The collective model from a Cartan-Weyl perspective

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1 The geometrical collective model
  - The basics of the geometrical collective model
  - The physics motivation

2 The collective variables in the Cartan-Weyl scheme
  - An algebraic description...
  - ... within the Cartan-Weyl scheme

3 Test application in quantum shape phase transitions

4 conclusions & outlook

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3. Test application in quantum shape phase transitions

4. conclusions & outlook

What's so geometrical about the geometrical model?

- Macroscopically, the atomic nucleus can be compared to a charged liquid drop.
- Deviations from the sphere are developed in multipole orders. Up to 2nd order

Radius

$$R(\theta,\phi) = R_0[1 + \alpha \cdot Y_2(\theta,\phi)]$$

- α^2_μ :: collective quadrupole[a] coordinates

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[a] $L = 2$ tensor

A gallery of shapes

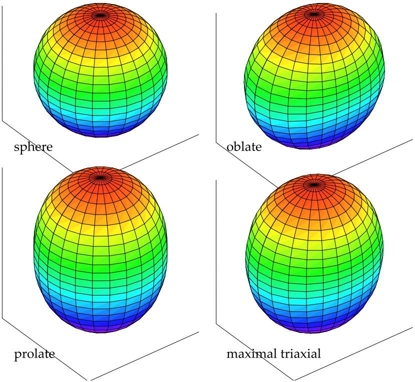
Radius

$$R(\theta,\phi) = R_0[1 + \alpha \cdot Y_2(\theta,\phi)]$$

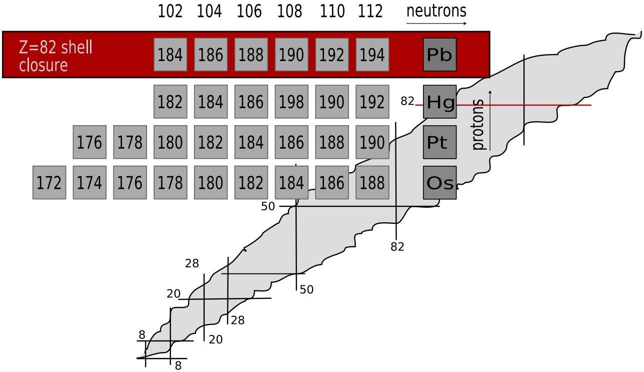
- Rotation to the intrinsic system

$$\begin{cases} \alpha'_0 = \beta \cos\gamma \\ \alpha'_2 = \alpha'_{-2} = \beta/\sqrt{2}\sin\gamma \\ \alpha'_1 = \alpha'_{-1} = 0 \end{cases}$$

- β is a measure for the deformation, γ for the triaxiality



The nuclear chart around Z=82 shell closure



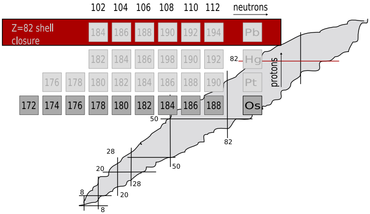
				102	104	106	108	110	112	neutrons
Z=82 shell closure				184	186	188	190	192	194	Pb
				182	184	186	198	190	192	Hg
		176	178	180	182	184	186	188	190	Pt
	172	174	176	178	180	182	184	186	188	Os

- Collective excitation modes are very important in the low-energy spectra
- Renewed interest due to the quantum shape phase transitions

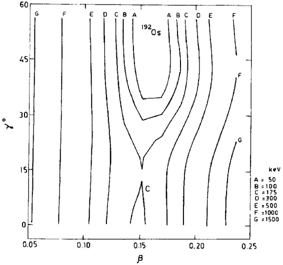
All kinds of shapes

neighbouring isotope chains

- Os :: triaxial nuclei
- Pt :: the γ-softie
- Pb :: 3 coexisting families



- Hartree-Fock mean field calculation
- A minimum can be found at $\gamma \neq 0$



