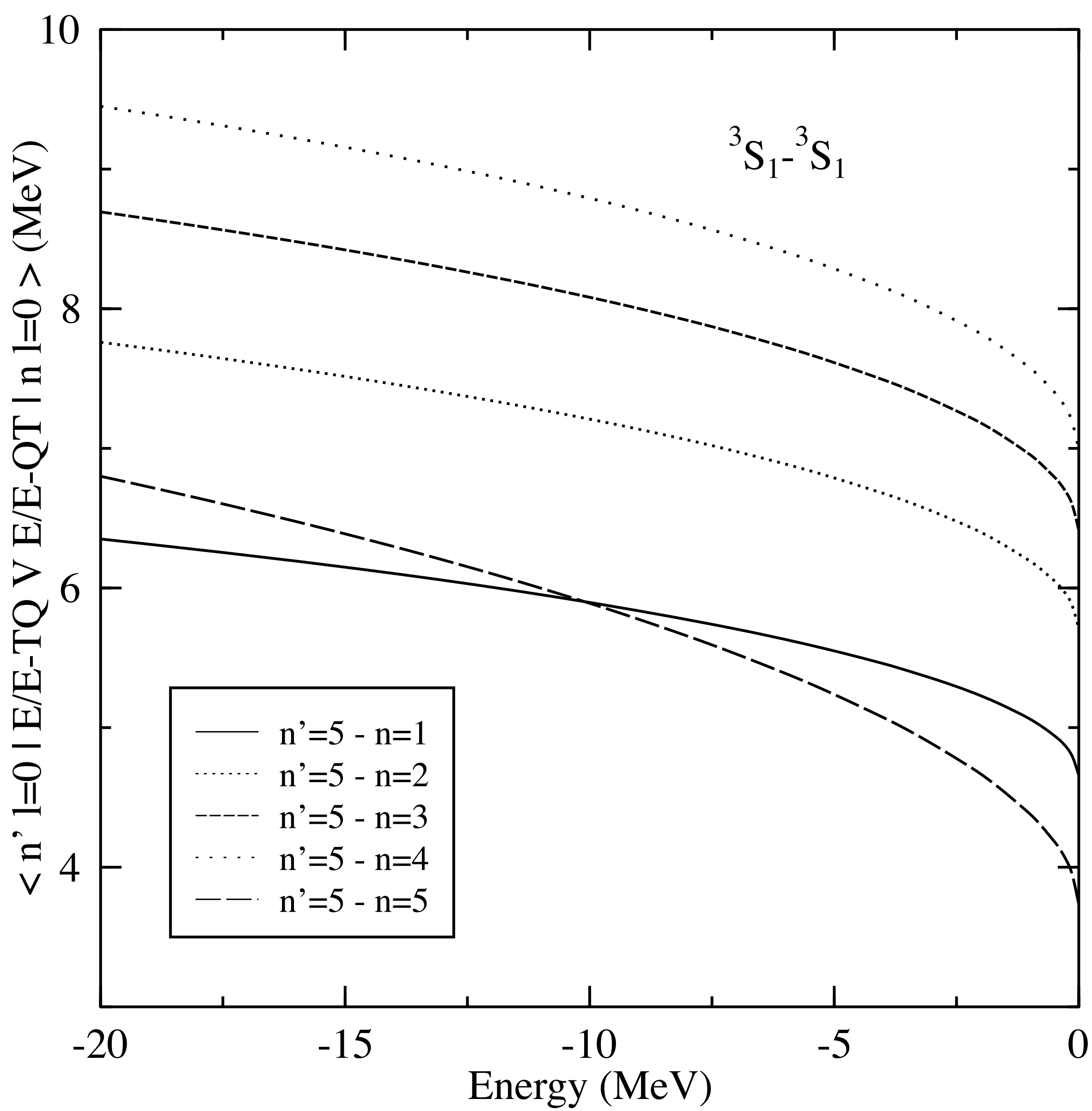
Effects isolated in double-edge matrix elements



Effects of QT to all orders on matrix elements linear in V:

