Solutions of the Nuclear Pairing Problem

A.B. Balantekin

University of Wisconsin–Madison, U.S.A.

INT Seminar, September 28, 2004

Fermion Pairing

- A salient property of multi-fermion systems.
- Already in 1950 Meyer suggested that short-range attractive N-N interaction yields J=0 nuclear ground states.
- Mean field calculations with effective interactions describe many nuclear properties, however they cannot provide a complete solution.
- After many years of investigations we now that the structure of low-lying collective states in medium heavy to heavy nuclei are determined by pairing correlations. This was exploited by many successful models of nuclear structure.

- Pairing in Nuclear Matter: Neutron superfluidity is present in the crust and the inner part of a neutron star. Pairing could significantly effect the thermal evolution of the neutron star by suppressing neutrino (and possibly exotics such as axions) emission.
- Charge symmetry: interactions between two protons and two neutrons are very similar.
- Isospin symmetry: proton-neutron interaction is also very similar.
- There is very little experimental information about np pairing in heavier nuclei. Radiative beam facilities will change this picture.
- Theory of pairing in nuclear physics has many parallels with the theory of ultrasmall metallic grains in condensed matter physics.

- First microscopic theory of pairing: Bardeen, Cooper, Schriffer (BCS) theory, 1957
- Applications of BCS theory to nuclear structure: Bohr, Belyaev, Migdal 1958-1959.
- Application of the BCS theory to nuclear structure has main drawback: BCS wave function is NOT an eigenstate of the number operator. Several solutions were offered:
  - Random-Phase Approximation (RPA)
  - Projection of the particle number after variation
  - Projection of the particle number before variation, Lipkin-Nogami technique

Quasi-Spin Algebra

$$\hat{S}_j^+ = \sum_{m>0} (-1)^{(j-m)} a_{j\,m}^\dagger a_{j\,-m}^\dagger,$$
$$\hat{S}_j^- = \sum_{m>0} (-1)^{(j-m)} a_{j\,-m} a_{j\,m}$$

$$\hat{S}_j^0 = \frac{1}{2} \sum_{m>0} \left(a_{j\,m}^\dagger a_{j\,m} + a_{j\,-m}^\dagger a_{j\,-m} - 1, \right)$$

Mutually commuting SU(2) algebras:

$$[\hat{S}_i^+, \hat{S}_j^-] = 2\delta_{ij} \hat{S}_j^0, \qquad [\hat{S}_i^0, \hat{S}_j^\pm] = \pm\delta_{ij} \hat{S}_j^\pm$$

$$\hat{S}_j^0 = \hat{N}_j - \frac{1}{2}\Omega_j.$$

$\Omega_j = j + \frac{1}{2}$ = the maximum number of pairs that can occupy the level j

$$\hat{N}_j = \frac{1}{2}\sum_{m>0}\left(a^\dagger_{j\,m}a_{j\,m} + a^\dagger_{j\,-m}a_{j\,-m}\right).$$

$0 < \hat{N}_j < \Omega_j \longrightarrow \frac{1}{2}\Omega_j$ representation

- Nucleons interacting with a pairing force:

$$\hat{H} = \sum_{jm} \epsilon_j a^\dagger_{j\,m} a_{j\,m} - |G| \sum_{jj'} c_{jj'} \hat{S}^+_j \hat{S}^-_{j'} .$$

- When the pairing strength is separable ($c_{jj'} = c^*_j c_{j'}$):

$$\hat{H} = \sum_{jm} \epsilon_j a^\dagger_{j\,m} a_{j\,m} - |G| \sum_{jj'} c^*_j c_{j'} \hat{S}^+_j \hat{S}^-_{j'} .$$

- If we assume that the NN interaction is determined by a single parameter (scattering length) and the single-particle energies are discrete we then get

$$\hat{H} = \sum_{jm} \epsilon_j a^\dagger_{j\,m} a_{j\,m} - |G| \sum_{jj'} \hat{S}^+_j \hat{S}^-_{j'} .$$

