The thermodynamic limit of the Lipkin model

Dieter Heiss
Hendrik Geyer
Frederik Scholtz

Stellenbosch University

The Lipkin model:

N Fermions occupying 2 degenerate levels, degeneracy at least *N*-fold.

Interaction lifts or lowers a Fermion pair

$$H = \sum a^{\dagger}_{k,m} a_{k,m} + \lambda \sum a^{\dagger}_{k,m} a^{\dagger}_{k',m'} a_{k',-m'} a_{k,-m}$$

as a consequence:

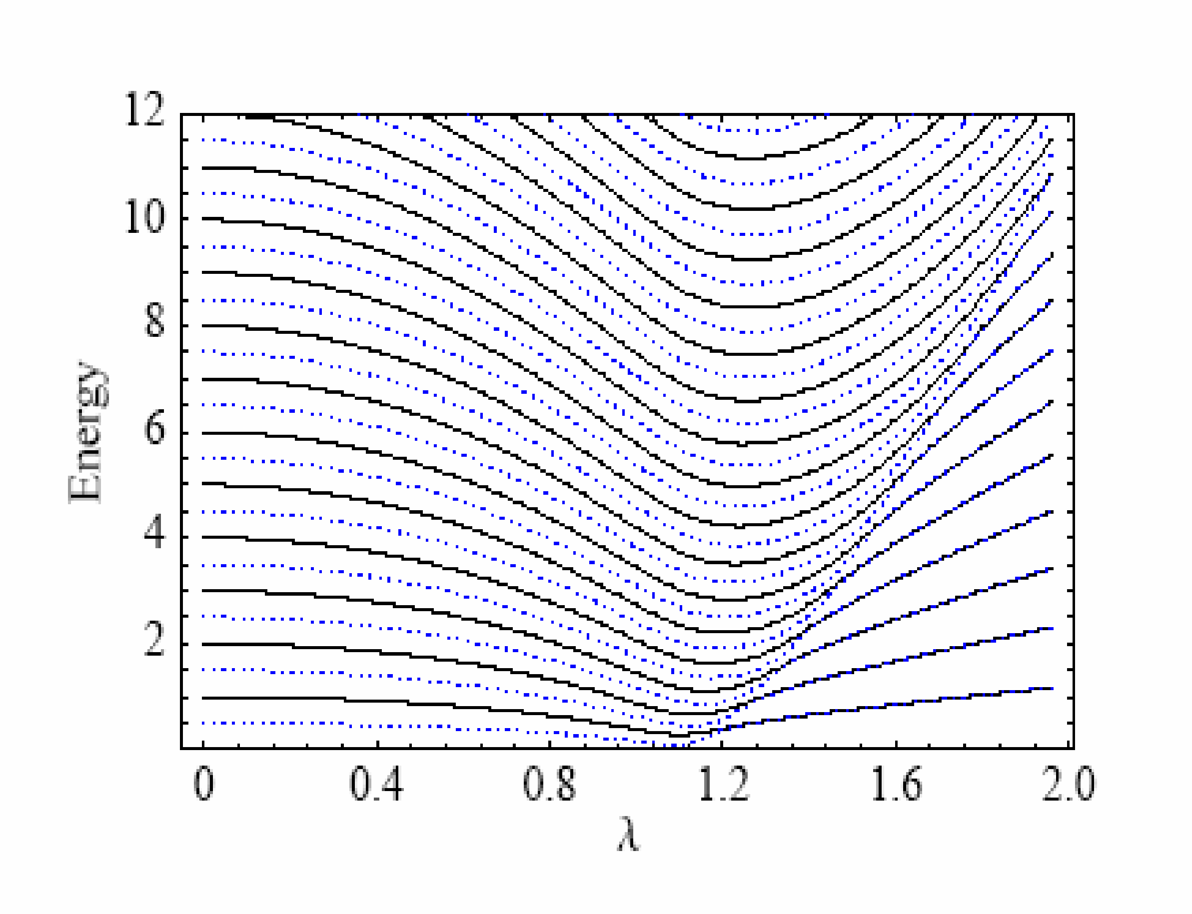
model is reducible into *even* or *odd N*

Hamiltonian conveniently rewritten after energy shift and rescaling:

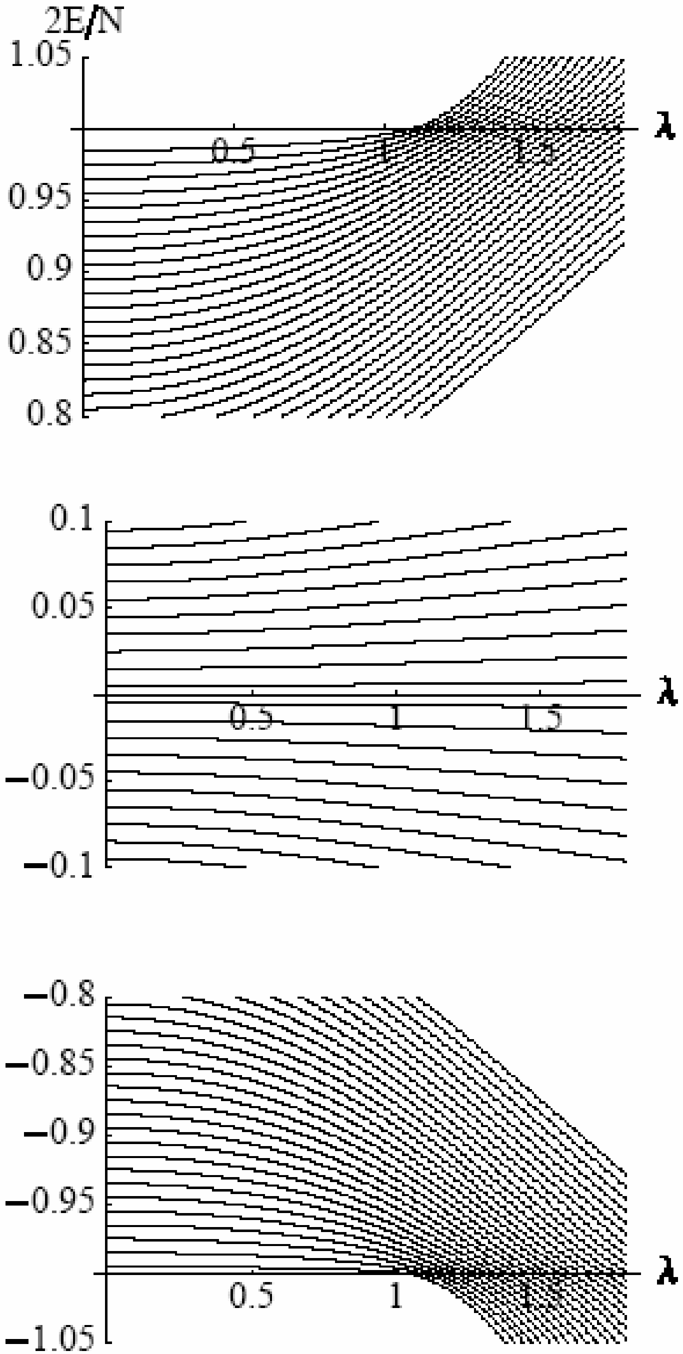
$$H = J_z + \frac{\lambda}{2N}\left(J_+^2 + J_-^2\right)$$