Shifts of typically 2-3 MeV over 20 MeV

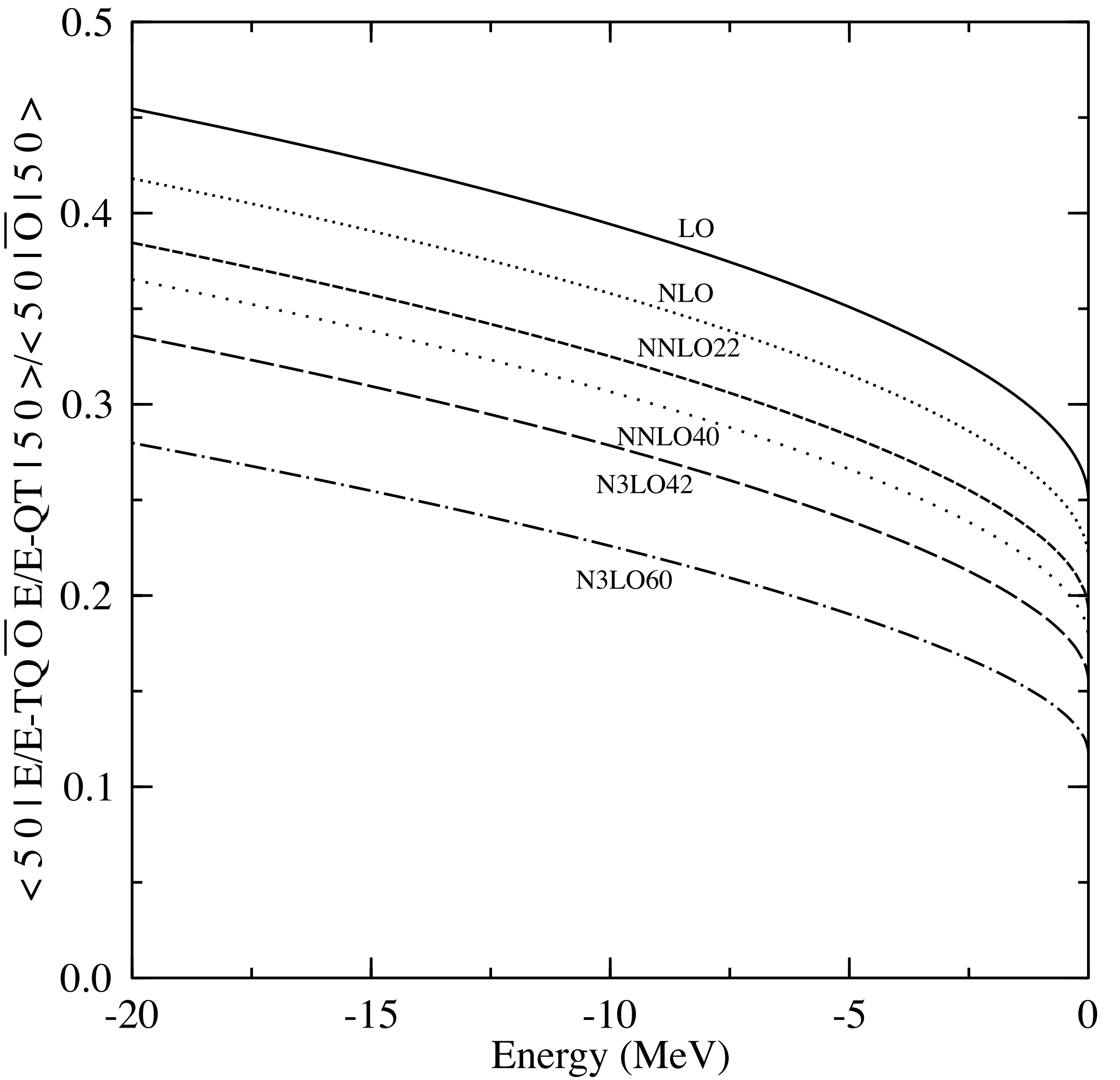
All edge-state matrix elements altered



Effects of QT to all orders on matrix elements of $\overline{O}$:

Factor-of-two renormalization of strong m.e.s over 20 MeV

Illustrated for double-edge s-wave matrix elements -- single-edge m.e. changes would be 50% as large