This case was solved by Richardson.

$$\hat{H} = \sum_{jm} \epsilon_j a^{\dagger}_{j\,m} a_{j\,m} - |G| \sum_{jj'} \hat{S}^+_j \hat{S}^-_{j'} .$$

If we assume that the energy levels are degenerate the first term is a constant for a given number of pairs. This can be solved by using quasispin algebra since $H \propto S^+ S^-$. (Kerman)

Exactly solvable cases:

- Quasi-spin limit (Kerman)

$$\hat{H} = -|G| \sum_{jj'} \hat{S}_j^+ \hat{S}_{j'}^- .$$

- Richardson's solution:

$$\hat{H} = \sum_{jm} \epsilon_j a_{j\,m}^\dagger a_{j\,m} - |G| \sum_{jj'} \hat{S}_j^+ \hat{S}_{j'}^- .$$

- Gaudin's model - closely related to Richardson's.
- The limit in which the energy levels are degenerate (the first term is a constant for a given number of pairs):

$$\hat{H} = -|G| \sum_{jj'} c_j^* c_{j'} \hat{S}_j^+ \hat{S}_{j'}^- .$$

  (Draayer, Pan, Balantekin, Pehlivan, de Jesus)
- Most general separable case with two shells (Balantekin and Pehlivan).

Gaudin Algebra

$$[J^+(\lambda), J^-(\mu)] = 2\frac{J^0(\lambda) - J^0(\mu)}{\lambda - \mu},$$

$$[J^0(\lambda), J^\pm(\mu)] = \pm\frac{J^\pm(\lambda) - J^\pm(\mu)}{\lambda - \mu},$$

$$[J^0(\lambda), J^0(\mu)] = [J^\pm(\lambda), J^\pm(\mu)] = 0$$

A possible realization:

$$J^0(\lambda) = \sum_{i=1}^{N} \frac{\hat{S}_i^0}{\epsilon_i - \lambda} \quad \text{and} \quad J^\pm(\lambda) = \sum_{i=1}^{N} \frac{\hat{S}_i^\pm}{\epsilon_i - \lambda}.$$

$$H(\lambda) = J^0(\lambda)J^0(\lambda) + \frac{1}{2}J^+(\lambda)J^-(\lambda) + \frac{1}{2}J^-(\lambda)J^+(\lambda)$$

Not the Casimir operator of the Gaudin algebra!

$$[H(\lambda), H(\mu)] = 0$$

Lowest weight vector

$$J^-(\lambda)|0\rangle = 0, \quad \text{and} \quad J^0(\lambda)|0\rangle = W(\lambda)|0\rangle$$

$$H(\lambda)|0\rangle = \left[W(\lambda)^2 - W'(\lambda)\right]|0\rangle$$

How do we find other eigenstates? Consider the state $|\xi\rangle \equiv J^+(\xi)|0\rangle$ for an arbitrary complex number ξ. Since

$$[H(\lambda), J^+(\xi)] = \frac{2}{\lambda - \xi}\left(J^+(\lambda)J^0(\xi) - J^+(\xi)J^0(\lambda)\right).$$

Hence if $W(\xi) = 0$, then $J^+(\xi)|0\rangle$ is an eigenstate of $H(\lambda)$ with the eigenvalue

$$E_1(\lambda) = \left[W(\lambda)^2 - W'(\lambda)\right] - 2\frac{W(\lambda)}{\lambda - \xi}.$$

Gaudin showed that this can be generalized.