A. Ansari, Phys. Rev. C 38 (1988) 953.

- Calculations in the framework of analytically solvable potentials

L. Fortunato et. al., Phys. Rev. C74 (2006) 014310.

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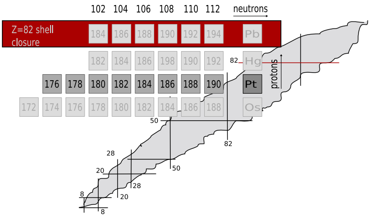
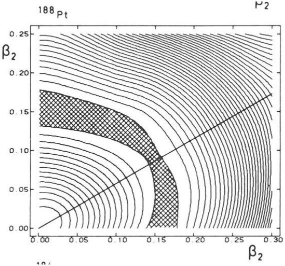


- Potential Energy Surface calculation
- Very flat γ dependence



R. Bengtsson et al., Phys. Lett. B 183 (1987) 1 (2004).

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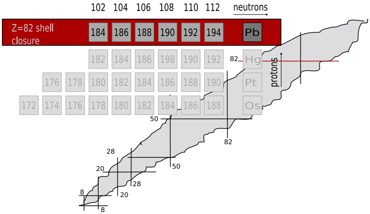
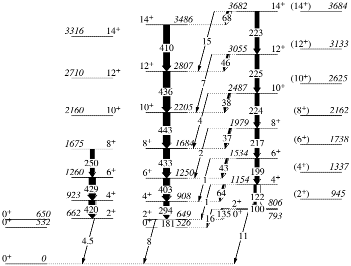


- Interacting Boson Model calculation
- Extension to coexisting configurations points towards spherical-oblate-prolate structure



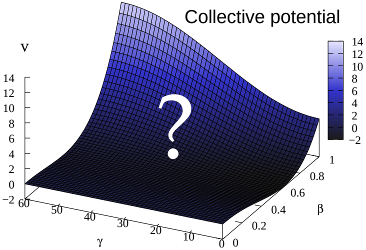
V. Hellemans, private communication (2007)

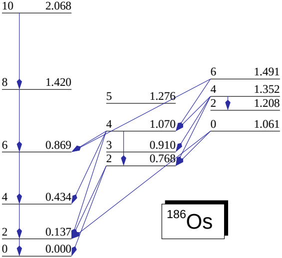
input/output of the geometrical model

The Bohr Hamiltonian

$$\hat{H} = \hat{T} + V(\alpha)$$

- kinetical egergy describes the stiffness of the surface
- $\hat{T} = \frac{1}{B_2}\pi \cdot \pi + B_3[\pi\alpha]^2 \cdot \pi + \ldots$





- α describes small deformations

☞ Use a Taylor expansion

$$V(\alpha) = c_2(\alpha \cdot \alpha) + c_3([\alpha\alpha]^2 \cdot \alpha) + c_4(\alpha \cdot \alpha)^2 + c_5([\alpha\alpha]^2 \cdot \alpha)(\alpha \cdot \alpha) + \ldots$$

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The rules of the game

Commutation relations fix the structure

$$[\pi_\mu, \alpha_\nu] = -i\hbar\delta_{\mu\nu}, \quad [\alpha_\mu, \alpha_\nu] = 0, \quad [\pi_\mu, \pi_\nu] = 0$$

- The algebraic structure of the geometrical model is contained in the following recoupling formula

$$\begin{aligned}(\alpha\cdot\alpha)(\pi^*\cdot\pi^*) &= (\alpha\cdot\pi^*)(\alpha\cdot\pi^*) + 3i\hbar(\alpha\cdot\pi^*) \\ &\quad - 2([\alpha\pi^*]^{(1)}\cdot[\alpha\pi^*]^{(1)} + [\alpha\pi^*]^{(3)}\cdot[\alpha\pi^*]^{(3)})\end{aligned}$$

- It comprises the generators of the direct product group

$$\left.\begin{aligned}\hat{Z}_1 &= \alpha\cdot\alpha \\ \hat{Z}_2 &= \pi^*\cdot\pi^* \\ \hat{Z}_3 &= \alpha\cdot\pi^*\end{aligned}\right\} SU(1,1)\times O(5)\left\{\begin{aligned}\tfrac{i\hbar}{\sqrt{10}}L_M &= [\alpha\pi^*]^{(1)}_M \\ \tfrac{i\hbar}{\sqrt{10}}O_M &= [\alpha\pi^*]^{(3)}_M\end{aligned}\right.$$

Why $SU(1,1) \times O(5)$?