The implicit dependence in $\overline{O}(E)$

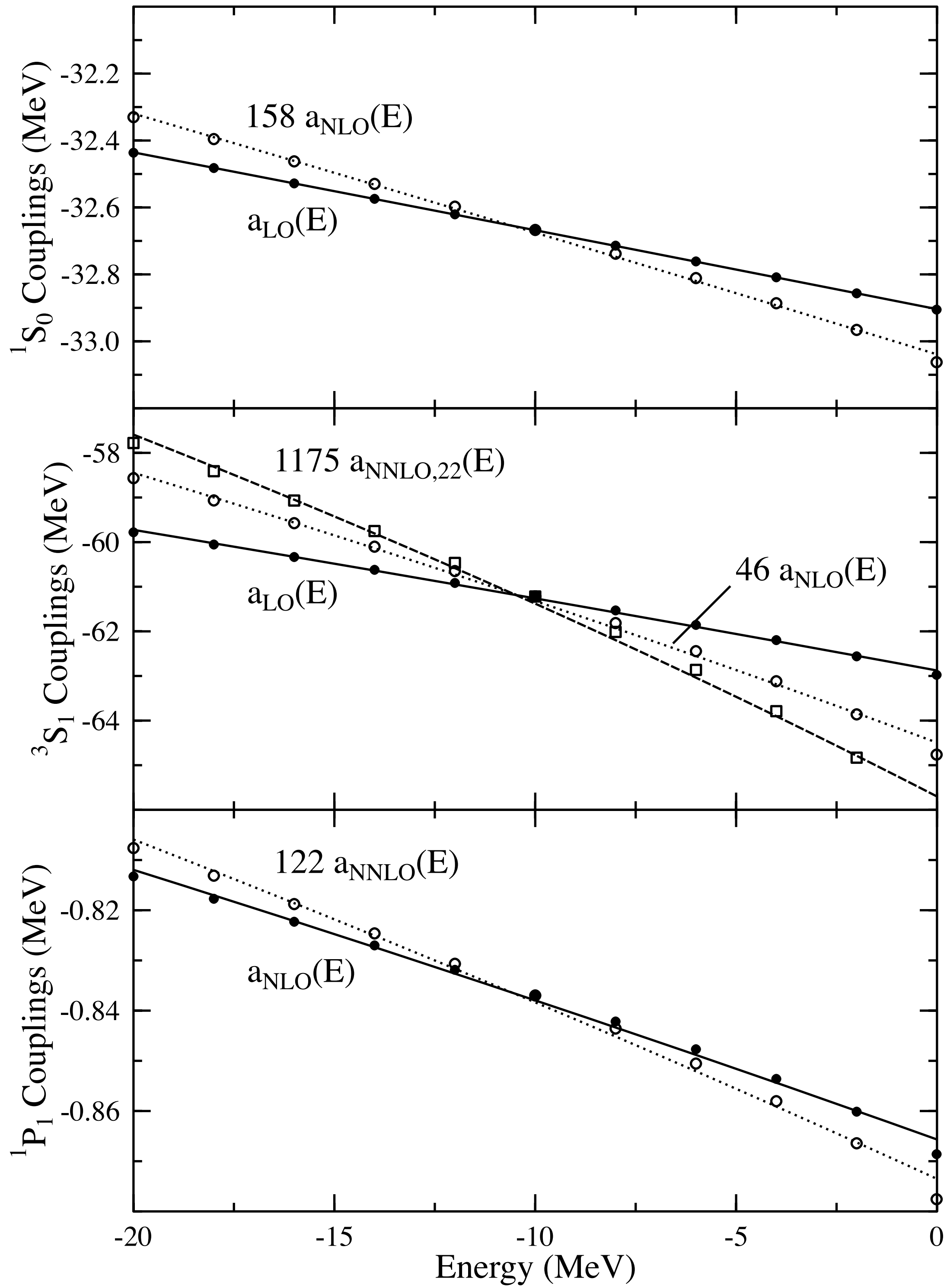
1-5% effects

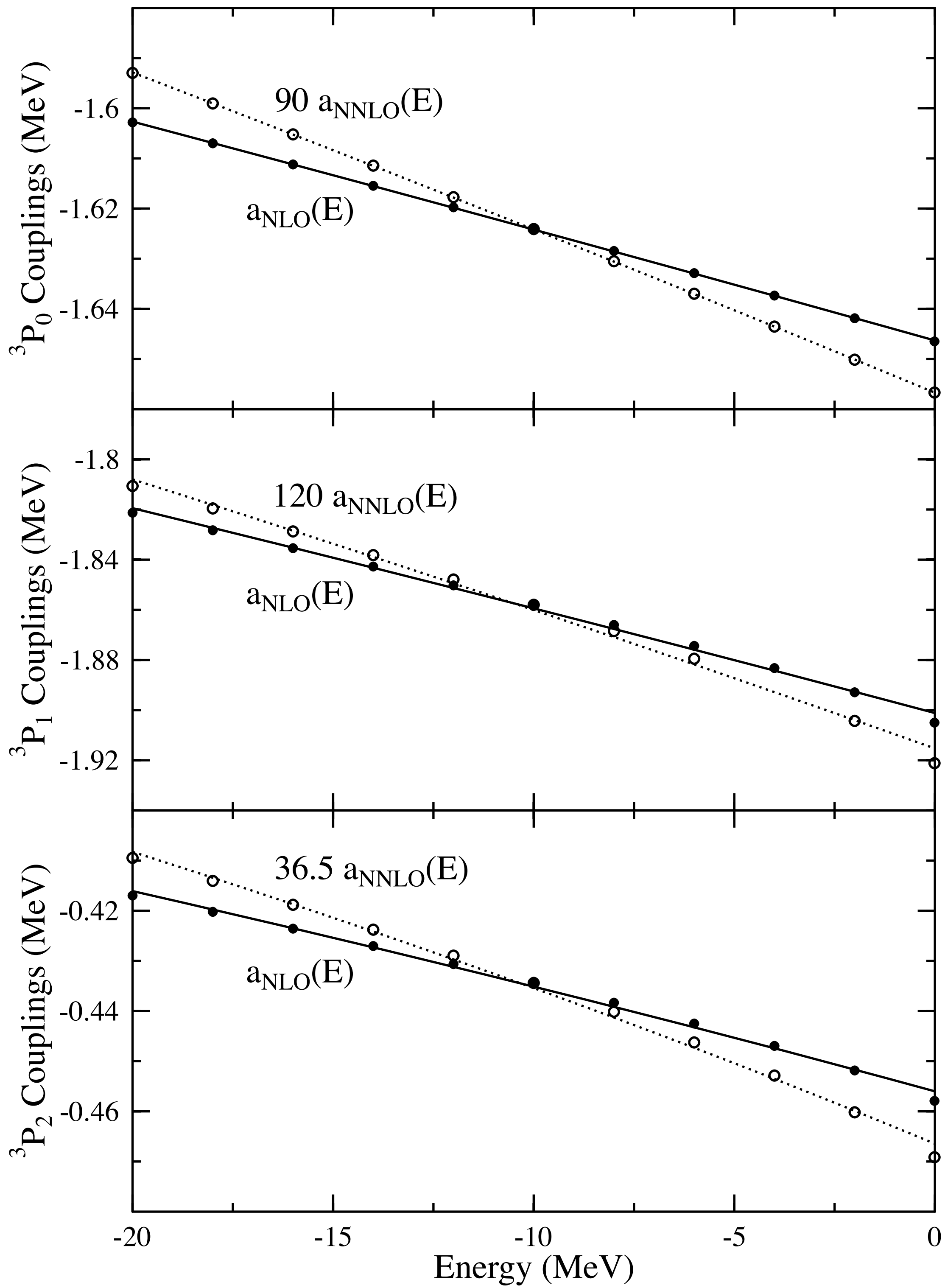
Correlates perfectly with convergence properties: slower convergence ⇒ larger implicit E-dependence

Higher order ⇒ larger spatial scale ⇒ lower E scale ⇒ stronger E dependence

Dependence linear

One concludes that 3S-3D channel is by far the most affected -- largest $\overline{O}$, slower convergence





90 $a_{NNLO}(E)$
$a_{NLO}(E)$
120 $a_{NNLO}(E)$
$a_{NLO}(E)$
36.5 $a_{NNLO}(E)$
$a_{NLO}(E)$
$^3P_0$ Couplings (MeV)
$^3P_1$ Couplings (MeV)
$^3P_2$ Couplings (MeV)
-1.6
-1.62
-1.64
-1.8
-1.84
-1.88
-1.92
-0.42
-0.44
-0.46
-20
-15
-10
-5
0
Energy (MeV)

We can fold these effects together to examine impact on 3S-3D channel

TABLE III: Spectral property variations in $H^{eff}(E)$ over 10 MeV

Term	Parameter	1st Moment Shift (MeV)	RMS Level Variation (MeV)	Wave Function Overlaps
$\langle\alpha\|T\|\widetilde{\beta}\rangle$	κ	2.554	1.107	95.75-99.74%
$\langle\widetilde{\alpha}\|V\|\widetilde{\beta}\rangle$	κ	0.272	0.901	99.35-99.82%
$\langle\widetilde{\alpha}\|\bar{O}\|\widetilde{\beta}\rangle$	κ	-0.239	0.957	99.51-99.99%
$\langle\alpha\|\bar{O}(E)\|\beta\rangle$	implicit	0.135	0.107	99.95-100%

- By our various spectral measures, 95% of the energy dependence in the 3S-3D channel is explicit, isolated in κ
  - ignoring implicit dependence induces ~100 keV drift over 20 MeV
  - $^1S_0$ effects would be ~ 20 keV

