Ward identities, O(a) improvement & twisted mass QCD (lecture IV)

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Some (more or less) pedagogical references

1. R. Sommer, "Non-perturbative renormalisation of QCD", Schladming Winter School lectures 1997, `hep-ph/9711243v1`;
   "Non-perturbative QCD: Renormalization, O(a) improvement and matching to heavy quark effective theory" Lectures at Nara, November 2005 `hep-lat/0611020`
2. M. Lüscher: "Advanced lattice QCD", Les Houches Summer School lectures 1997 `hep-lat/9802029`
3. S. Capitani, "Lattice perturbation theory" Phys. Rept. 382 (2003) 113-302 `hep-lat/0211036`
4. S. Sint "Nonperturbative renormalization in lattice field theory" Nucl. Phys. (Proc. Suppl.) 94 (2001) 79-94, `hep-lat/0011081`

For twisted mass QCD and chirally twisted Schrödinger functional see

1. A. Shindler "Twisted mass lattice QCD" review article, July 2007 (`arXiv:0707.4093 [hep-lat]`)
2. S. Sint, "Lattice QCD with a chiral twist" Lectures at Nara, November 2005 `hep-lat/0702008`

Contents

- Symmetries and Ward identities
- Wilson quarks and chiral Ward identities
- Chiral symmetry and O(a) improvement
- Wilson quarks with a chirally twisted mass term
- Equivalence to standard QCD
- By-passing lattice specific renormalisation problems

Continuum vs. lattice symmetries

On the lattice symmetries are typically reduced with respect to the continuum. Examples are

1. Space-Time symmetries: the Euclidean O(4) rotations are reduced to the O(4,$\mathbb{Z}$) group of the hypercubic lattice. Other lattice geometries are possible, even random lattices have been tried.
2. Supersymmetry: only partially realisable on the lattice (cf. lectures by S. Catterall)
3. Chiral and Flavour symmetries:
   - staggered quarks: only a U(1)×U(1) symmetry remains
   - Wilson quarks: an exact $SU(N_\mathrm{f})_\mathrm{V}$
   - twisted mass Wilson quarks: various U(1) symmetries (both axial and vector)
   - overlap/Neuberger quarks: complete continuum symmetries!
   - Domain Wall quarks: (negligibly ?) small violations of axial symmetries; consequences are analysed like for Wilson quarks

In the following: chiral and flavour symmetries with Wilson like quarks

Exact lattice Ward identities (1)

Euclidean action $S = S_{\rm f} + S_{\rm g}$:

$$S_{\rm f} = a^4 \sum_x \overline{\psi}(x)\,(D_W + m_0)\,\psi(x), \qquad S_{\rm g} = \frac{1}{g_0^2}\sum_{\mu,\nu} {\rm tr}\,\{1 - P_{\mu\nu}(x)\}$$

$$D_W = \tfrac{1}{2}\left\{\left(\nabla_\mu + \nabla_\mu^*\right)\gamma_\mu - a\nabla_\mu^*\nabla_\mu\right\}$$

Non-singlet vector transformations ($N_{\rm f} = 2$, $\tau^{1,2,3}$ are the Pauli matrices):

$$\psi(x) \rightarrow \psi'(x) = \exp\left(i\theta(x)\tfrac{1}{2}\tau^a\right)\psi(x) = \left(1 + \delta_{\rm V}^a(\theta) + {\rm O}(\theta^2)\right)\psi(x),$$
$$\overline{\psi}(x) \rightarrow \overline{\psi}'(x) = \overline{\psi}(x)\exp\left(-i\theta(x)\tfrac{1}{2}\tau^a\right)\psi(x) = \left(1 + \delta_{\rm V}^a(\theta) + {\rm O}(\theta^2)\right)\overline{\psi}(x)$$

Perform change of variables in the functional integral and expand in θ

$$\langle O[\psi,\overline{\psi},U]\rangle = Z^{-1}\int D[\psi,\overline{\psi}]D[U]{\rm e}^{-S}O[\psi,\overline{\psi},U].$$

Due to $D[\psi,\overline{\psi}] = D[\psi',\overline{\psi}']$ one finds the vector Ward identity

$$\langle \delta_{\rm V}^a(\theta) O\rangle = \langle O\delta_{\rm V}^a(\theta)S\rangle$$

