Renormalization of Singular Potentials

— Power counting of EFT for nuclear forces

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Outline

Introduction

What is EFT?   Why EFT?

NN potentials from chiral EFT

Power counting of nuclear forces ←→ renormalization

Renormalization of singular potentials

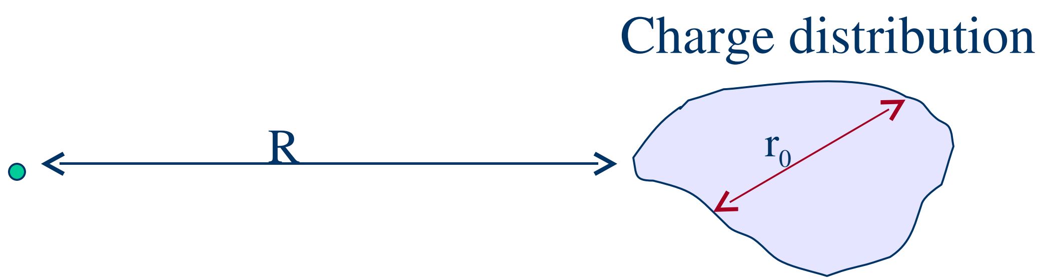
Toy theory $-1/r^2 + 1/r^4$

Renormalization of LO & NLO

Conclusion

A classical example of EFT

— multipole expansions in classical E&M



$$V = \frac{q}{R} + \frac{d \cdot \hat{R}}{R^2} + \frac{Q_{ij}\hat{R}_i\hat{R}_j}{R^3} + \cdots \qquad R >> r_0$$

Low energy constants (LECs)

$$|d| \sim q r_0$$

$$|Q_{ij}| \sim q r_0^2$$

Power counting

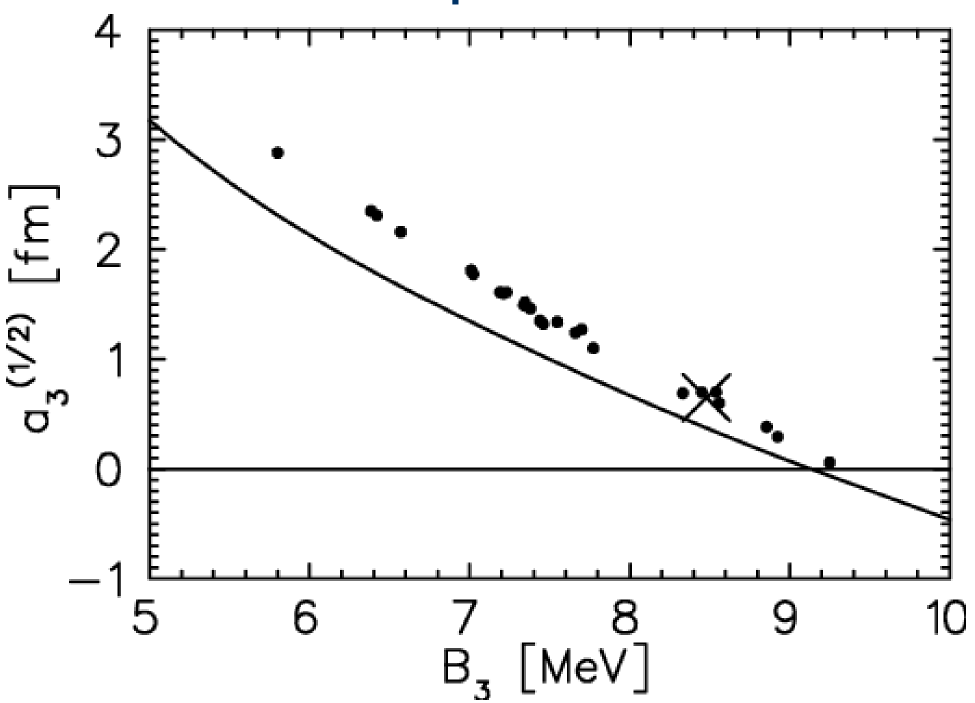
$$V = \frac{q}{R}\left[1 + O(\frac{r_0}{R}) + O(\frac{r_0^2}{R^2}) + \cdots\right]$$

Why EFT ?

"Pion-less" EFT of 3N system (without explicit pions)

Bedaque, Hammer & van Kolck (1999)

Phillips line



Renormalization-group invariance promotes 3N force

Not same as Bloch-Horowitz, etc.

Nd scattering length (S-wave, j=1/2) v.s. triton binding energy B_d

Points are predictions of different models

EFT of nuclear physics

A low-energy EFT of strong interaction (Q¿ M_{QCD} ~ 1GeV)

Generic external momentum

Degrees of freedom: pions, nucleons, delta isobars, etc.