- If one reaches a numerical accuracy where corrections are desired,

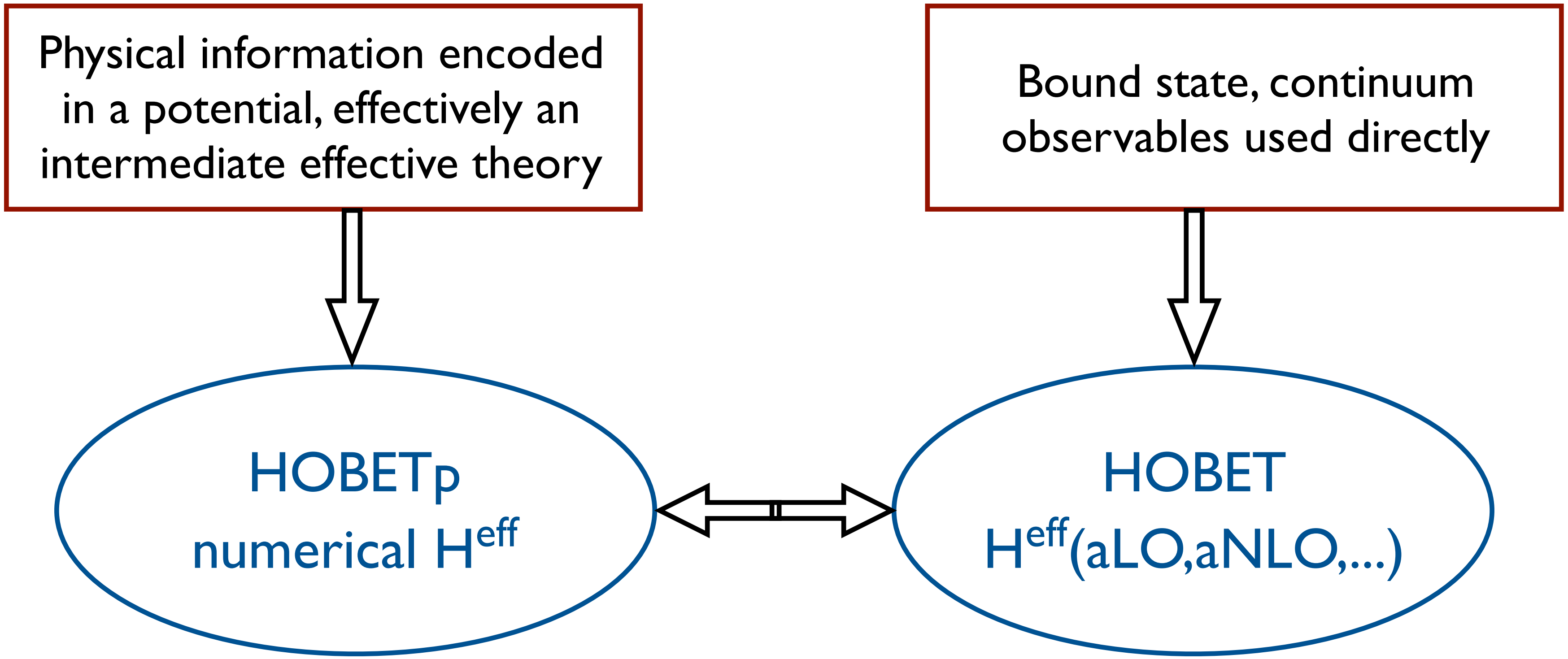
$$V\frac{1}{E-QH}QV \sim V\frac{1}{E_0-QH}QV + V\frac{1}{E_0-QH}(E_0-E)\frac{1}{E_0-QH}QV + \ldots$$

$$\sim \bar{O}(E_0) + (E_0-E)\bar{O}'(E_0)$$

Remarkably simple result: The long-range physics imbedded in κ that is needed in constructing a systematic expansion for the HOBET H^{eff} in the case of an isolated state, operationally also defines the state dependence

Not at all surprising: QT is the only source of strong nearest-shell coupling of P and Q

Short-range physics in Q corresponds to large excitation scales, thus making binding energy differences largely irrelevant