A state of the form

$$|\xi_1, \xi_2, \ldots, \xi_n > \equiv J^+(\xi_1) J^+(\xi_2) \ldots J^+(\xi_n)|0>$$

is an eigenvector of $H(\lambda)$ if the numbers $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{C}$ satisfy the so-called Bethe Ansatz equations:

$$W(\xi_\alpha) = \sum_{\substack{\beta=1 \\ (\beta \neq \alpha)}}^{n} \frac{1}{\xi_\alpha - \xi_\beta} \quad \text{for} \quad \alpha = 1, 2, \ldots, n.$$

Corresponding eigenvalue is

$$E_n(\lambda) = \left[W(\lambda)^2 - W'(\lambda) \right] - 2 \sum_{\alpha=1}^{n} \frac{W(\lambda) - W(\xi_\alpha)}{\lambda - \xi_\alpha}.$$

$\mathcal{R}$-operators:

$$\lim_{\lambda \to \epsilon_k} (\lambda - \epsilon_k) H(\lambda) = \mathcal{R}_k$$

$$\mathcal{R}_k = -2 \sum_{j \neq k} \frac{\mathbf{S}_k \cdot \mathbf{S}_j}{\epsilon_k - \epsilon_j}$$

$$[H(\lambda), H(\mu)] = 0 \quad \Rightarrow \quad \begin{array}{l} [H(\lambda), \mathcal{R}_k] = 0 \\ [\mathcal{R}_j, \mathcal{R}_k] = 0 \end{array}$$

One can also show that

$$\sum_i \mathcal{R}_i = 0$$

and

$$\sum_i \epsilon_i \mathcal{R}_i = -2 \sum_{i \neq j} \mathbf{S}_i \cdot \mathbf{S}_j$$

Recall the Gaudin Algebra

$$[J^+(\lambda), J^-(\mu)] = 2\frac{J^0(\lambda) - J^0(\mu)}{\lambda - \mu},$$

$$[J^0(\lambda), J^\pm(\mu)] = \pm\frac{J^\pm(\lambda) - J^\pm(\mu)}{\lambda - \mu},$$

$$[J^0(\lambda), J^0(\mu)] = [J^\pm(\lambda), J^\pm(\mu)] = 0$$

Not only the operators $\mathbf{J}(\lambda)$, but also the operators $\mathbf{J}(\lambda) + \mathbf{c}$ satisfy this algebra for a constant $\mathbf{c}$. In this case

$$H(\lambda) = \mathbf{J}(\lambda) \cdot \mathbf{J}(\lambda) \Rightarrow H(\lambda) + 2\mathbf{c} \cdot \mathbf{J}(\lambda) + \mathbf{c}^2$$

which has the same eigenstates.

Richardson operators:

$$\lim_{\lambda \to \epsilon_k} (\lambda - \epsilon_k) \left(H(\lambda) + 2\mathbf{c} \cdot \mathbf{S} \right) = R_k$$

$$R_k = -2\mathbf{c} \cdot \mathbf{S}_k - 2 \sum_{j \neq k} \frac{\mathbf{S}_k \cdot \mathbf{S}_j}{\epsilon_k - \epsilon_j}$$

$$[H(\lambda) + 2\mathbf{c} \cdot \mathbf{S}, R_k] = 0 \quad [R_j, R_k] = 0$$

and

$$\sum_i R_i = -2\mathbf{c} \cdot \sum_k \mathbf{S}_k$$

$$\sum_i \epsilon_i R_i = -2 \sum_i \epsilon_i \mathbf{c} \cdot \mathbf{S}_i - 2 \sum_{i \neq j} \mathbf{S}_i \cdot \mathbf{S}_j$$

$$\hat{H} = \sum_{jm} \epsilon_j a^\dagger_{j\,m} a_{j\,m} - |G| \sum_{jj'} \hat{S}^+_j \hat{S}^-_{j'} .$$

$$\Rightarrow H = \sum_j \epsilon_j S^0_j - |G| \left((\sum_i \mathbf{S}_i) \cdot (\sum_i \mathbf{S}_i) - (\sum_i S^0_i)^2 + (\sum_i S^0_i) \right) + \text{constant terms}$$

Choose

$$\mathbf{c} = (0, 0, -1/2|G|)$$

then

$$\frac{H}{|G|} = \sum_i \epsilon_i R_i + |G|^2 (\sum_i R_i)^2 - |G| \sum_i R_i + \cdots$$