Symmetries: inherit all the symmetries of QCD

Lorentz invariance, P, T, baryon #…

color gauge symmetry : trivially preserved in EFT

all hadrons are color singlets

chiral symmetry: not trivial at all

Chiral EFT in terms of N & π

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \boldsymbol{\pi})^2 - \frac{1}{2}m_\pi^2 \boldsymbol{\pi}^2 + N^\dagger \left[i\partial_0 - \frac{1}{4f_\pi^2}\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\dot{\boldsymbol{\pi}}) \right] N + \frac{g_A}{2f_\pi} N^\dagger \left(\boldsymbol{\tau}\cdot\vec{\sigma}\cdot\vec{\nabla}\boldsymbol{\pi} \right) N$$
$$-\frac{1}{2}C_S(N^\dagger N)^2 - \frac{1}{2}C_T(N^\dagger \vec{\sigma} N)^2 + \cdots \qquad g_A = 1.26 \quad f_\pi = 92 \text{ MeV}$$

Operators involving pions are derivative, $\propto m^2_\pi$

→ π N interactions $\propto$ Q, m^2_π

Nuclear interactions expanded in powers of Q/M_{QCD}

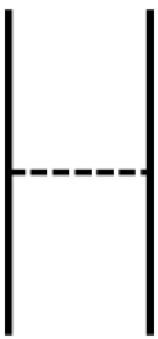
How to calculate cross-section, binding energy etc. ?

Weinberg's approach

1. Derive potentials in perturbation theory
2. Iterate potentials to all orders by solving Lippman-Schwinger eqn.
   or Schrodinger eqn.

Other approaches: Kaplan, Savage & Wise (KSW scheme, 1998)…

(LO)

$V_{1\pi} =$  $= -\left(\frac{g_A}{2f_\pi}\right)^2 \tau_1 \cdot \tau_2 \frac{\vec{\sigma}_1 \cdot \vec{q}\,\vec{\sigma}_2 \cdot \vec{q}}{q^2 + m_\pi^2} \sim \frac{1}{f_\pi^{\,2}}\left(\frac{Q}{M_{\mathrm{QCD}}}\right)^0$

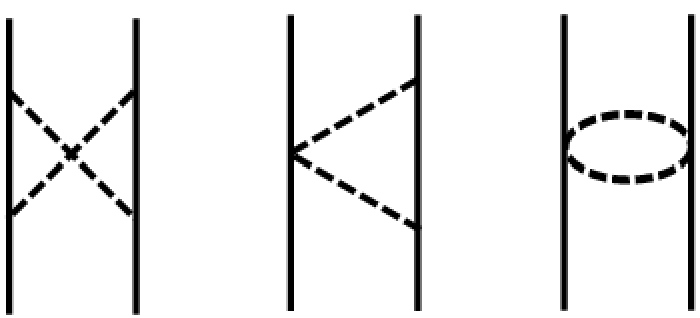
momentum transfer

$Q \sim m_\pi$

$M_{\mathrm{QCD}} \sim 1\mathrm{GeV}$

Power counting

(N^2LO)

$V_{2\pi} =$  $\cdots \sim \frac{1}{f_\pi^{\,2}}\left(\frac{Q}{M_{\mathrm{QCD}}}\right)^2$

$V_{1\pi}$ in coordinate space

$$V_{1\pi}(\vec{r}) = \frac{m_\pi^3}{12\pi}\left(\frac{g_A}{2f_\pi}\right)^2 \tau_1\cdot\tau_2[T(r)S_{12} + Y(r)\vec{\sigma}_1\cdot\vec{\sigma}_2]$$

Tensor force: $T(r) = \frac{e^{-m_\pi r}}{m_\pi r}\left[1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2}\right]$ $\quad Y(r) = \frac{e^{-m_\pi r}}{m_\pi r}$

Tensor operator: $S_{12} = 3(\sigma_1\cdot\hat{r})(\sigma_2\cdot\hat{r}) - \sigma_1\cdot\sigma_2$

$T(r) \sim \frac{1}{r^3}$

- attractive: $^3P_0$,… $\quad -1/r^3$ Singular potentials
- repulsive: $^3P_1$,…

short-distance interactions

Besides pion-exchange forces (long-distance),

there are contact NN interactions (short-distance)

Ignoring spin structure for a moment,

$$V_S = \frac{1}{4\pi}[c_0 + c_2(\vec{p}'^2 + \vec{p}^2) + c'(\vec{p}' \cdot \vec{p}) + \cdots]$$

S wave — P wave

coordinate space: $V_S(\vec{r}) = \frac{1}{4\pi}[c_0\delta^{(3)}(\vec{r}) + \cdots]$

Naive dimensional analysis

$$\frac{c_2}{c_0} \sim \frac{c'}{c_0} \sim \frac{1}{M_{\rm QCD}^2}$$ → higher wave cont. int. less important