model shows phase transition at $\lambda = 1$

including ***symmetry breaking*** in that for $\lambda>1$ a ‘deformed’ phase occurs where even and odd *N* become degenerate



spectrum with respect to ground state: ground state at 0

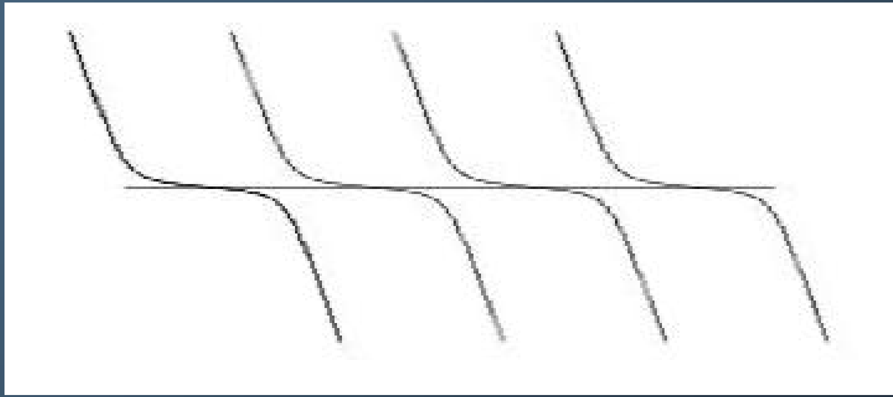


Spectrum as function of λ

nothing interesting in middle, symmetry around E = 0

phase transition for all $\lambda>1$ at $2E/N = -1$ (and $2E/N = +1$)