Degenerate Solution

Define

$$\hat{S}^+(0) = \sum_j c_j^* \hat{S}_j^+ \quad \text{and} \quad \hat{S}^-(0) = \sum_j c_j \hat{S}_j^-,$$

$$\hat{H} = -|G|\hat{S}^+(0)\hat{S}^-(0).$$

In the 1970's Talmi showed that under certain assumptions, a state of the form

$$\hat{S}^+(0)|0\rangle = \sum_j c_j^* \hat{S}_j^+ |0\rangle, \quad |0\rangle\text{: particle vacuum}$$

is an eigenstate of a class of Hamiltonians including the one above. Indeed

$$\hat{H}\hat{S}^+(0)|0\rangle = \left(-|G|\sum_j \Omega_j |c_j|^2\right)\hat{S}^+(0)|0\rangle$$

What about other one-pair states?

For example for two levels j_1 and j_2, the orthogonal state

$$\left(\frac{c_{j_2}}{\Omega_{j_1}} \hat{S}^+_{j_1} - \frac{c_{j_1}}{\Omega_{j_2}} \hat{S}^+_{j_2} \right) |0\rangle,$$

is also an eigenstate with E=0.

Energy/$(-\|G\|)$	State
0	$\left(-\frac{c_{j_2}}{\Omega_{j_1}}\hat{S}_{j_1}^{+} + \frac{c_{j_1}}{\Omega_{j_2}}\hat{S}_{j_2}^{+}\right)\|0\rangle$
$\Omega_{j_1}\|c_{j_1}\|^2 + \Omega_{j_2}\|c_{j_2}\|^2$	$\left(c_{j_1}^{*}\hat{S}_{j_1}^{+} + c_{j_2}^{*}\hat{S}_{j_2}^{+}\right)\|0\rangle$

States with N=1 for two shells

What about other one-pair states?

For example for two levels j_1 and j_2, the orthogonal state

$$\left(\frac{c_{j_2}}{\Omega_{j_1}} \hat{S}_{j_1}^{+} - \frac{c_{j_1}}{\Omega_{j_2}} \hat{S}_{j_2}^{+} \right) |0\rangle,$$

is also an eigenstate with E=0.

Is there a systematic way to derive these states?

Yes, as showed by Pan, et al. for particle pair states.

Define

$$\hat{S}^+(x) = \sum_j \frac{c_j^*}{1 - |c_j|^2 x} \hat{S}_j^+ \quad \text{and} \quad \hat{S}^-(x) = \sum_j \frac{c_j}{1 - |c_j|^2 x} \hat{S}_j^- .$$

Then eigenstates are of the form

$$\hat{S}^+(x)\hat{S}^+(y)\cdots\hat{S}^+(z)|0\rangle$$

F. Pan, J.P. Draayer, W.E. Ormand, Phys. Lett. B **422**, 1 (1998)

$$\hat{S}^+(x) = \sum_j \frac{c_j^*}{1 - |c_j|^2 x} \hat{S}_j^+ \quad \text{and} \quad \hat{S}^-(x) = \sum_j \frac{c_j}{1 - |c_j|^2 x} \hat{S}_j^-$$

Introduce the operator

$$\hat{K}^0(x) = \sum_j \frac{1}{1/|c_j|^2 - x} \hat{S}_j^0$$

$$[\hat{S}^+(x), \hat{S}^-(0)] = [\hat{S}^+(0), \hat{S}^-(x)] = 2K^0(x)$$

$$[\hat{K}^0(x), \hat{S}^\pm(y)] = \pm \frac{\hat{S}^\pm(x) - \hat{S}^\pm(y)}{x - y}$$

This is very similar to Gaudin algebra!

$$\hat{S}^{+}(0)\hat{S}^{+}(z_1^{(N)})\dots\hat{S}^{+}(z_{N-1}^{(N)})|0\rangle$$

is an eigenstate if the following Bethe ansatz equations are satisfied:

$$\sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - z_m^{(N)}} = \frac{1}{z_m^{(N)}} + \sum_{k=1(k\neq m)}^{N-1} \frac{1}{z_m^{(N)} - z_k^{(N)}} \qquad m = 1, 2, \dots N-1.$$

$$E_N = -|G| \left(\sum_j \Omega_j |c_j|^2 - \sum_{k=1}^{N-1} \frac{2}{z_k^{(N)}} \right)$$

Pan et al did not note but this is an eigenstate if the shell is at most half full.