The Hamiltonian

$$\hat{H} = \frac{1}{2B_2}\pi\cdot\pi + B_3\pi\cdot[\alpha\pi]^2 + c_2(\alpha\cdot\alpha) + c_3([\alpha\alpha]^2\cdot\alpha) + c_4(\alpha\cdot\alpha)^2$$
$$+ c_5([\alpha\alpha]^2\cdot\alpha)(\alpha\cdot\alpha) + c_6(\alpha\cdot\alpha)^3 + d_6([\alpha\alpha]^2\cdot\alpha)^2 + \dots$$

$SU(1,1)$ basic block

$$\alpha\cdot\alpha = \beta^2$$

- The "radial" dependence
- Basis is known

$O(5)$ basic block

$$[\alpha\alpha]^2\cdot\alpha = \sqrt{\tfrac{2}{7}}\beta^3\cos 3\gamma$$

- The "angular" dependence
- Cartan-Weyl basis

The Cartan-Weyl basis of $O(5)$

Cartan's theorem

Every semi simple algebra of dimension n and rank r can be rotated to a natural basis for which

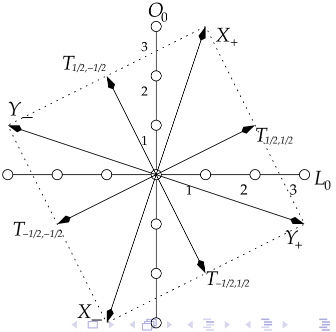
$$\begin{aligned} &[H_i, H_j] = 0, \quad [H_i, E_\alpha] = \alpha_i E_\alpha, && \{i,j\} \in r \\ &[E_\alpha, E_\beta] = N_{\alpha+\beta} E_{\alpha+\beta} \quad [E_\alpha, E_{-\alpha}] = \alpha^i H_i && \{\alpha, \beta\} \in n - r \end{aligned}$$

- Rotation
  $\{L_m, O_{m'}\} \rightarrow \{X_i, Y_j, T_{\mu\nu}\}$
- Group reduction is clear

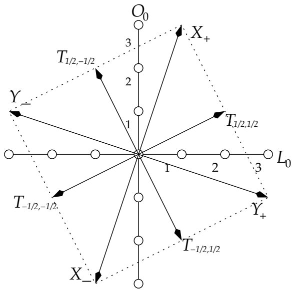
$$\underbrace{O(5)}_{v} \supset \underbrace{O(4)}_{X} \cong \underbrace{SU(2)}_{(X, M_X)} \times \underbrace{SU(2)}_{(X, M_Y)}$$

- A natural Cartan basis emerges

$$|vX(M_X, M_Y)\rangle$$



Action of the $O(5)$ generators on the natural basis (i)



- $\{X_\pm, X_0\}$ and $\{Y_\pm, Y_0\}$ span standard $SU(2)$ algebras
- According Racah: $T_{\mu,\nu}$ acts as a bispinor of character $1/2$ in the $SU(2) \times SU(2)$ space

$$[X_0, T^{\frac12\frac12}_{\mu\nu}] = \mu T^{\frac12\frac12}_{\mu\nu}$$

$$[X_\pm, T^{\frac12\frac12}_{\mu\nu}] = \sqrt{(\tfrac12 \mp \mu)(\tfrac12 \pm \mu + 1)}\, T^{\frac12\frac12}_{\mu\pm1\nu}$$

- The action of $T_{\mu,\nu}$ on a basis state $|vX(M_X, M_Y)\rangle$ is thus

$$\begin{aligned} T^{\frac12\frac12}_{\mu\nu}|vXM_XM_Y\rangle &= a_+|v, X+\tfrac12, M_X+\mu, M_Y+\nu\rangle \\ &\quad + a_-|v, X-\tfrac12, M_X+\mu, M_Y+\nu\rangle \end{aligned}$$

- Applying the Wigner Eckart theorem twice

$$a_\pm = (-)^k \begin{pmatrix} X\pm\frac12 & \frac12 & X \\ -M_X-\mu & \mu & M_X \end{pmatrix} \begin{pmatrix} X\pm\frac12 & \frac12 & X \\ -M_Y-\nu & \nu & M_Y \end{pmatrix} \langle vX\pm\tfrac12||T||vX\rangle$$