Exact lattice Ward identities (2)

Variation of the action:

$$\delta_{\mathrm{V}}^{a}(\theta)S = -ia^4 \sum_x \partial_\mu^* \widetilde{V}_\mu^a(x)$$

Noether current:

$$\widetilde{V}_\mu^a(x) = \overline{\psi}(x)(\gamma_\mu - 1)\frac{\tau^a}{4}U(x,\mu)\psi(x+a\hat{\mu}) + \overline{\psi}(x+a\hat{\mu})(\gamma_\mu+1)\frac{\tau^a}{4}U(x,\mu)^\dagger\psi(x)$$

Choose region R and θ:

$$R = \{x:\ t_1 \leq x_0 \leq t_2\}, \qquad \theta(x) = \begin{cases} 1 & \text{if } x \in R \\ 0 & \text{otherwise} \end{cases}$$

if $O = O_{\mathrm{ext}}$ is localised outside R:

$$0 = \langle O_{\mathrm{ext}} \delta_{\mathrm{V}}^a(\theta) S\rangle = -ia \sum_{x_0=t_1}^{t_2} a^3 \sum_{\mathbf{x}} \langle O_{\mathrm{ext}} \partial_\mu^* \widetilde{V}_\mu^a(x)\rangle = a \sum_{x_0=t_1}^{t_2} \partial_0^* \langle O_{\mathrm{ext}} Q_{\mathrm{V}}^a(x_0)\rangle$$

There is a conserved charge, $Q_{\mathrm{V}}^a(t_1) = Q_{\mathrm{V}}^a(t_2)$ reflecting the exact vector symmetry on the lattice

Exact lattice Ward identities (3)

Choosing $O = O_{\rm ext}\widetilde{V}_\mu^b(y)$, with $y \in R$:

$$i\varepsilon^{abc}\left\langle O_{\rm ext}\widetilde{V}_k^c(y)\right\rangle = \left\langle O_{\rm ext}\widetilde{V}_k^b(y)\left[Q_{\rm V}^a(t_2) - Q_{\rm V}^a(t_1)\right]\right\rangle$$

$$i\varepsilon^{abc}\left\langle O_{\rm ext}Q_{\rm V}^c(y_0)\right\rangle = \left\langle O_{\rm ext}Q_{\rm V}^b(y_0)\left[Q_{\rm V}^a(t_2) - Q_{\rm V}^a(t_1)\right]\right\rangle$$

Euclidean version of charge algebra!

- implies that the Noether current $\widetilde{V}_\mu^a$ is protected against renormalisation; if we admit a renormalisation constant $Z_{\tilde{\rm V}}$ it follows that $Z_{\tilde{\rm V}}^2 = Z_{\tilde{\rm V}}$ hence $Z_{\tilde{\rm V}} = 1$; its anomalous dimension vanishes!
- Any other definition of a lattice current, e.g. the local current

  $$V_\mu^a(x) = \overline{\psi}(x)\gamma_\mu\gamma_5\psi(x), \qquad (V_{\rm R})_\mu^a = Z_{\rm V}V_\mu^a$$

  can be renormalised by comparing with the conserved current. Its anomalous dimension must vanish, i.e.

  $$Z_{\rm V} = Z_{\rm V}(g_0) \quad \overset{g_0\to 0}{\sim} \quad 1 + \sum_{n=1}^{\infty} Z_{\rm V}^{(n)} g_0^{2n}.$$

Continuum chiral WI's as normalisation conditions

- For chiral symmetry there is no conserved current with Wilson quarks.
- However: expect that chiral symmetry can be restored in the continuum limit!
- [Bochicchio et al '85]: use continuum chiral Ward identities and impose them as normalisation condition at finite a
- Define chiral variations:

$$\delta_{\mathrm{A}}^{a}(\theta)\psi(x) = i\gamma_5 \tfrac{1}{2}\tau^a \theta(x)\psi(x), \qquad \delta_{\mathrm{A}}^{a}(\theta)\overline{\psi}(x) = \overline{\psi}(x) i\gamma_5 \tfrac{1}{2}\tau^a \theta(x)$$