LO cont. int. $V_S^{(0)} = \frac{1}{4\pi}[C_S P_S + C_T P_T]$ Weinberg (1990)

P_S, P_T: projectors onto $^1S_0$ & $^3S_1$

Iterating effective potentials

cutoff in momentum space

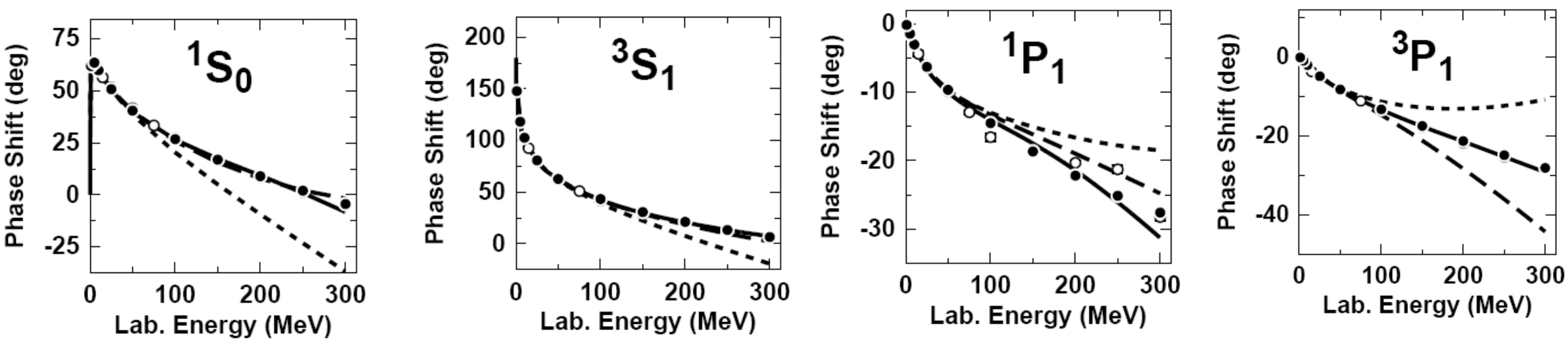
NN scattering amplitude by iterating $V_{LO} = V_{1\pi} + V_S^{(0)}$

$$T_{LO}(\vec{p}\,', \vec{p}, E) = V_{LO}(\vec{p}\,', \vec{p}) + \int^{\Lambda} \frac{d^3 l}{(2\pi)^3} V_{LO}(\vec{p}\,', \vec{l}) \frac{1}{E - l^2/m_N + i\epsilon} T_{LO}(\vec{l}, \vec{p}, E)$$

RG invariance

Choosing $C_S(\Lambda)$, $C_T(\Lambda)$ $\Rightarrow$ T_{LO} independent of Λ at $\Lambda \rightarrow 1$

IF we have done it right



Entem & Machleidt(2003), Epelbaum, Glöckle & Meißner(2004)

What could go wrong?

$$V_S^{(0)} = \frac{1}{4\pi}[C_S P_S + C_T P_T]$$

Promoted by RG analysis

- $$V_S^{(0)} = \frac{1}{4\pi}[\left(C_S + m_\pi^2 D_2\right)P_S + C_T P_T]$$ KSW; Beane, Bedaque, Savage & van Kolck

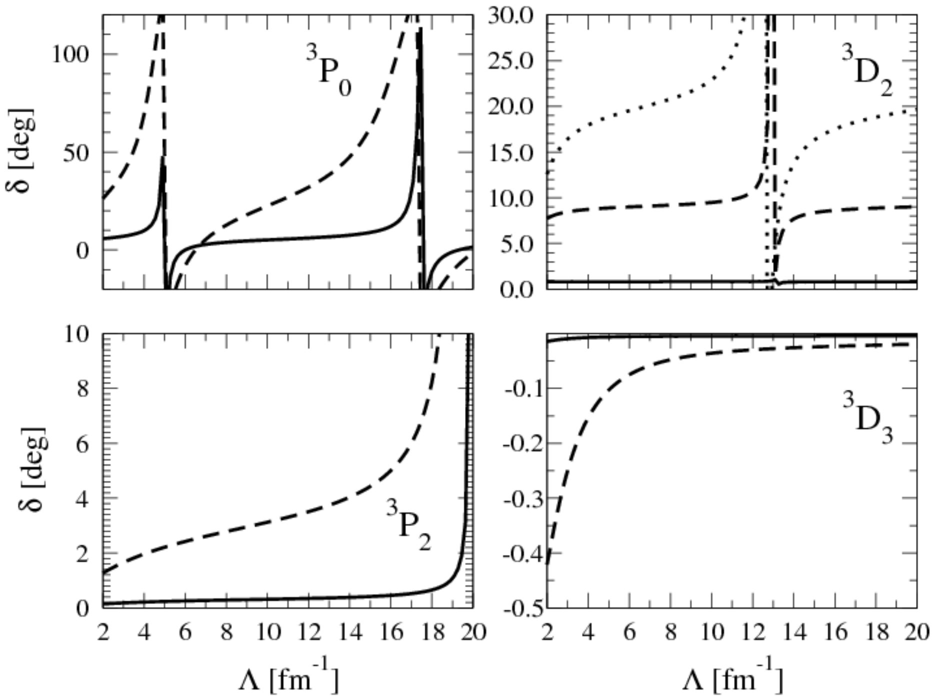
- Renormalization *not* needed for P, D,-waves ?