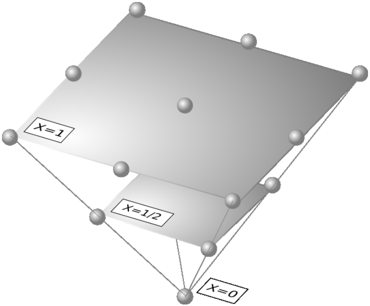
Action of the $O(5)$ generators on the natural basis (ii)

- For every v and X :: two unknown matrix elements ☞ two conditions needed

$$\mathcal{C}_2[O(5)] = 2(X^2 + Y^2 - 2[TT]^{(00)})$$

$$[T_{\mu\nu}, T_{\mu'\nu'}] = c^X_m X_0 + c^Y_m Y_0 \quad m = \{\mu\nu, \mu'\nu'\}$$

- norm ☞ selection rules $(X = 0 \dots \frac{v}{2})$
- intermediate state method renders double reduced matrix elements



Example: Action of $T_{1/2,1/2}$

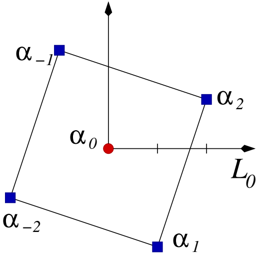
$$T_{\frac{1}{2}\frac{1}{2}}|vXM_XM_Y\rangle = \frac{\sqrt{(X+M_X+1)(X+M_Y+1)(v-2X)(v+2X+3)}}{2\sqrt{(2X+1)(2X+2)}}|vX+\tfrac{1}{2}M_X+\tfrac{1}{2}M_Y+\tfrac{1}{2}\rangle$$

$$-\frac{\sqrt{(X-M_X)(X-M_Y)(v-2X+1)(v+2X+2)}}{2\sqrt{(2X)(2X+1)}}|vX-\tfrac{1}{2}M_X+\tfrac{1}{2}M_Y+\tfrac{1}{2}\rangle$$

Tensor character of the α variables

basis block of the potential

$$\beta^3 \cos(3\gamma) \sim [\alpha\alpha]^{(2)} \cdot \alpha$$



☞ Need for matrix elements

$$\langle vX(M_X, M_Y)|\alpha_\mu|v'X'(M'_X, M'_Y)\rangle$$

- $\{\alpha_0\}$ and $\{\alpha_{-2}, \alpha_{-1}, \alpha_1, \alpha_2\}$ have bispinor 0 and $\frac{1}{2}$ character respectively

$$\alpha_0 \to \alpha^{00}_{00}, \qquad \{\alpha_{-2}, \alpha_{-1}, \alpha_1, \alpha_2\} \to \alpha^{\frac{1}{2}\frac{1}{2}}_{\mu\nu}$$

- Wigner Eckart

$$\langle vXM_XM_Y|\alpha^{\lambda\lambda}_{\mu\nu}|v'X'M'_XM'_Y\rangle = (-)^k \begin{pmatrix} X & \lambda & X' \\ -M_X & \mu & M'_X \end{pmatrix} \begin{pmatrix} X & \lambda & X' \\ -M_Y & \nu & M'_Y \end{pmatrix} \langle vX||\alpha^\lambda||v'X'\rangle$$

Closed expressions for the α matrix elements

What is used

$$[T_{\mu\nu}, \alpha^{\frac{1}{2}\frac{1}{2}}_{\mu'\nu'}] = \frac{(-)^{\mu-\nu}}{\sqrt{2}}\delta_{-\mu\mu'}\delta_{-\nu\nu'}\alpha^{00}_{00}$$

$$[T_{\mu\nu}, \alpha^{00}_{00}] = \frac{1}{\sqrt{2}}\alpha^{\frac{1}{2}\frac{1}{2}}_{\mu,\nu}$$

$$[\alpha^{\lambda\lambda}_{\mu,\nu}, \alpha^{\lambda'\lambda'}_{\mu'\nu'}] = 0$$

$$\alpha\cdot\alpha = \beta^2 = Z_1$$

- Seniority selection rules :: α is a $v = 1$ tensor
- Bispinor $\{\frac{1}{2}\frac{1}{2}\}$ can be expressed in terms of biscalar $\{00\}$ matrix elements
- Closed expressions for the matrix elements result

What is obtained: matrix elements of α

$$\langle vXM_XM_Y|\alpha^{00}_{00}|v+1, XM_XM_Y\rangle = \beta\sqrt{\frac{(v-2X+1)(v+2X+3)}{(2v+3)(2v+5)}}$$