Similarly

$$\hat{S}^+(x_1^{(N)})\hat{S}^+(x_2^{(N)})\dots\hat{S}^+(x_N^{(N)})|0\rangle$$

is an eigenstate with zero energy if the following Bethe ansatz equations are satisfied:

$$\sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - x_m^{(N)}} = \sum_{k=1(k\neq m)}^{N} \frac{1}{x_m^{(N)} - x_k^{(N)}} \quad \text{for every} \quad m = 1, 2, \dots, N$$

Again this is a state if the shell is at most half full.

What if the available states are more than half full? There are degeneracies:

No. of Pairs	**Energy/$(-\|G\|)$**	**State**
1	$\sum_j \Omega_j \|c_j\|^2$	$\hat{S}^+(0)\|0\rangle$
N_{max}	$\sum_j \Omega_j \|c_j\|^2$	$\|\bar{0}\rangle$

$|0\rangle$: particle vacuum

$|\bar{0}\rangle$: state where all levels are completely filled

If the shells are more than half full then the state

$$\hat{S}^-(z_1^{(N)})\hat{S}^-(z_2^{(N)})\ldots\hat{S}^-(z_{N-1}^{(N)})|\bar{0}\rangle$$

is an eigenstate with energy

$$E = -G\left(\sum_j \Omega_j |c_j|^2 - \sum_{k=1}^{N-1} \frac{2}{z_k^{(N)}}\right)$$

if the following Bethe ansatz equations are satisfied

$$\sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - z_m^{(N)}} = \frac{1}{z_m^{(N)}} + \sum_{k=1(k\neq m)}^{N-1} \frac{1}{z_m^{(N)} - z_k^{(N)}}$$

Here $N_{max} + 1 - N =$ number of particle pairs

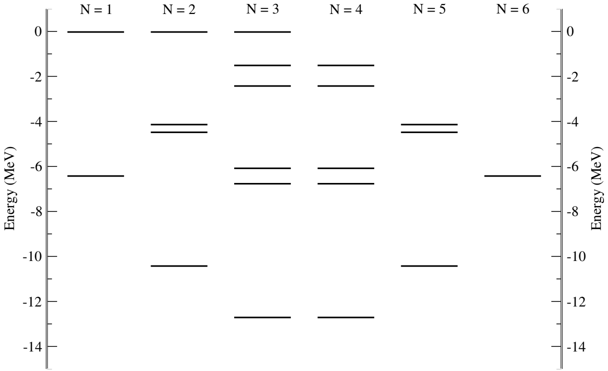
A.B. Balantekin, J. de Jesus, and Y. Pehlivan, Phys. Rev. C **75**, 064304 (2007)

Particle-hole degeneracy:

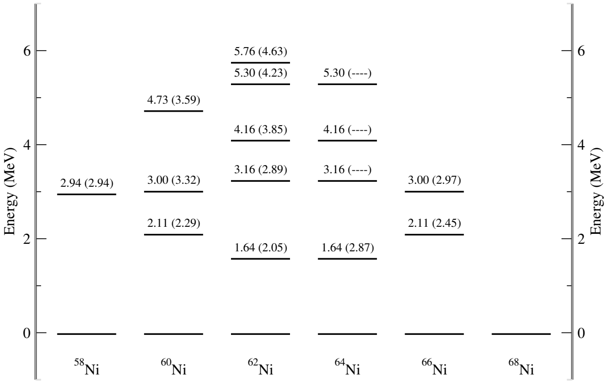
No. of Pairs	**State**
N	$\hat{S}^{+}(0)\hat{S}^{+}(z_1^{(N)})\ldots\hat{S}^{+}(z_{N-1}^{(N)})\|0\rangle$
$N_{max}+1-N$	$\hat{S}^{-}(z_1^{(N)})\hat{S}^{-}(z_2^{(N)})\ldots\hat{S}^{-}(z_{N-1}^{(N)})\|\bar{0}\rangle$