Yes, Renormalization is needed

in attractive triplet channels (P,D…waves)

Renormalization of V_{LO}

Nogga, Timmermans & van Kolck (2005)



← Cutoff dependence of phase shifts in attractive triplet channels at T_{lab}=10 (solid), 50 (dashed) & 100 (dotted) MeV

→ Higher-wave contact interactions promoted

→ Naive dimensional analysis fails in nonperturbative interactions

Disputed by Meißner, Epelbaum → Λ too high?

Singular potentials

Typical example: $-1/r^n, n \geq 2$

OPE, TPE $-1/r^3, 1/r^5$

van der Waals forces

Case (1950), Frank, Land & Spector (1971)

Singularity : observables not uniquely determined

→ short-distance physics not accounted for correctly

→ Renormalization needed (Beane et al. 2001, Hammer & Swingle 2005)

Ren. of singular potentials → ren. of nuclear forces → power counting

How to treat NLO ?

A toy theory

$$V_L^{(0)}(r) = -\frac{\lambda}{2mr^2}$$

Analytically tractable
Realistic applications: 3-boson system…

$$V_L^{(1)}(r) = \frac{g}{2mM^2 r^4}$$

$\lambda>0$, g : dimensionless, ~ O(1)

$Q << M$ Q: generic momentum such as k, $(2B_d m)^{1/2}$

$$V_L^{(1)}(q) \sim V_L^{(0)}(q) \cdot \left(\frac{Q}{M}\right)^2$$

M: scale of underlying theory

Partial-wave decomposition in momentum space

$$V_{Ll}^{(0)}(p',p) = -\frac{\pi^2 \lambda}{m(2l+1)} \frac{p_<^l}{p_>^{l+1}} \qquad p_> \equiv \max\{p',p\} \qquad p_< \equiv \min\{p',p\}$$

$$V_{Ll}^{(1)}(p',p;R) = 4\pi \int_R^\infty dr r^2 j_l(p'r) \frac{g}{2mM^2 r^4} j_l(pr)$$

$$= -\frac{\pi^2 g}{2mM^2}\{\frac{4}{\pi R}\delta_{l0} + \frac{1}{(2l+1)(2l-1)} \frac{p_<^l}{p_>^{l-1}}(1-\frac{2l-1}{2l+3}\frac{p_<^2}{p_>^2}) + O(Rp^2)\}$$

Absorbed into short-range interactions

Singularity of 1/ r^2

Dimensionless t-matrix $t_l(p', p; E = \frac{p^2}{2m}) \equiv \frac{mp}{\pi^2} T_l(p', p; E)$

$$t_l(px, p) = -\frac{\lambda}{2l+1} \left\{ x^{-(l+1)} \left[\theta(x-1) - \int_0^x dy \frac{y^{l+2}}{y^2 - 1 - i\epsilon} t_l(py, p) \right] + x^l \left[\theta(1-x) - \int_x^{\Lambda/p} dy \frac{y^{1-l}}{y^2 - 1 - i\epsilon} t_l(py, p) \right] \right\}$$

$$x = \frac{p'}{p}$$

Int. eqn. for k matrix : principle-value replacing iε prescription

Born expansions in l-wave

$$t_l = -\frac{\lambda}{2l+1} \left[1 + \frac{\lambda}{2l+1} \left(\frac{1}{2l+1} + i \frac{\pi}{2} \right) \right] + \cdots$$

$$l \leq l_p \equiv \frac{\lambda\pi}{4} - \frac{1}{2}$$

l_p :The angular momentum beyond which perturbation theory starts to be valid

Singularity of 1/ r^2

Interesting property of $1/r^2$ potential

$$-\nabla'^2 V_L^{(0)}(\vec{p}', \vec{p}) = -\frac{\lambda}{2m}(2\pi)^3 \delta^3(\vec{p}' - \vec{p})$$

L-S eqn. $\Rightarrow x\frac{\partial^2}{\partial x^2}\left(x t_l\left(px, p\right)\right) + \left[\frac{\lambda x^2}{x^2 - 1 - i\varepsilon} - l(l+1)\right] t_l\left(px, p\right) = \lambda\delta(1-x)$