- Derive formal continuum Ward identities assuming that the functional integral can be treated like an ordinary integral:

$$\begin{aligned}
\langle \delta_{\mathrm{A}}^{a}(\theta) O \rangle &= \langle O \delta_{\mathrm{A}}^{a}(\theta) S \rangle, \\
\delta_{\mathrm{A}}^{a}(\theta) S &= -i \int \mathrm{d}^4 x \theta(x) \left(\partial_\mu A_\mu^a(x) - 2m P^a(x) \right) \\
A_\mu^a(x) &= \overline{\psi}(x)\gamma_\mu\gamma_5 \tfrac{1}{2}\tau^a \psi(x), \qquad P^a(x) = \overline{\psi}(x)\gamma_5 \tfrac{1}{2}\tau^a\psi(x)
\end{aligned}$$

Simplest chiral WI: the PCAC relation

- Shrink the region R to a point:

$$\begin{aligned} \langle O_{\rm ext}\delta_{\rm A}^a(\theta)S\rangle &= 0 \\ \Rightarrow \quad \left\langle \partial_\mu A_\mu^a(x) O_{\rm ext}\right\rangle &= 2m\left\langle P^a(x) O_{\rm ext}\right\rangle \end{aligned}$$

- The PCAC relation implies that chiral symmetry is restored in the chiral limit.
- Impose PCAC on Wilson quarks at fixed a: define a bare PCAC mass:

$$m = \frac{\left\langle \partial_\mu A_\mu^a(x) O_{\rm ext}\right\rangle}{\left\langle P^a(x) O_{\rm ext}\right\rangle}$$

- A renormalised quark mass can thus be written in two ways

$$m_{\rm R} = Z_{\rm A} Z_{\rm P}^{-1} m = Z_m(m_0 - m_{\rm cr}) \quad \Rightarrow \quad m = Z_m Z_{\rm P} Z_{\rm A}^{-1}(m_0 - m_{\rm cr})$$

⇒ The critical mass can be determined by measuring the bare PCAC mass m as a function of m_0 and extra/interpolation to $m = 0$.

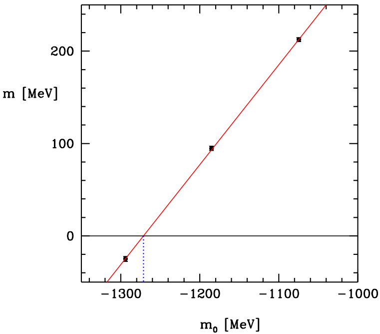
- Note: m is only defined up to O(a); any change in $O_{\rm ext}$ will lead to O(a) differences.

Determination of the critical mass

PCAC quark mass from SF correlation functions:

$$m = \frac{\partial_0 f_{\rm A}(x_0)}{2 f_{\rm P}(x_0)}$$

$8^3 \times 16$ lattice, quenched QCD, $a = 0.1\,\mathrm{fm}$



More chiral WI's: axial current normalisation

- At $m = 0$ we can derive the Euclidean charge algebra :

$$i\varepsilon^{abc}\left\langle O_{\rm ext}\, Q_{\rm V}^{c}(y_0)\right\rangle = \left\langle O_{\rm ext}\, Q_{\rm A}^{b}(y_0)\,[Q_{\rm A}^{a}(t_2) - Q_{\rm A}^{a}(t_1)]\right\rangle$$

- Imposing this continuum identity on the lattice (at $m = 0$) fixes the normalisation of the axial current

$$(A_{\rm R})_{\mu}^{a} = Z_{\rm A}(g_0) A_{\mu}^{a}, \qquad Z_{\rm A}(g_0) \overset{g_0 \rightarrow 0}{\sim} \quad 1 + \sum_{n=1}^{\infty} Z_{\rm A}^{(n)} g_0^{2n}.$$

- Note: When changing the external fields $O_{\rm ext}$, the result for $Z_{\rm A}$ will change by terms of $\mathrm{O}(a)$.
- The PCAC relation and the charge algebra become operator identities in Minkowski space. Changing $O_{\rm ext}$ corresponds to looking at different matrix elements of these operator identities. On the lattice these must be equal up to $\mathrm{O}(a)$ terms.