$$E = -G\left(\sum_j \Omega_j |c_j|^2 - \sum_{k=1}^{N-1}\frac{2}{z_k^{(N)}}\right)$$

$$\sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - z_m^{(N)}} = \frac{1}{z_m^{(N)}} + \sum_{k=1(k\neq m)}^{N-1}\frac{1}{z_m^{(N)} - z_k^{(N)}}$$



Results for the sd shell with $0d_{5/2}$, $0d_{3/2}$, and $1s_{1/2}$



theory (experiment)

Exact solutions with two shells

Consider the most general pairing Hamiltonian with only two shells:

$$\frac{\hat{H}}{|G|} = \sum_j 2\varepsilon_j \hat{S}_j^0 - \sum_{jj'} c_j^* c_{j'} \hat{S}_j^+ \hat{S}_{j'}^- + \sum_j \varepsilon_j \Omega_j,$$

with $\varepsilon_j = \epsilon_j/|G|$.
States can be written using the step operators

$$J^+(x) = \sum_j \frac{c_j^*}{2\varepsilon_j - |c_j|^2 x} S_j^+$$

as

$$J^+(x_1)J^+(x_2)\dots J^+(x_N)|0\rangle.$$

Balantekin and Pehlivan, submitted for publication.

$$J^+(x_1)J^+(x_2)\dots J^+(x_N)|0\rangle.$$

Defining

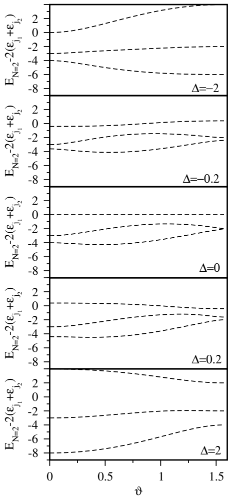
$$\beta = 2\frac{\varepsilon_{j_1} - \varepsilon_{j_2}}{|c_{j_1}|^2 - |c_{j_2}|^2} \qquad \delta = 2\frac{\varepsilon_{j_2}|c_{j_1}|^2 - \varepsilon_{j_1}|c_{j_2}|^2}{|c_{j_1}|^2 - |c_{j_2}|^2}.$$

we obtain

$$E_N = -\sum_{n=1}^{N} \frac{\delta x_n}{\beta - x_n}.$$

If the parameters x_k satisfy the Bethe ansatz equations

$$\sum_j \frac{\Omega_j |c_j|^2}{2\varepsilon_j - |c_j|^2 x_k} = \frac{\beta}{\beta - x_k} + \sum_{n=1(\neq k)}^{N} \frac{2}{x_n - x_k}.$$



Exact Energy eigenvalues for $j_1 = 3/2$ and $j_2 = 5/2$. $\cos\vartheta = c_1$ and $\sin\vartheta = c_2$ $\Delta = \epsilon_1 - \epsilon_2$.