in fact, magnification along the line $2E/N = -1$ looks like

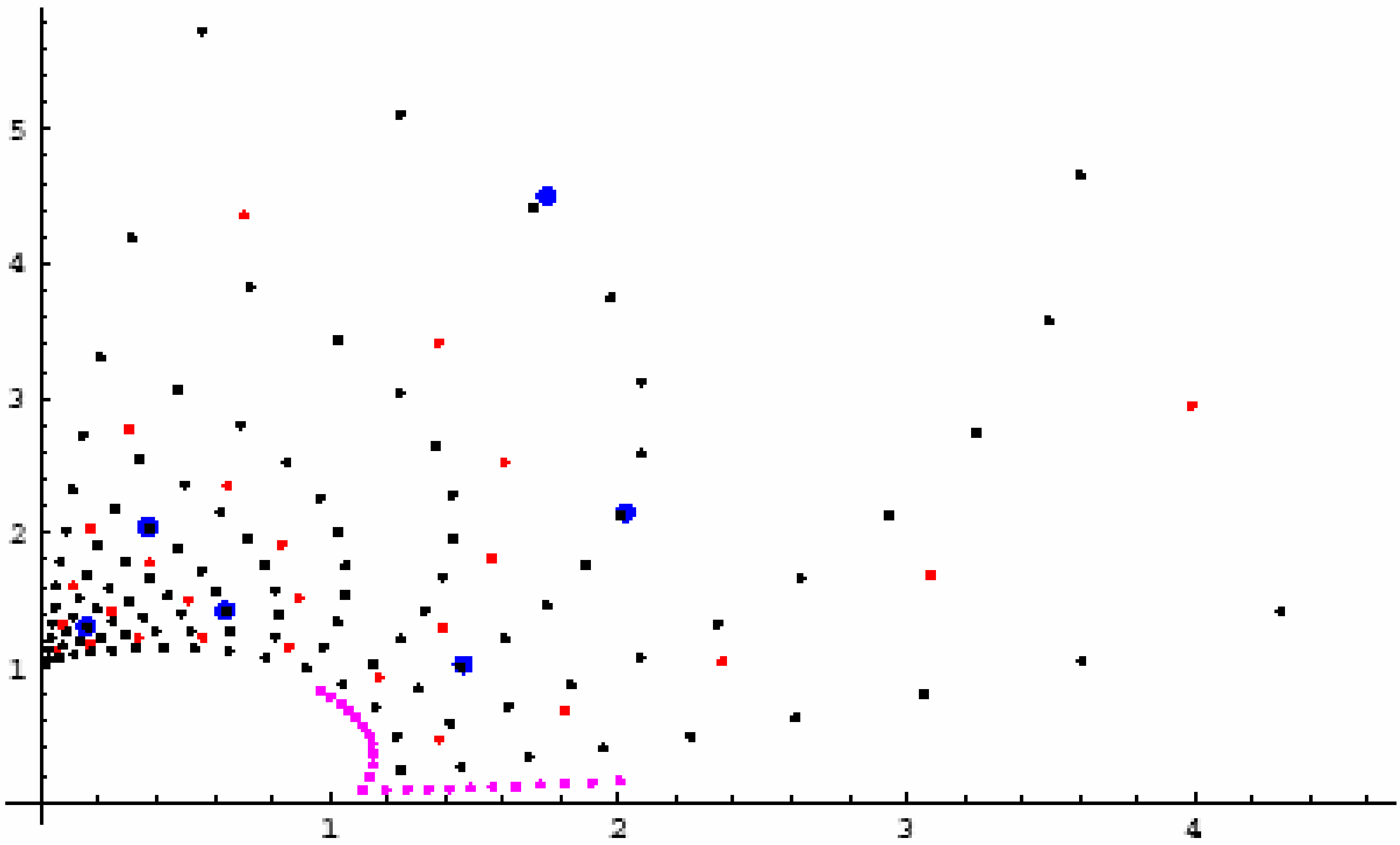


level repulsion – watch EP!

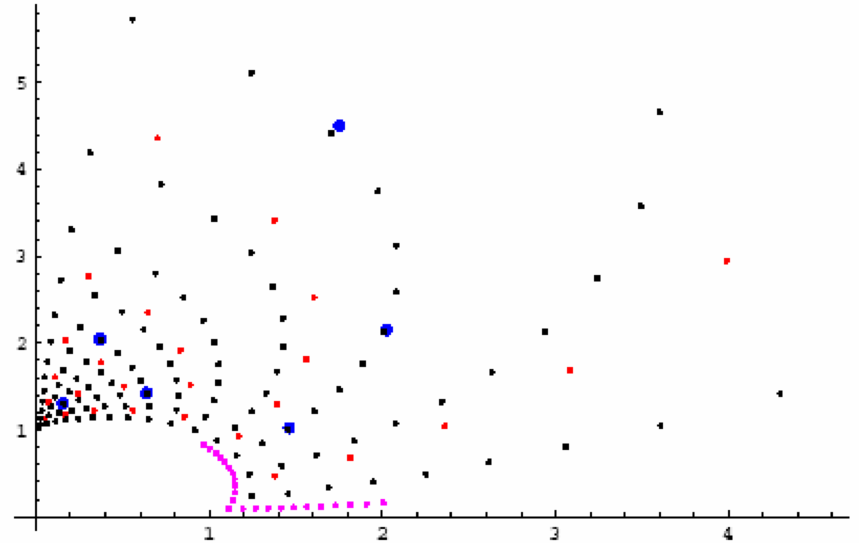
EPs in complex λ - plane for various N

Exceptional Points are square root singularities where two levels *and* their eigenfunctions coalesce. They occur in the vicinity of level repulsions for complex values of the parameter which gives rise to level repulsion. For a finite N-dimensional problem all levels are analytically connected at the EPs; there are N(N-1) EPs.
The EPs give rise to the structure of the spectrum (level repulsion), yielding among others to phase transitions and/or chaos.

EPs in complex λ - plane for various N



N=8 (blue), =16(red), =32(black), =96(pink)



The inner circle

$$|\lambda| < 1$$

remains free of singularities

In contrast, for increasing *N*, EPs accumulate in particular along the real λ - axis for λ>1

If the EPs retain their character in the thermodynamic limit

$$N \to \infty$$

the Hamilton-op cannot have

1) an ‘obvious’ self-adjoint limit

‘obvious’: not at all or not unique.

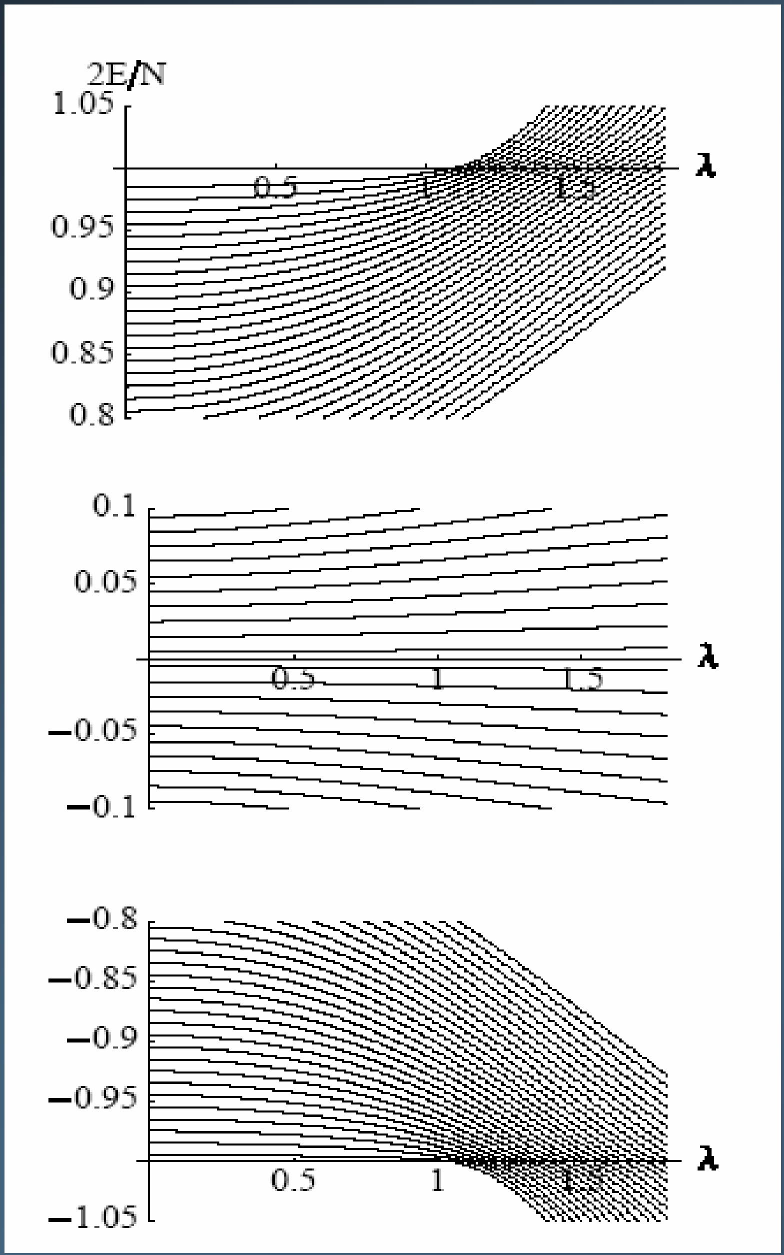
A self-adjoint op cannot have an EP on the real line.