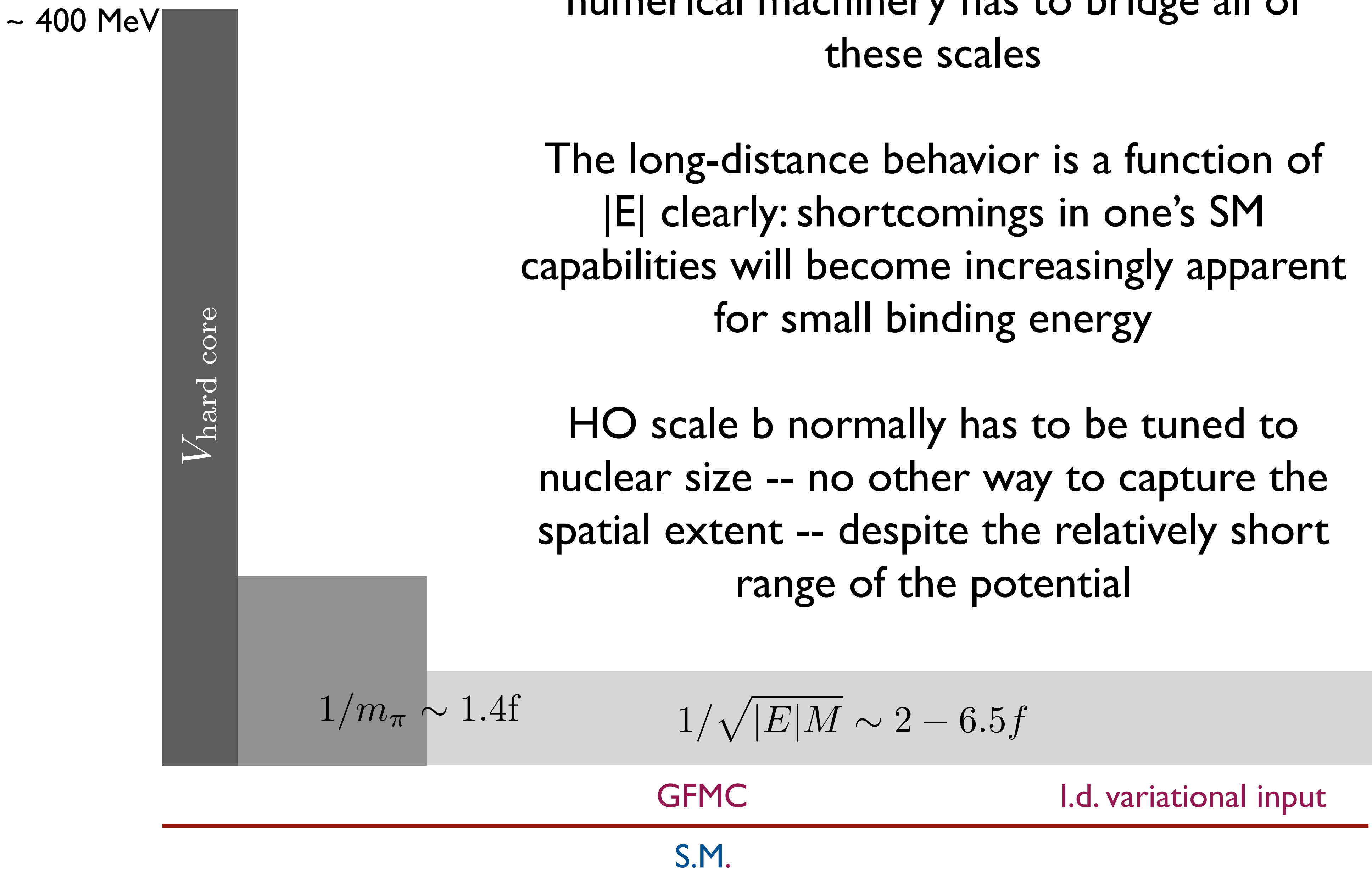


Scales and Calculations

Very difficult problem numerically if one's numerical machinery has to bridge all of these scales

The long-distance behavior is a function of |E| clearly: shortcomings in one's SM capabilities will become increasingly apparent for small binding energy

HO scale b normally has to be tuned to nuclear size -- no other way to capture the spatial extent -- despite the relatively short range of the potential