$x >> 1$ Extremely off-shell

$$x\frac{\partial^2}{\partial x^2}\left(x t_l\left(px, p\right)\right) + \left[\lambda - l(l+1)\right] t_l\left(px, p\right) = O\left(\frac{t_l}{x^2}\right)$$

Two solutions when λ strong enough

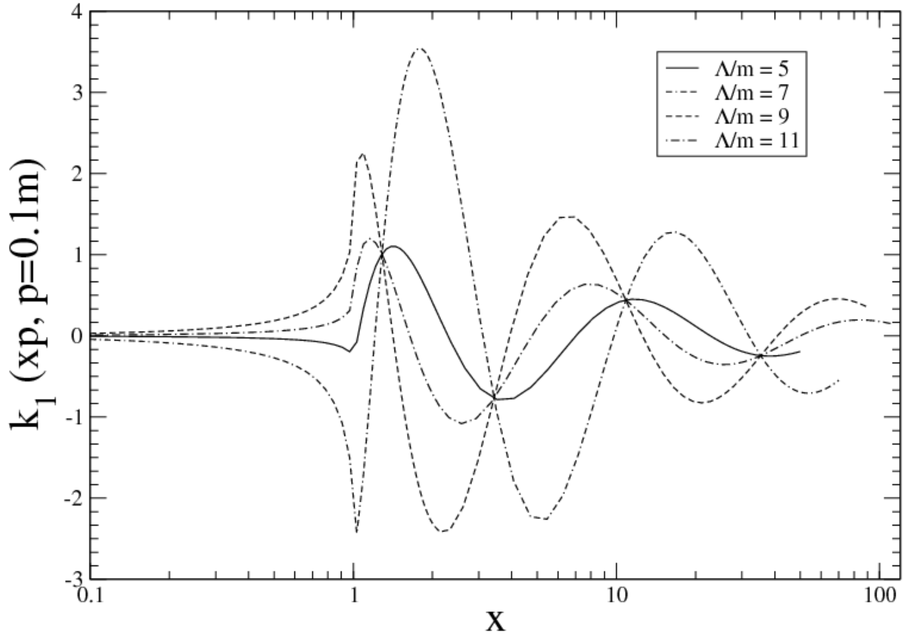
$x^{-1/2+i\nu_l}$ or $x^{-1/2-i\nu_l}$ $\quad \nu_l = \sqrt{\lambda - (l+1/2)^2}$

Singularity exists for $l < l_c \equiv \sqrt{\lambda} - 1/2$

$$t_l(px, p) \sim x^{-1/2}\cos\left(\nu_l \ln x + \theta_l\right)$$

θ_l dependent on Λ

Singularity of 1/ r²



P-wave off-shell k matrix $k_1(px, p)$ as function of x for p/m = 0.1 and various cutoffs. $\lambda = 4.25$ → P-wave nonperturbative and singular

Singularity in higher waves (same issue in OPE potential)

Renormalization-group invariance requires short-range interactions V_S in P-wave

Renormalization of $-1/r^2$

$C_0^{(0)} \sim \lambda / Q$

$$V_{Sl}^{(0)}(p',p) = \frac{\pi^2 C_l^{(0)}(\Lambda)}{m(2l+1)} p'^l p^l$$

l-wave contact int. with fewest derivatives in coordinate space

$$\begin{aligned} t_l^{(0)}(px,p) &= -\frac{1}{2l+1}\Big\{\lambda x^{-(l+1)}\left[\theta(x-1) - \int_0^x dy\, \frac{y^{l+2}}{y^2-1-i\epsilon} t_l^{(0)}(py,p)\right] \\ &\quad + \lambda x^l \left[\theta(1-x) - \int_x^{\Lambda/p} dy\, \frac{y^{1-l}}{y^2-1-i\epsilon} t_l^{(0)}(py,p)\right] \\ &\quad + C_l^{(0)} p^{2l+1} x^l \left[1 - \int_0^{\Lambda/p} dy\, \frac{y^{l+2}}{y^2-1-i\epsilon} t_l^{(0)}(py,p)\right]\Big\} \end{aligned}$$

RG inv. → on-shell t independent of Λ for Λ → 1

$$\Lambda \frac{d}{d\Lambda} t_l^{(0)}\left(p, p; \Lambda, C_l^{(0)}(\Lambda)\right) = O\left(\frac{p^2}{\Lambda^2}\right)$$

But for –1/r^2, off-shell t is also RG inv. (accidental)

$$\Lambda \frac{d}{d\Lambda} t_l^{(0)}\left(px, p; \Lambda, C_l^{(0)}(\Lambda)\right) = O\left(\frac{p^2}{\Lambda^2}\right) \quad \text{for any x}$$