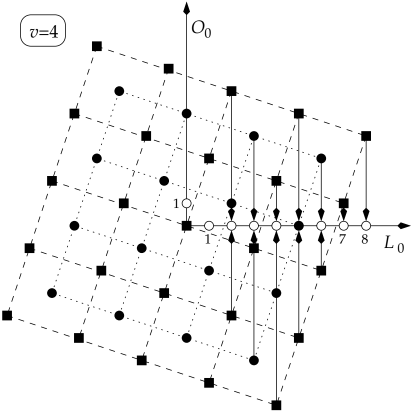
$$\langle vXM_XM_Y|\alpha^{00}_{00}|v-1, XM_XM_Y\rangle = \beta\sqrt{\frac{(v-2X)(v+2X+2)}{(2v+1)(2v+3)}}$$

Projection to the physical basis

- Experimental spectra have good angular momentum quantum number L
- The Hamiltonian is a scalar with respect to the angular momentum algebra $O(3)$

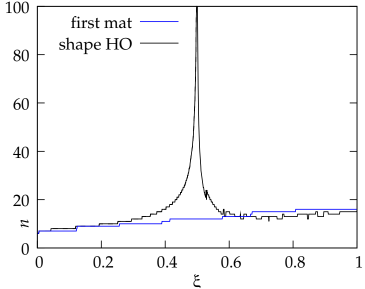
$$[L \cdot L, \mathcal{C}_G] \neq 0$$
$$[L_0, \mathcal{C}_G] = 0$$

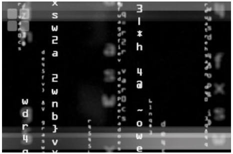
- Only the angular momentum projection is a good quantum number in the Cartan basis
- A rotation brings the natural- to the physical basis



All ingredients are ready

- α Matrix elements in $O(5)$ basis are derived
- Inclusion of $SU(1,1)$ basis is straightforward in a similar fashion
- Diagonalising = choosing a basis
- Harmonic oscillator = choosing $\hbar\omega$
- $H = \frac{1}{2B_2}\pi \cdot \pi + \xi V_{\text{vib}} + (1-\xi) V_{\gamma\text{-ind}}$
- Margetan & Williams Phys. Rev. C25 (1982) 1602





- Computer code is now under continuous development to diagonalise general collective potentials.
- Present status: upto β^4

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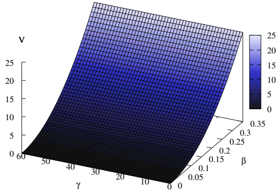
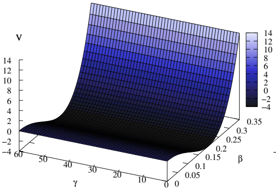
Test application in quantum shape phase transitions

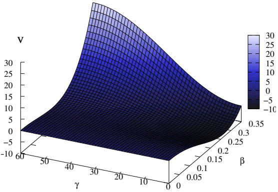
- Quantum shape phase transitions cover a large part of the model Hilbert space
- Ideal testground for the method
- Upto β^4, 3 meaningful limits result

3 limits

- vibrational limit
- γ-independent rotor
- axial deformed rotor







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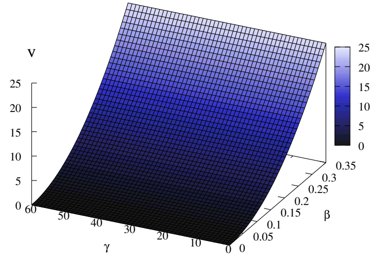
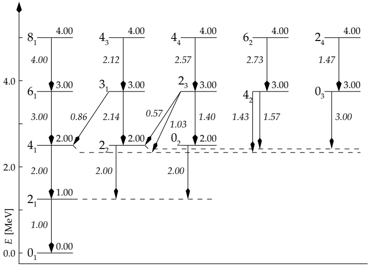