2) the dense population of EPs could forbid analytic connectedness;

for finite *N*, all levels are analytically connected.

A dense set of singularities on a line/curve constitutes a natural boundary of analytic domain

Once more a look at the spectra:



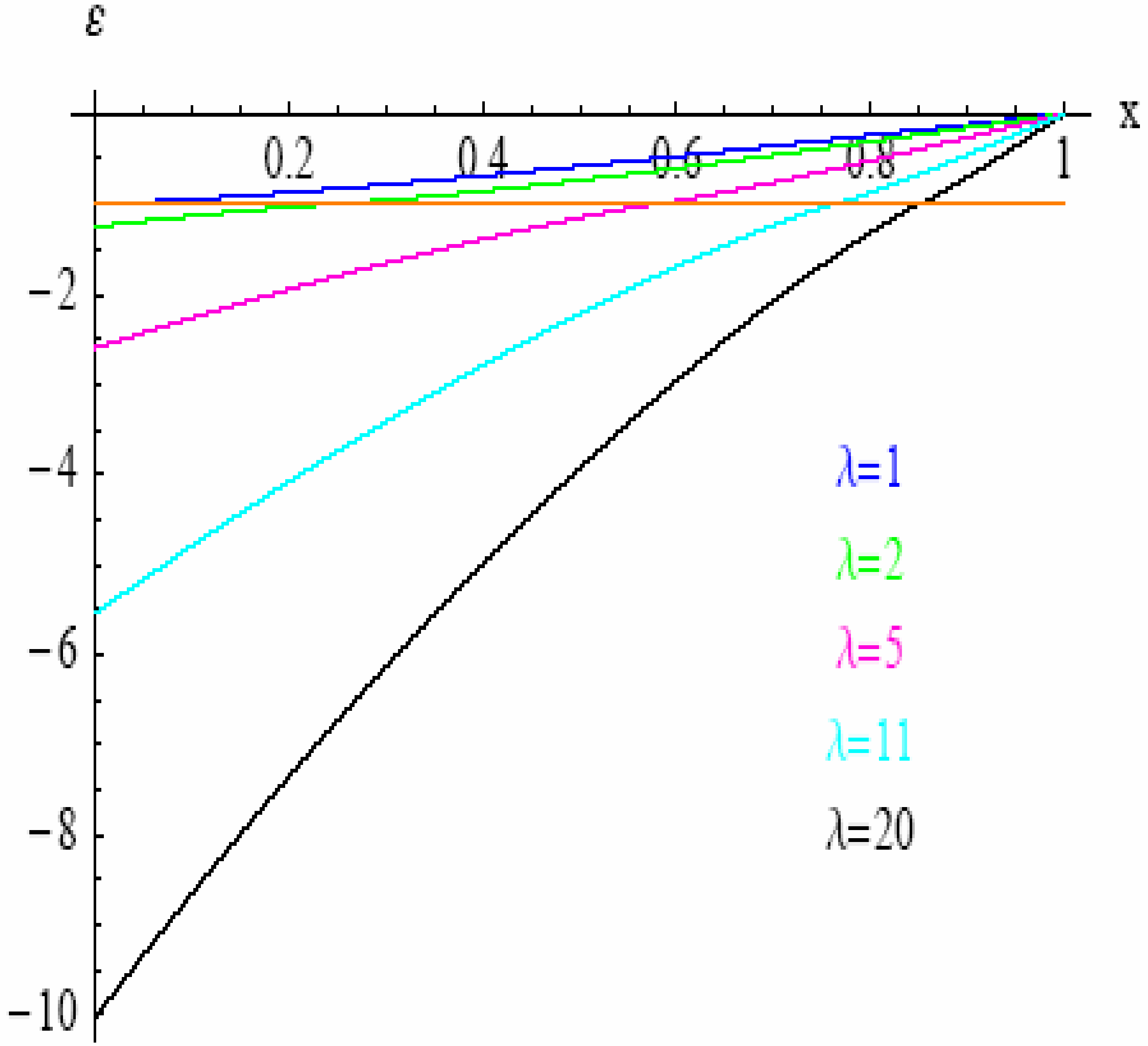
We take cuts for various

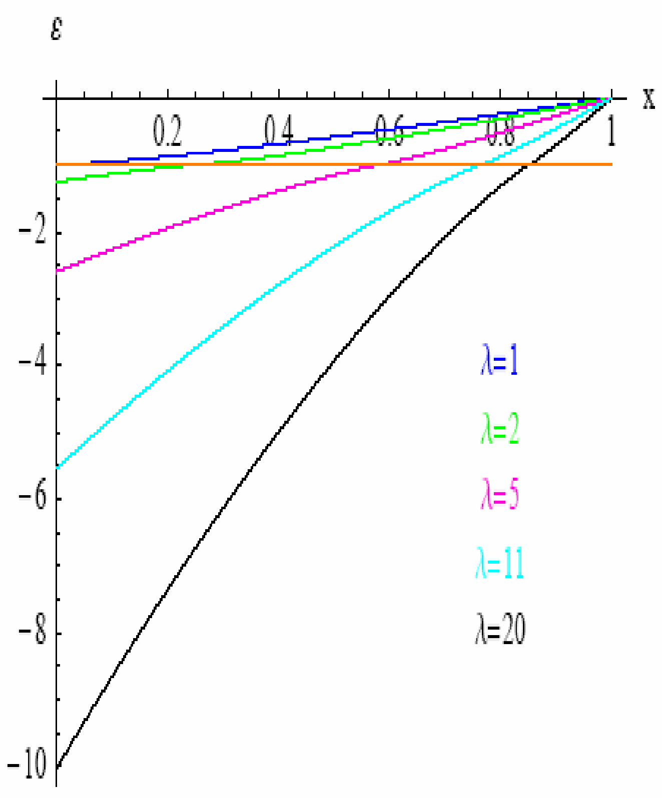
$$\lambda \geq 1$$

and 'enumerate' the lower part of the levels $2E_k/N = \varepsilon(x)$ by the 'continuous' label $0<x<1$