Renormalization of $-1/r^2$

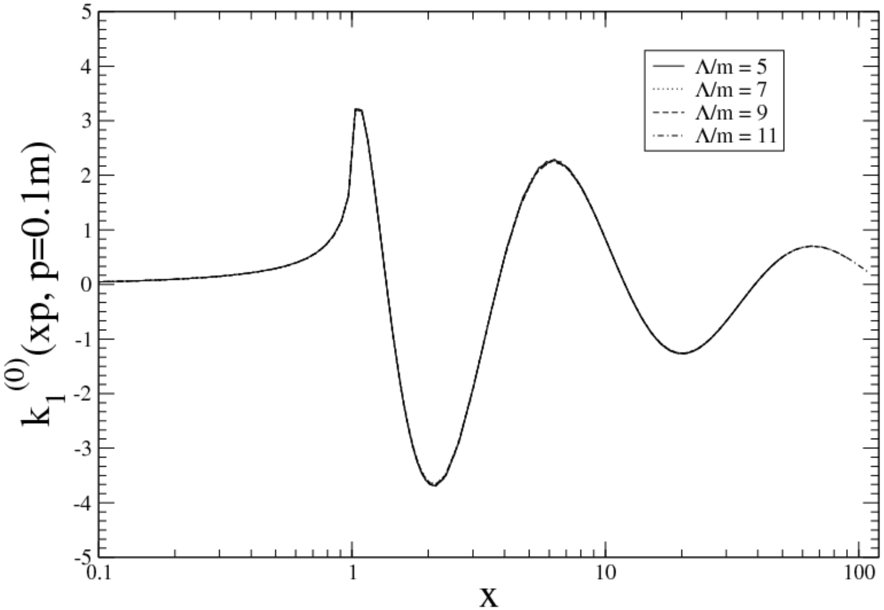
$$\left(\frac{xp}{\Lambda}\right)^l \frac{t_l^{(0)}(\Lambda,p)}{\lambda - C_l^{(0)}\Lambda^{2l+1}}\left[\frac{1}{2l+1}\left(\lambda - C_l^{(0)}\Lambda^{2l+1}\right)^2 - \Lambda^{2(l+1)}\frac{dC_l^{(0)}}{d\Lambda}\right] = O\left(\frac{p^2}{\Lambda^2}t^{(0)}\right)$$

Renormalization-group equation of $C^{(0)}_l$

Limit cycle-like

$$\Lambda\frac{dC_l^{(0)}}{d\Lambda} \equiv \beta\left(C_l^{(0)}\right) = \frac{1}{(2l+1)\Lambda^{2l+1}}\left(\lambda - \Lambda^{2l+1}C_l^{(0)}\right)^2 \rightarrow C_l^{(0)}(\Lambda) = -\frac{\lambda}{\Lambda^{2l+1}}\frac{2l+1-2\nu_l\tan[\nu_l\ln(\Lambda/\Lambda_{*l})]}{2l+1+2\nu_l\tan[\nu_l\ln(\Lambda/\Lambda_{*l})]}$$

Λ_{*l} determined by fitting to l-wave observable



← LO P-wave $k_1^{(0)}(xp, x)$ at various cutoffs, with $V_{S\,1}^{(0)}$ introduced for renormalization.

$\lambda = 4.25$, $\Lambda_{*1}/m = 0.2$

How to deal with NLO potential (1/r^4)

Two options

(1) Iterate LO and NLO to all orders

$$(T^{(0)}+T^{(1)})=(V^{(0)}+V^{(1)})+(V^{(0)}+V^{(1)})G_0(T^{(0)}+T^{(1)})$$

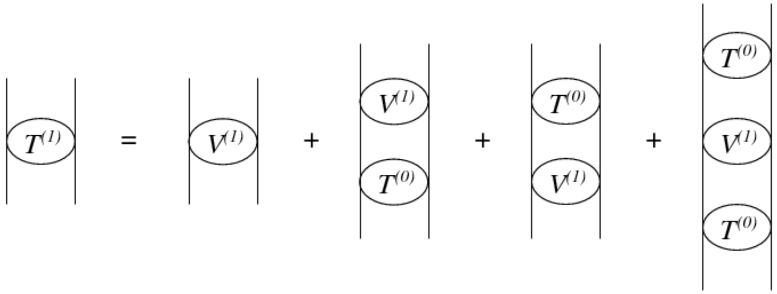
(2) NLO as perturbation (distorted wave expansion)

$$T^{(1)}=V^{(1)}+V^{(1)}G_0T^{(0)}+T^{(0)}G_0V^{(1)}+T^{(0)}G_0V^{(1)}G_0T^{(0)}$$

#2 chosen

More tractable to analytical calculation

Naive dimensional analysis expected to hold



$$V_L^{(0)} \sim \frac{\pi^2\lambda}{mQ} \qquad V_L^{(1)} \sim \frac{\pi^2 gQ}{mM^2}$$

$$V_L^{(1)} \sim V_L^{(0)} \cdot \left(\frac{Q^2}{M^2}\right) \overset{?}{\rightarrow} V_S^{(1)} \sim V_S^{(0)} \cdot \left(\frac{Q^2}{M^2}\right)$$