- Trivial limit
- Large degeneracies
- $B(E2)$ addition rule



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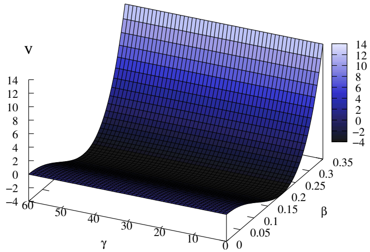
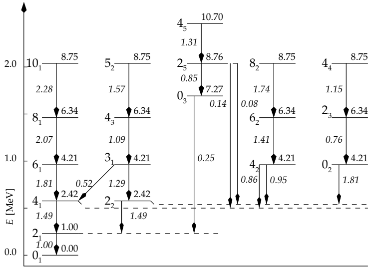


- Remaining seniority symmetry
- β-excitation band



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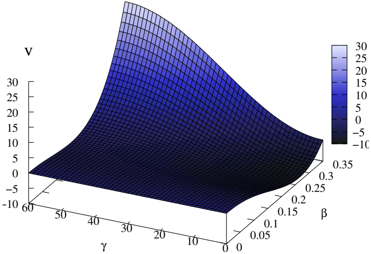
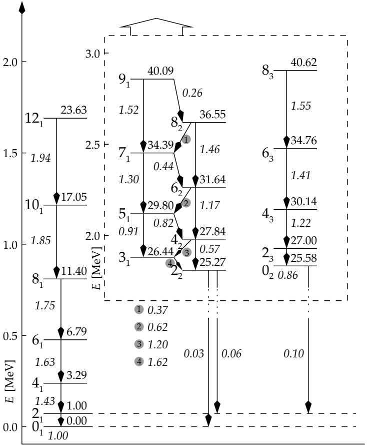
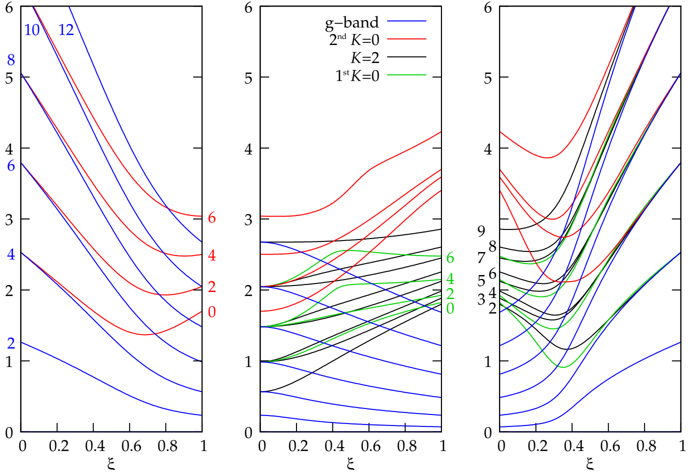


- Highly pronounced bands
- Rotation-Vibration model



Along the transition path: energy spectrum



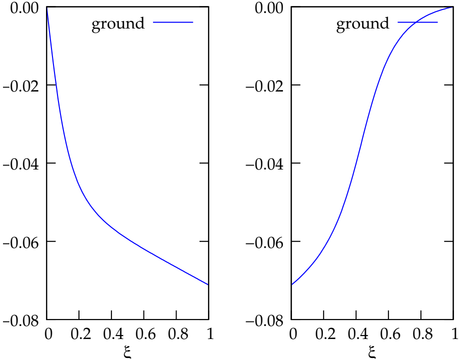
vibrator → γ-independent rotor → axial rotor → vibrator

Along the transition path: quadrupole moments

Quadrupole moments

$$Q = \langle \hat{Q}_\mu \rangle_{2_1} = \frac{3ZR_0^2}{4\pi} \langle \alpha_\mu \rangle_{2_1}$$

- α_μ is seniority $v = 1$ tensor
- ☞ $Q \equiv 0$ for vib. → γ-ind. rotor transition
- Other observables ($B(E2)$, $\rho(E0)$)



γ-ind. rotor → axial rotor → vibrator

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Conclusions & outlook

- Collectivity accounts for a lot of the physics around $Z = 82$ shell closure
- All necessary matrix elements of a more general type of potential can be calculated in a Cartan-Weyl scheme
- First test applications in the framework of quantum shape phase transitions renders reliable results
- Further terms need to be included to study more general collective structures (e.g. triaxiality, shape coexistence), needed for the collectivity around the $Z = 82$ closed shell
- Possible extension to higher rank algebras ($O(7)$ octupole degrees of freedom and beyond)
- ...

Thank you for you attention!