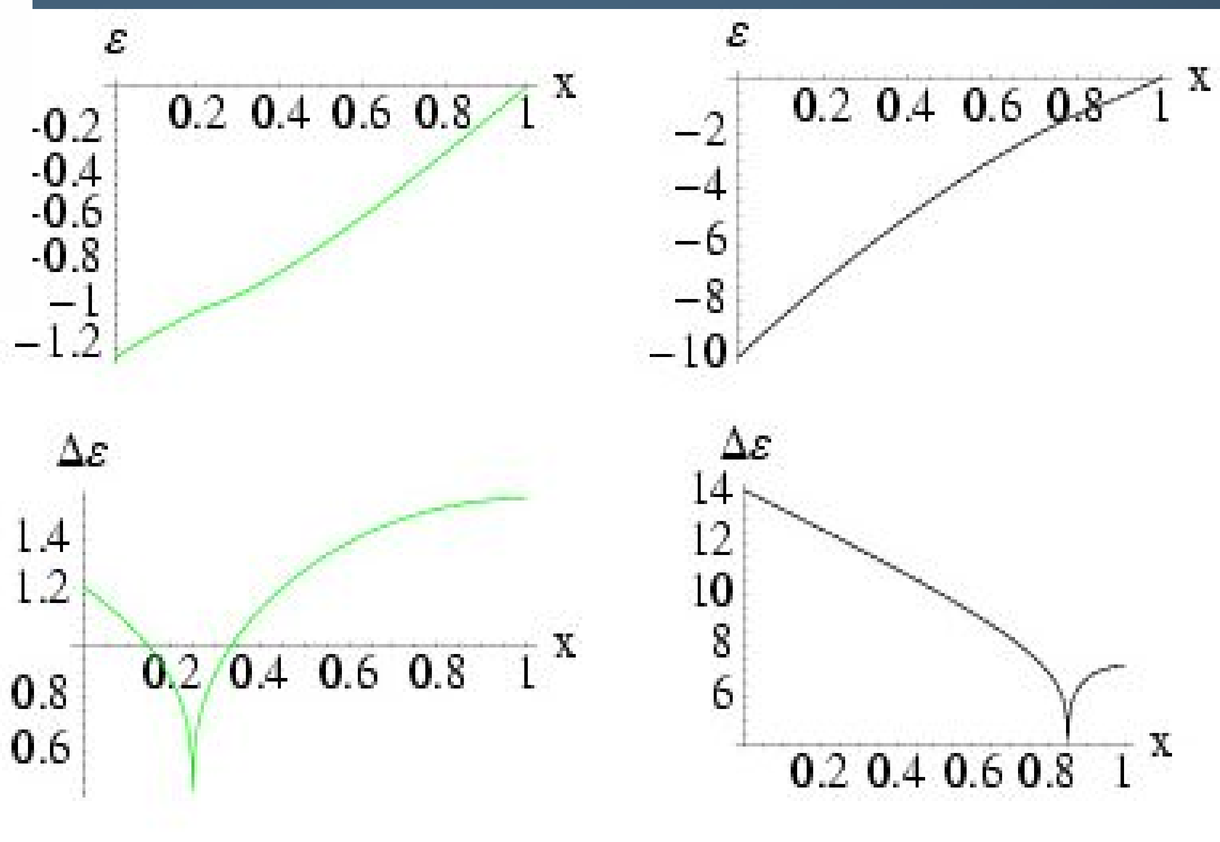
$$k = 1 \quad \Leftrightarrow x = 0$$

$$k = N/2 \Leftrightarrow x = 1$$

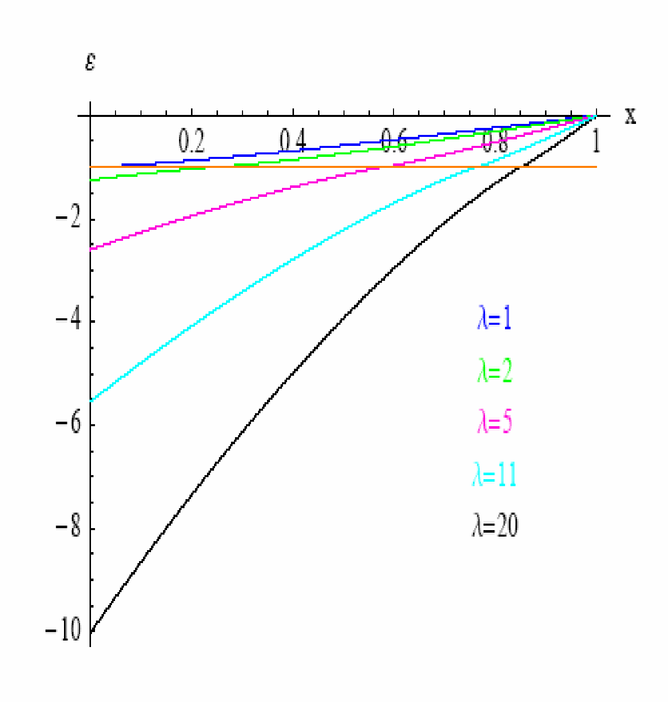

ε
x
0.2
0.4
0.6
0.8
1
−2
−4
−6
−8
−10
λ=1
λ=2
λ=5
λ=11
λ=20

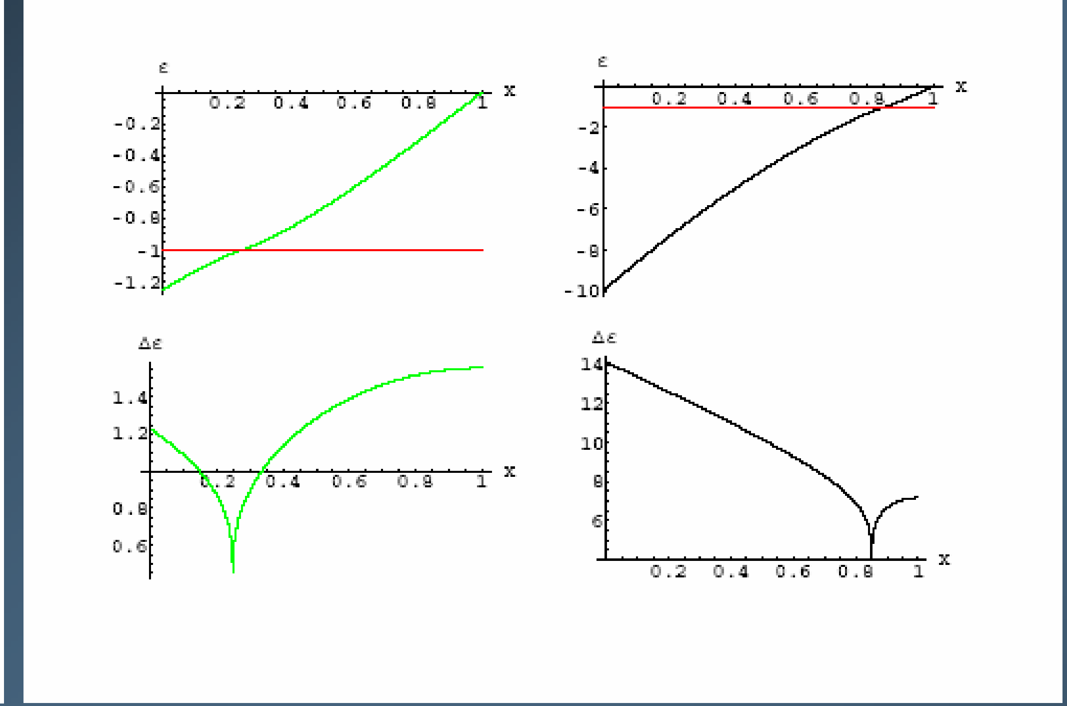


The red line at $\varepsilon = -1$ separates the normal (above) from the deformed (below) phase. Note again the



A closer look at special role of th





When the spectrum passes through the red line it shows – for *N* infinity – a point of inflection with a vanishing derivative while the second derivative is infinity, it is a **singularity.**