Renormalization of NLO

Cutoff dependence of NLO, without $V_S^{(1)}$

S-wave $$\Lambda \frac{d}{d\Lambda} t_0(p,p) = -\frac{g}{M^2} N_0^2\left(\frac{p}{\Lambda_{*0}}\right)\left[\Lambda^2 G\left(\frac{\Lambda}{\Lambda_{*0}}\right) + p^2 H\left(\frac{\Lambda}{\Lambda_{*0}}\right)\right]$$

$N_0\left(\frac{p}{\Lambda_{*0}}\right)$ Pre-factor of LO t-matrix $t_0^{(0)}(p,p)$

$G\left(\frac{\Lambda}{\Lambda_{*0}}\right), H\left(\frac{\Lambda}{\Lambda_{*0}}\right)$ Exact forms unknown

guess

$$\rightarrow \quad V_{S0}^{(1)}(p',p) = \frac{\pi^2}{m}\left[C_0^{(1)} + D_0^{(1)}(p'^2 + p^2)\right] \sim \frac{\pi^2}{m}\left(\frac{gQ}{M^2} + \frac{g}{QM^2}(p'^2+p^2)\right) \sim V_S^{(0)}\left(\frac{Q^2}{M^2}\right)$$

$$C_0(\Lambda) = C_0^{(0)}(\Lambda) + C_0^{(1)}(\Lambda)$$

LO running (already determined)

NLO running yet to be determined

Renormalization of NLO

Put in NLO counterterms $V_{S0}^{(1)}(p',p)=\frac{\pi^2}{m}\left[C_0^{(1)}+D_0^{(1)}(p'^2+p^2)\right]$

$$\Lambda\frac{d}{d\Lambda}t_0^{(1)}(p,p)=-N_0^2\left[\left(\frac{p}{\Lambda_{*0}}\right)\right]\frac{\Lambda^2}{M^2}\left[R(C_0^{(1)}(\Lambda),D_0^{(1)};\Lambda)+\frac{p^2}{\Lambda^2}S(C_0^{(1)}(\Lambda),D_0^{(1)};\Lambda)+O\left(\frac{p^4}{\Lambda^4}\right)\right]$$

RG invariance fulfilled by

$$R(C_0^{(1)}(\Lambda),D_0^{(1)};\Lambda)=0 \qquad S(C_0^{(1)}(\Lambda),D_0^{(1)};\Lambda)=0$$

But exact forms of R, S unkown

Numerical experiment needs to confirm

RG invariance

NLO *is* perturbative

“Data”

$$k_0(0.1m) = -1.05 \qquad k_0(0.15m) = -0.34$$

chosen such that NLO corrections at 0.1m and 0.15m are 0 and 5%.

NLO correction

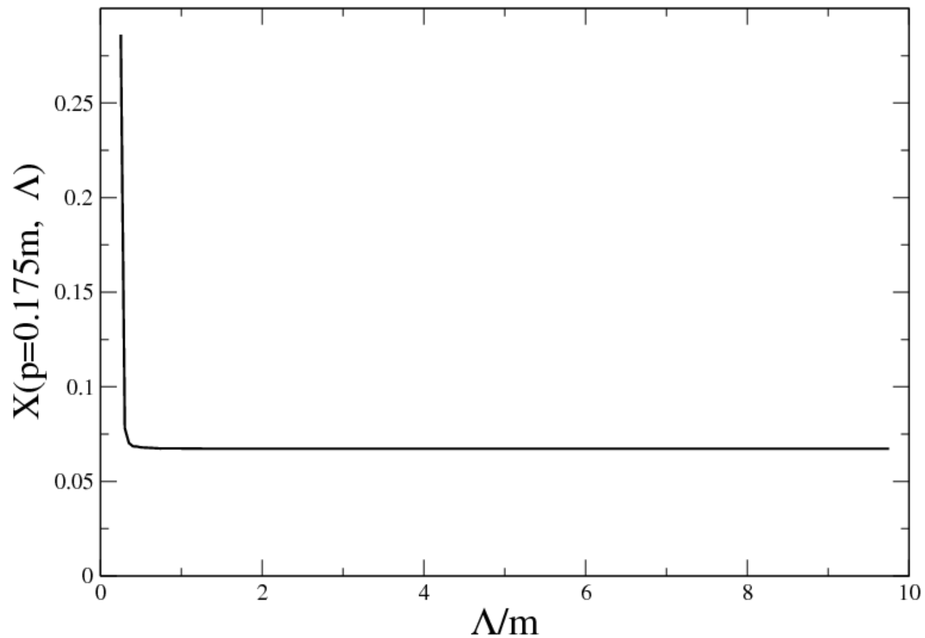
$$X(p,\Lambda) = \left| \frac{k_0^{(1)}(p,p;\Lambda)}{k_0^{(0)}(p,p;\Lambda)} \right|$$