Solutions of Bethe Ansatz equations

$$x_i^{(N)} = \frac{1}{|c_{j_2}|^2} + \eta_i^{(N)} \left(\frac{1}{|c_{j_1}|^2} - \frac{1}{|c_{j_2}|^2} \right)$$

$$\sum_{k=1(k\neq i)}^{N} \frac{1}{\eta_i^{(N)} - \eta_k^{(N)}} - \frac{\Omega_{j_2}/2}{\eta_i^{(N)}} + \frac{\Omega_{j_1}/2}{1 - \eta_i^{(N)}} = 0$$

In 1914 Stieltjes showed that the polynomial

$$p_N(z) = \prod_{i=1}^{N} (z - \eta_i^{(N)})$$

satisfies the hypergeometric equation

$$z(1-z)p_N'' + \left[-\Omega_{j_2} + \left(\Omega_{j_1}\Omega_{j_2}\right) z\right] p_N' + N\left(N - \Omega_{j_1} - \Omega_{j_2} - 1\right) p_N = 0$$

Supersymmetric Quantum Mechanics

Consider two Hamiltonians

$$H_1 = G^\dagger G, \; H_2 = GG^\dagger,$$

where G is an arbitrary operator. The eigenvalues of these two Hamiltonians

$$\begin{aligned} G^\dagger G|1,n\rangle &= E_n^{(1)}|1,n\rangle \\ GG^\dagger|2,n\rangle &= E_n^{(2)}|2,n\rangle \end{aligned}$$

are the same:

$$E_n^{(1)} = E_n^{(2)} = E_n$$

and that the eigenvectors are related:

$$|2,n\rangle = G\left[G^\dagger G\right]^{-1/2}|1,n\rangle.$$

This works for all cases except when $G|1,n\rangle = 0$, which should be the ground state energy of the positive-definite Hamiltonian H_1.

Why is this called supersymmetry? Define

$$Q^\dagger = \begin{pmatrix} 0 & 0 \\ G^\dagger & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix},$$

Then

$$H = \left\{ Q, Q^\dagger \right\} = \begin{pmatrix} H_2 & 0 \\ 0 & H_1 \end{pmatrix}.$$

with

$$[H, Q] = 0 = [H, Q^\dagger].$$

An Applications of SUSY QM to Nuclear Structure Physics

Separable pairing with degenerate single-particle spectra:

$$\hat{H}_{SC} \sim -|G|\hat{S}^+(0)\hat{S}^-(0),$$

$$\hat{S}^+(0) = \sum_j c_j^* \hat{S}_j^+ \quad \text{and} \quad \hat{S}^-(0) = \sum_j c_j \hat{S}_j^- .$$

Introduce the operator

$$\hat{T} = \exp\left(-i\frac{\pi}{2}\sum_i (\hat{S}_i^+ + \hat{S}_i^-)\right)$$

This operator transforms the empty shell, $|0\rangle$, to the fully occupied shell, $|\bar{0}\rangle$:

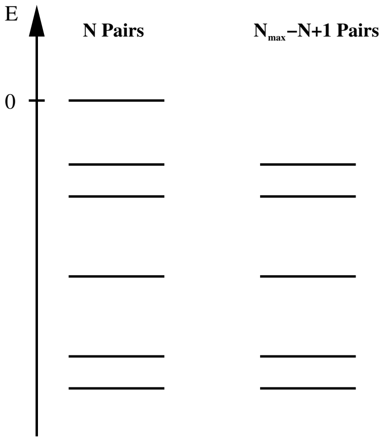
$$\hat{T}|0\rangle = |\bar{0}\rangle$$

Next define

$$\hat{B}^- = \hat{T}^\dagger \hat{S}^-(0), \qquad \hat{B}^+ = \hat{S}^+(0)\hat{T}.$$

- Supersymmetric quantum mechanics tells us that the partner Hamiltonians $\hat{H}_1 = \hat{B}^+\hat{B}^-$ and $\hat{H}_2 = \hat{B}^-\hat{B}^+$ have identical spectra except for the ground state of $\hat{H}_1$
- Here two Hamiltonians $\hat{H}_1$ and $\hat{H}_2$ are actually identical and equal to the pairing Hamiltonian. Hence the role of the supersymmetry is to connect the states $|\Psi_2\rangle$ and $|\Psi_1\rangle$.
- This supersymmetry connects particle and hole states.

A.B. Balantekin and Y. Pehlivan, J. Phys. G **34**, 1783 (2007).



Spectra of Nuclear pairing exhibiting supersymmetry