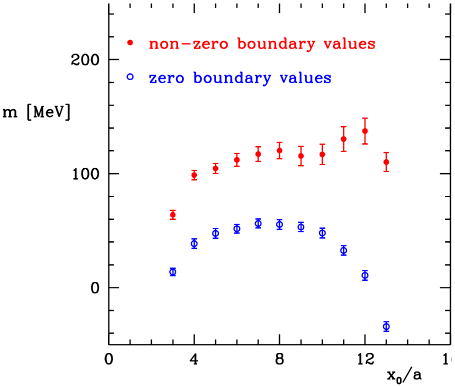
Need for O(a) improvement of Wilson quarks

O(a) artefacts can be quite large with Wilson quarks:

PCAC quark mass from SF correlation functions:

$$m = \frac{\partial_0 f_{\mathrm{A}}(x_0)}{2 f_{\mathrm{P}}(x_0)}$$

$8^3 \times 16$ lattice, quenched QCD, $a = 0.1\,\mathrm{fm}$, 2 different gauge background fields.



On-shell O(a) improvement

Recall Symanzik's effective continuum theory from lecture 1

$$S_{\rm eff} = S_0 + aS_1 + a^2 S_2 + \ldots, \qquad S_0 = S_{\rm QCD}^{\rm cont}$$
$$S_k = \int {\rm d}^4 x\, \mathcal{L}_{\rm k}(x)$$

where $\mathcal{L}_1$ is a linear combination of the fields:

$$\overline{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi, \qquad \overline{\psi}D_\mu D_\mu\psi, \qquad m\,\overline{\psi}\not{D}\psi, \quad m^2\overline{\psi}\psi, \qquad m\,{\rm tr}\,\{F_{\mu\nu}F_{\mu\nu}\}$$

The action S_1 appears as insertion in correlation functions

$$G_n(x_1,\ldots,x_n) = \langle\phi_0(x_1)\ldots\phi_0(x_n)\rangle_{\rm con} + a\int {\rm d}^4 y\, \langle\phi_0(x_1)\ldots\phi_0(x_n)\mathcal{L}_1(y)\rangle_{\rm con} + a\sum_{k=1}^{n}\langle\phi_0(x_1)\ldots\phi_1(x_k)\ldots\phi_0(x_n)\rangle_{\rm con} + {\rm O}(a^2)$$

On-shell O(a) improvement (1)

Basic idea:

- Introduce counterterms to the action and composite operators such that S_1 and ϕ_1 are cancelled in the effective theory
- As all physics can be obtained from on-shell quantitities (spectral quantitities like particle energies or correlation function where arguments are kept at non-vanishing distance) one may use the equations of motion to reduce the number of counterterms
- The contact terms which arise from having $y \approx x_i$ can be analysed in the OPE and are found to be of the same structure as the counterterms anyway contained in ϕ_1; this amounts to a redefinition of the counterterms in ϕ_1.
- After using the equations of motion one remains with:

$$\overline{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi, \qquad m^2\overline{\psi}\psi, \qquad m\,\mathrm{tr}\left\{F_{\mu\nu}F_{\mu\nu}\right\}$$

On-shell O(a) improvement (2)

1. On-shell O(a) improved Lattice action
   - The last two terms are equivalent to a rescaling of the bare mass and coupling ($m_{\rm q} = m_0 - m_{\rm cr}$):

$$\tilde{g}_0^2 = g_0^2(1 + b_g(g_0)am_{\rm q}), \qquad \tilde{m}_{\rm q} = m_{\rm q}(1 + b_{\rm m}(g_0)am_{\rm q})$$

   - The first term is the Sheikholeslami-Wohlert or clover term

$$S_{Wilson} \rightarrow S_{\rm Wilson} + iac_{\rm sw}(g_0)a^4 \sum_x \overline{\psi}(x)\sigma_{\mu\nu}\hat{F}_{\mu\nu}(x)\psi(x)$$

2. On-shell O(a) improved axial current and density:

$$(A_{\rm R})_\mu^a = Z_{\rm A}(\tilde{g}_0{}^2)(1 + b_{\rm A}(g_0)am_{\rm q})\left\{A_\mu^a + c_{\rm A}(g_0)\tilde{\partial}_\mu P^a\right\}$$

$$(P_{\rm R})^a = Z_{\rm P}(\tilde{g}_0{}^2, a\mu)(1 + b_{\rm P}(g_0)am_{\rm q})P^a$$

On-shell O(a) improvement (3)

- There are 2 counterterms in the massless theory $c_{\rm sw}, c_{\rm A}$, the remaining ones $(b_g, b_{\rm m}, b_{\rm A}, b_{\rm P})$ come with $am_{\rm q}$.
- Note: all counterterms are absent in chirally symmetric regularisations!