For the energy at $\varepsilon = -1$ as well as for the state vector we do understand the independence of λ

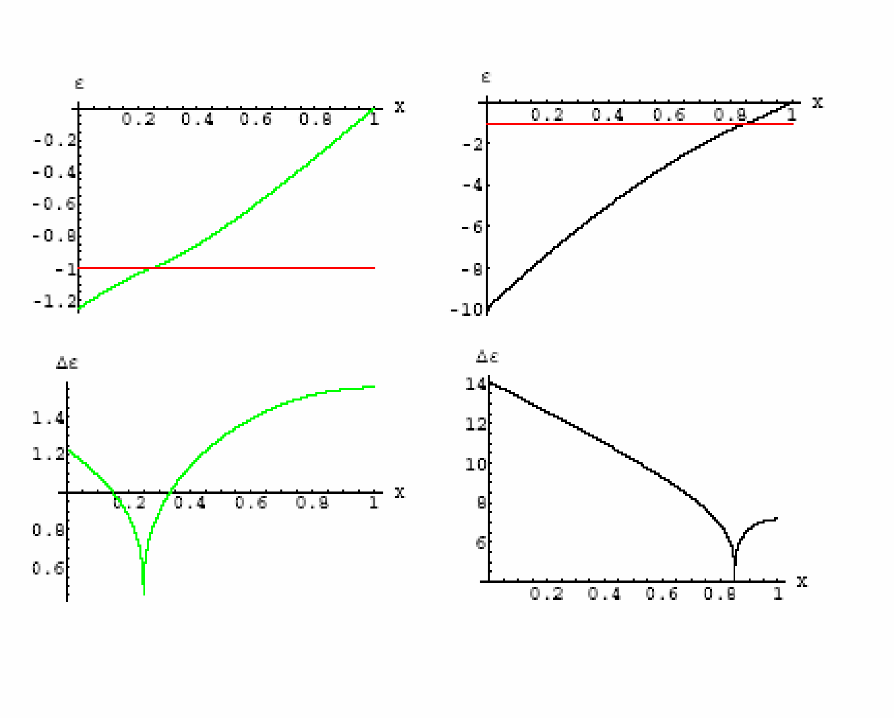
$$\frac{2}{N}\left[J_z + \frac{\lambda}{2N}(J_+^2 + J_-^2)\right]|j,-j\rangle =$$

$$= -|j,-j\rangle + \lambda|j,-j+2\rangle \times O(\frac{1}{N})$$

where $j=N/2$. The second term vanishes in limit.

Recall: for finite N all states are analytically connected.

Note: this implies an optimal localisation for this special state.