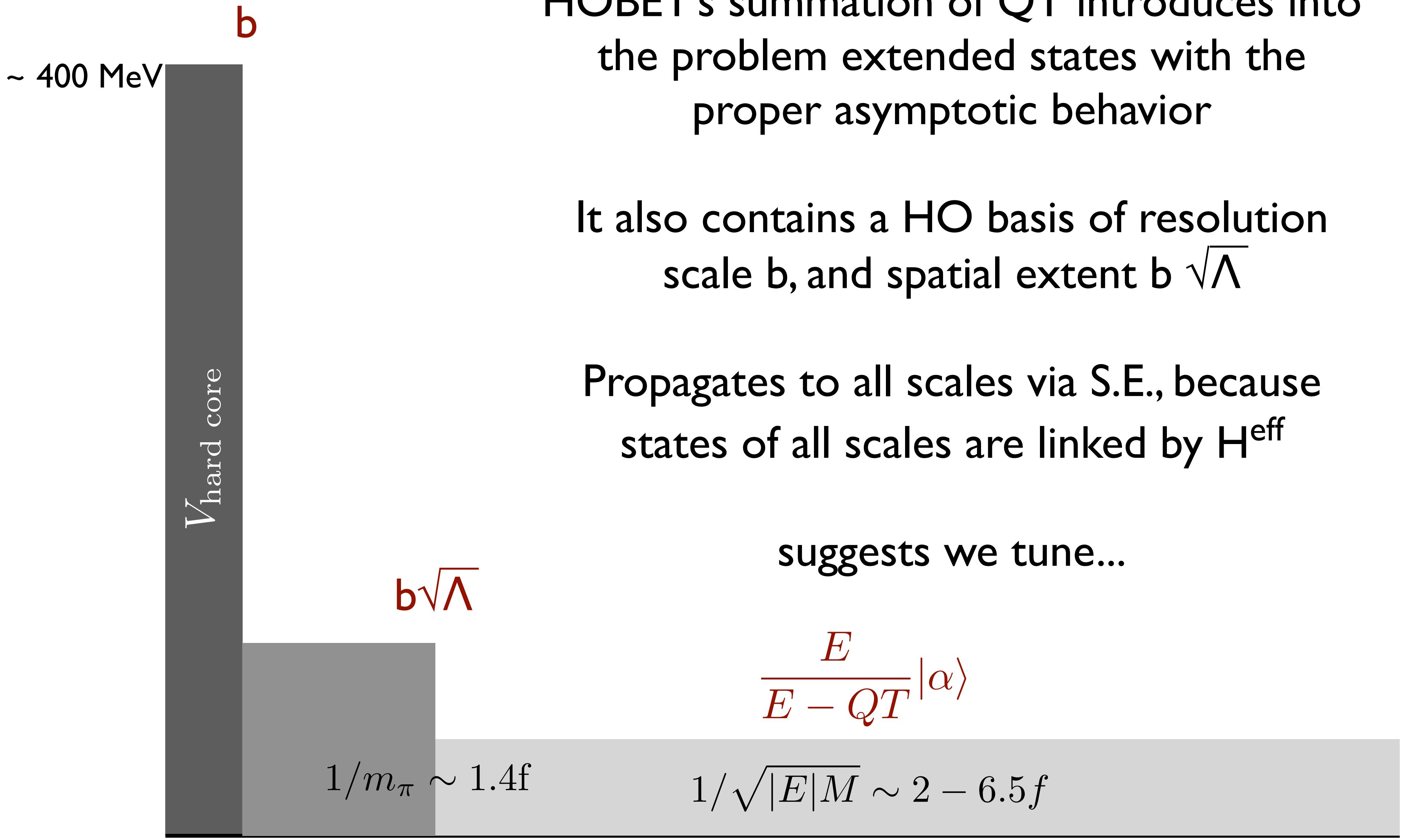
HOBET's summation of QT introduces into the problem extended states with the proper asymptotic behavior

It also contains a HO basis of resolution scale b, and spatial extent b $\sqrt{\Lambda}$