Note: $k_0^{(0)}$ already renormalized

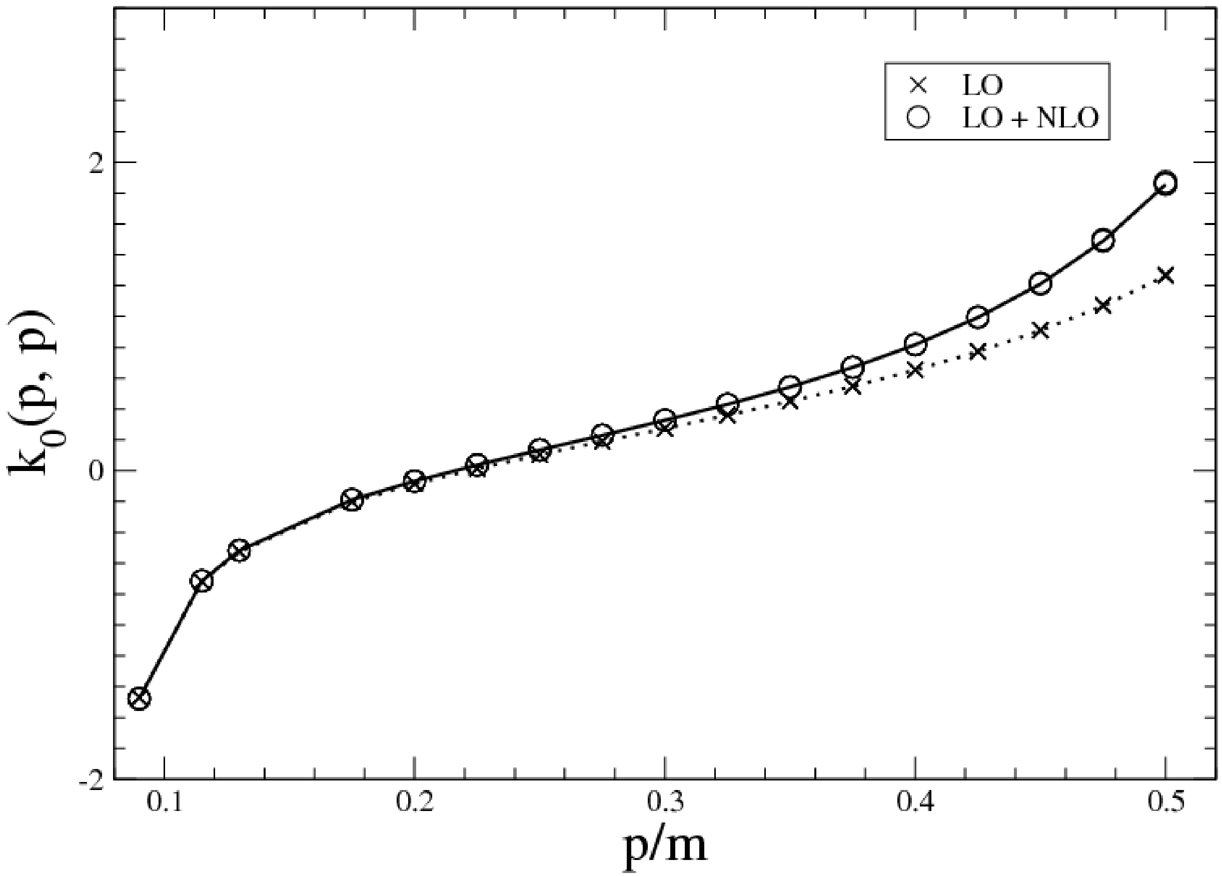
λ=2, g=1, M=0.5m

Cutoff dependence of NLO correction at p=0.175m

→



NLO numerics



LO and NLO S-wave $k_0(p,p)$ as a function of p/m, with Λ=5.5m, 6.5m, 7.5m & 8.5m

Conclusion

Renormalization of singular potentials

→ power counting of EFT of nuclear force

→ Model-independent descriptions of some atomic potentials

Toy theory $-1/r^2 + 1/r^4$

→ Naive dimensional analysis (NDA) fails in power counting of LO

→ NLO potential as perturbation, NDA works at NLO

Outlook

OPE + TPE in singular channels ~ $-1/r^3 + 1/r^5$