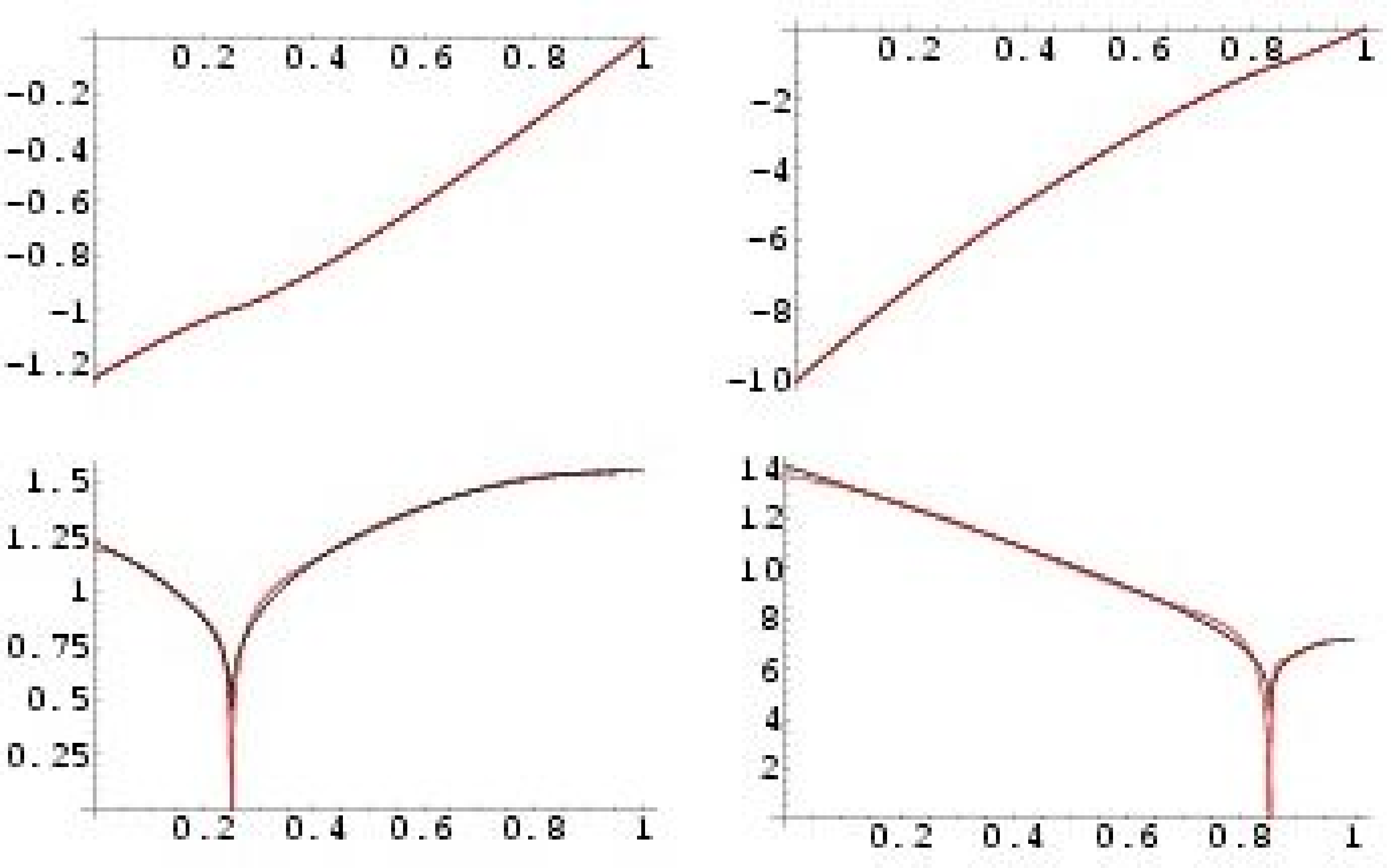
Trying to describe these curves, one must catch the singular behaviour. Denoting by $x_c(\lambda)$ the point of inflection, the best fit is obtained by

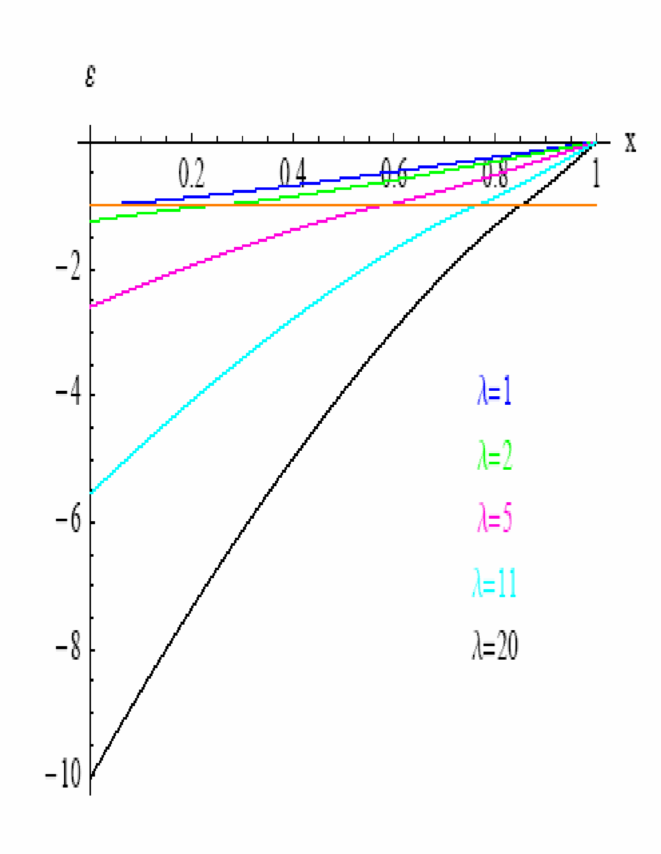
$$(x - x_c(\lambda))^2 \sum_{k=0} a_k(\lambda)(\log|x - x_c(\lambda)|)^k$$

where, however, the $a_k(\lambda)$ are different below and above the red line:

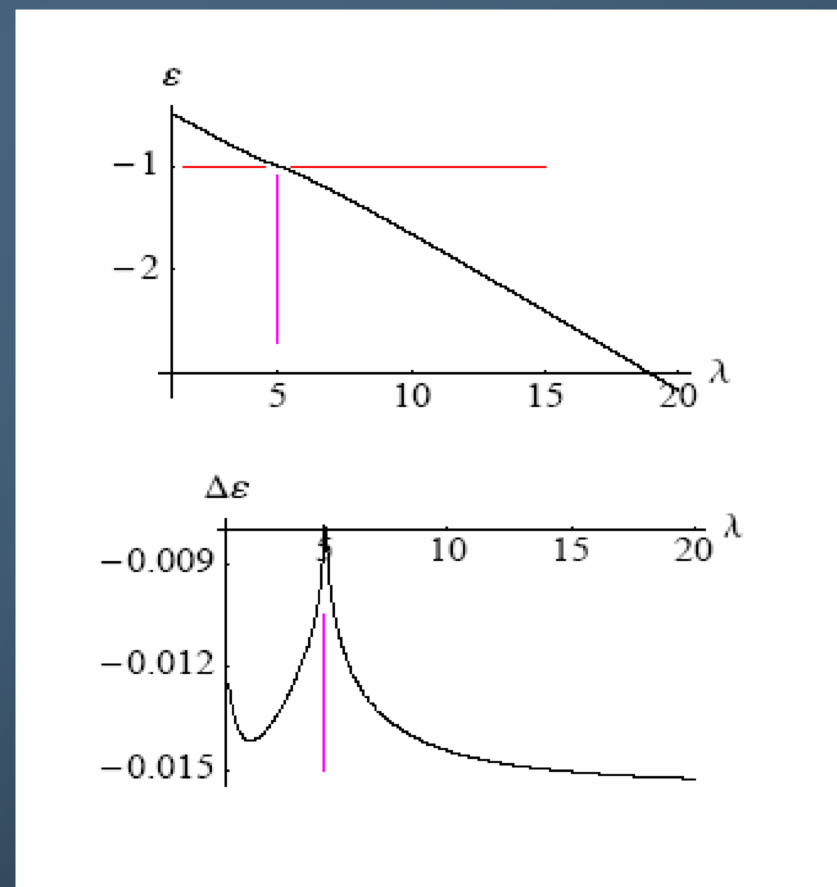
the two regimes are disconnected analytically!

Examples of the quality of the fits, k=3; the respective derivatives compare the derivative of the data with that of the primary fit.





In this figure we can look at one particular level (x fixed) and study its behaviour as a function of λ.



A typical example is the transition at $\lambda=5$ for $x=0.58$

again the same notorious cusp with behaviour

$$(\lambda-\lambda_c)^2 \log|\lambda-\lambda_c| + \ldots$$

Summary:

for $N \to \infty$

1. The EPs accumulate densely including the real λ – axis for λ > 1 evoking a dense set of log-singularities .

2. For real λ the two phase regimes become analytically disconnected.

3. There are two limits for the operator: the normal phase and the deformed phase

Questions left (at this stage)

Do the eigenvectors of each phase form a complete set?

Is each spectrum an analytic function of λ?

While the two phases are seemingly disconnected for real λ, is there a path in the λ – plane that connects them?

Future developments:

use time dependent

interaction parameter λ:

switch λ on – off or just on

can – for $N \to \infty$ –

a transition occur when λ switches

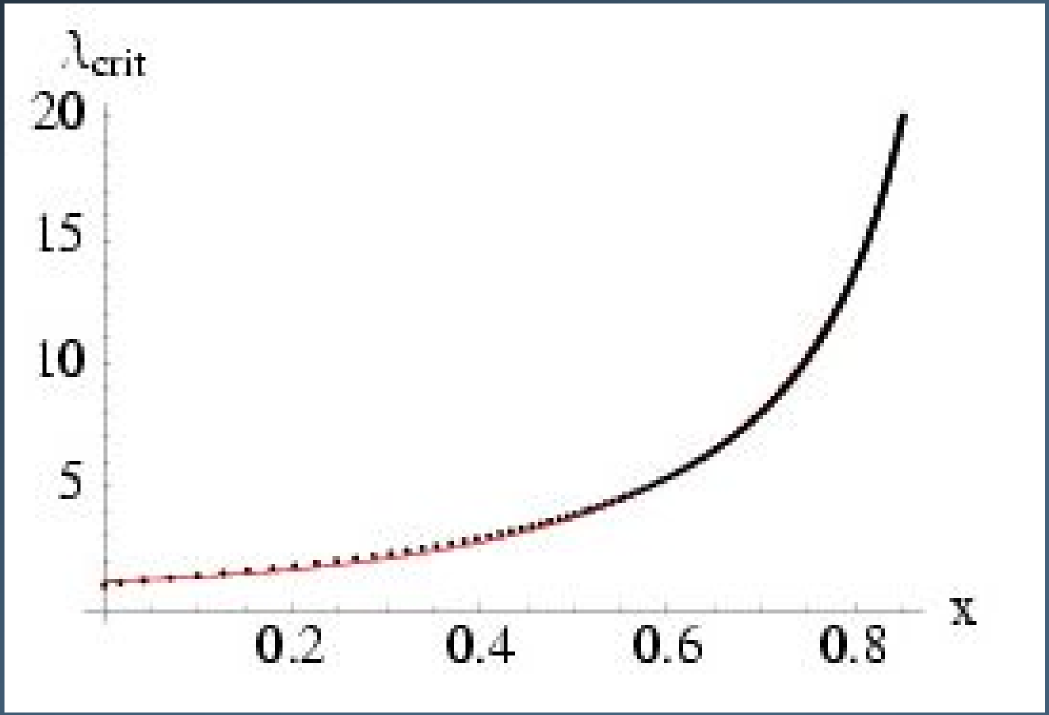
from λ<1 (normal phase)

to λ>1 (deformed phase)?

state: off-equilibrium ?

The End

thank you for your attention



λ_c versus x: seems to obey

$$\lambda_{crit} = A + B\frac{x}{\log x}$$

energy gap at the transition point, for large but finite *N*

$$\Delta E \sim \frac{1}{N^{1/3}} \quad \text{for } \lambda = 1$$

$$\Delta E \sim \frac{\sqrt{\lambda^2 - 1}}{\log N} \quad \text{for } \lambda > 1$$