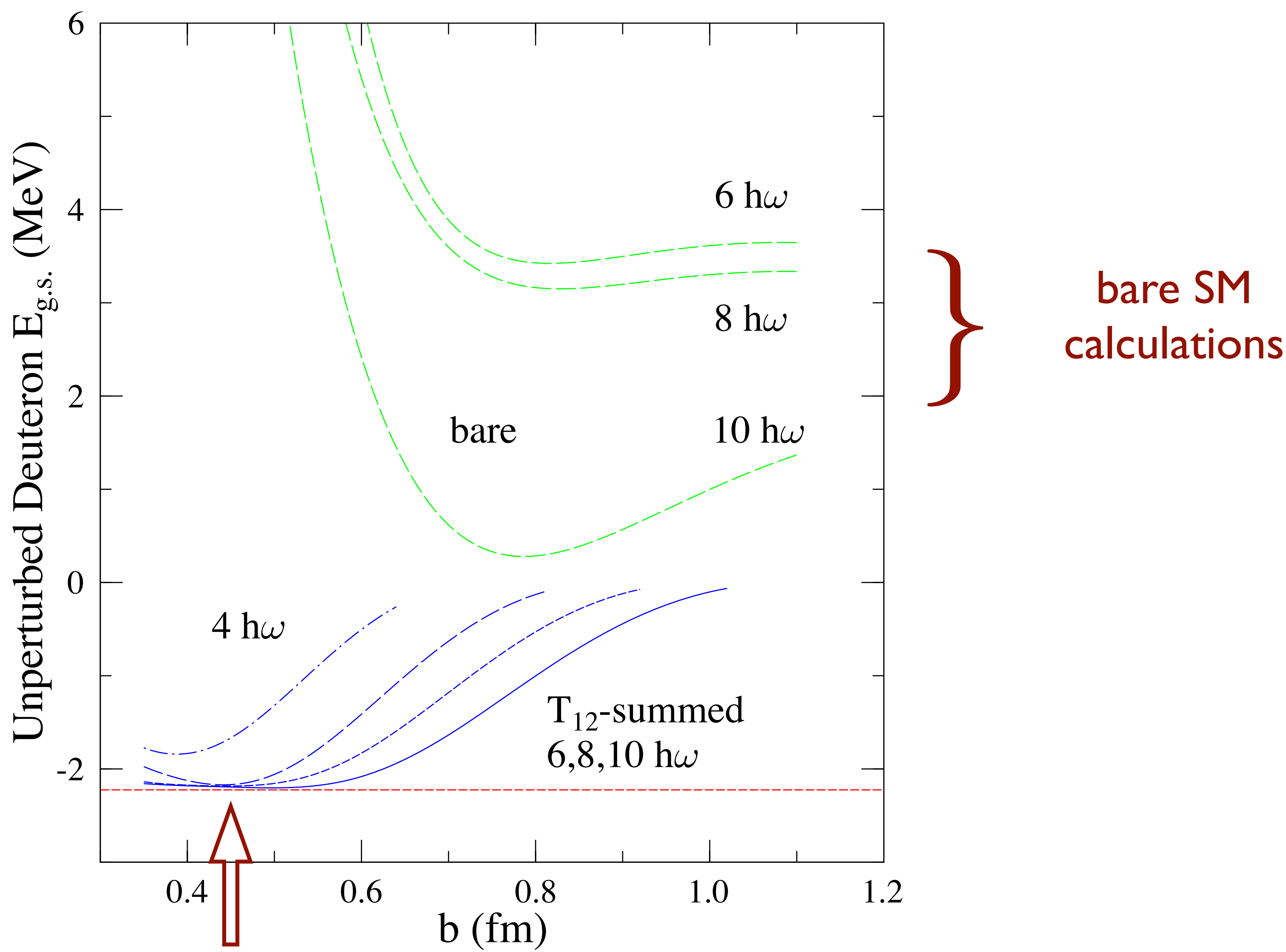
Propagates to all scales via S.E., because states of all scales are linked by H^{eff}

suggests we tune...

$$\frac{E}{E-QT}|\alpha\rangle$$



Could one thus absorb the strong physics into P as well, if Λ is sufficient?



bare HOBET, tuned b, but asymptotically correct extended states

- That is, we have a result in bare-order: this is the HOBET analog of usual potential theory

- HOBETp defines $P|\Psi\rangle$ as the restriction of the true wave function to the chosen HO basis. What is $\frac{E}{E-QT}P|\Psi\rangle \equiv |\Psi'\rangle$?

  - it is a solution of a S.E. with the full T, and exact eigenvalue E

$$\left[T + P\frac{E}{E-QT}(V+\bar{O})\right]|\Psi'\rangle = E|\Psi'\rangle$$

  - to the extent that we tune b and Λ to make effective contributions very small, by calculating the norm of $P|\Psi\rangle$ find

$$|\Psi'\rangle \rightarrow |\Psi\rangle \text{ as } \bar{O}(b,\Lambda_P) \rightarrow 0$$

the full, normalized wf of HOBETp in bare limit

Summary

- Demonstrated that a systematic expansion exists for the HOBET H^{eff} consisting of a set of short-range coefficients augmented by κ, a variable that links a ET parameter b with an observable |E|

- This expansion, through κ, also isolates simply the state-dependence that is important to proper ET behavior -- $P|\Psi\rangle$

- The summing of QT to all orders introduces extended states with proper asymptotic behavior, which mix with compact HO states ⇒ focus machinery of direct diagonalization on the scales relevant to the internucleon potential, rather than the nuclear size
  - The next HW problem for HOBETp is to repeat the bare deuteron calculation for other light nuclei

- Our extended states include the continuum (E>0) : our next HOBET HW problem is scattering via the free-T equation, fitting the strong coefficients of H^{eff} directly to experiment, eliminating the potential