$\Rightarrow$ turn this around: impose chiral symmetry to determine $c_{\rm sw}, c_{\rm A}$ non-perturbatively:

- define bare PCAC quark masses from SF correlation functions

$$m_{\rm R} = \frac{Z_{\rm A}(1 + b_{\rm A}\, am_{\rm q})}{Z_{\rm P}(1 + b_{\rm P}\, am_{\rm q})} m, \qquad m = \frac{\tilde{\partial}_0 f_{\rm A}(x_0) + c_{\rm A}\, a\partial_0^*\partial_0 f_{\rm P}(x_0)}{f_{\rm P}(x_0)}$$

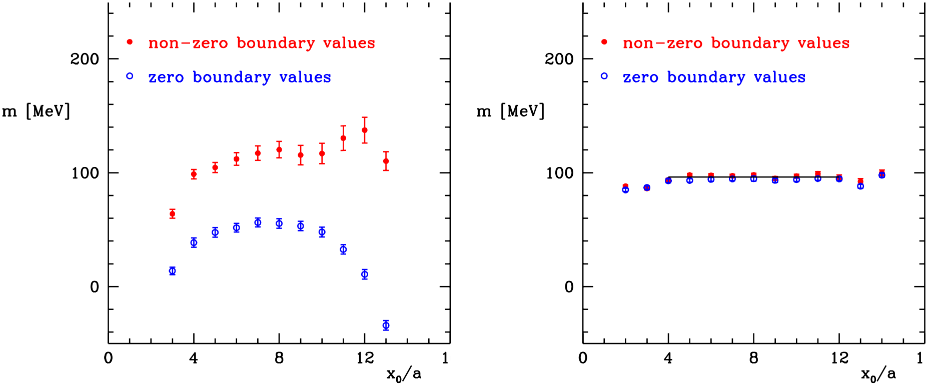
- At fixed g_0 and $am_{\rm q} \approx 0$ define 3 bare PCAC masses $m_{1,2,3}$ (e.g. by varying the gauge boundary conditions) and impose

$$m_1(c_{\rm sw}, c_{\rm A}) = m_2(c_{\rm sw}, c_{\rm A}), \qquad m_1(c_{\rm sw}, c_{\rm A}) = m_3(c_{\rm sw}, c_{\rm A}) \Rightarrow c_{\rm sw}, c_{\rm A}$$

SF b.c.'s $\Rightarrow$ high sensitivity to $c_{\rm sw}$ & simulations near chiral limit

On-shell O(a) improvement (4)

Before and after O(a) improvement (PCAC masses from SF correlation functions, $8^3 \times 16$ lattice)



Quenched result for the charm quark mass [ALPHA '02]

- The RGI charm quark mass can be defined in various ways
  - starting from the subtracted bare quark mass $m_{\rm q,c} = m_{0,c} - m_{\rm cr}$
  - starting from the average strange-charm PCAC mass m_{sc}
  - starting from the PCAC mass m_{cc} for a hypothetical mass degenerate doublet of quarks.
- Tune the bare charm quark masses to match the D_s meson mass
- Obtain the corresponding O(a) improved RGI masses:

$$
\begin{aligned}
r_0 M_c|_{m_{sc}} &= Z_M \Big\{ 2 r_0 m_{sc} \left[1 + (b_{\rm A} - b_{\rm P})\tfrac{1}{2}(a m_{\rm q,c} + a m_{\rm q,s})\right] \\
&\qquad - r_0 m_{\rm s} \left[1 + (b_{\rm A} - b_{\rm P}) a m_{\rm q,s})\right] \Big\}, \\
r_0 M_c|_{m_{\rm c}} &= Z_M r_0 m_{\rm c} \left[1 + (b_{\rm A} - b_{\rm P}) a m_{\rm q,c}\right], \\
r_0 M_c|_{m_{\rm q,c}} &= Z_M Z r_0 m_{\rm q,c} \left[1 + b_{\rm m} a m_{\rm q,c}\right].
\end{aligned}
$$

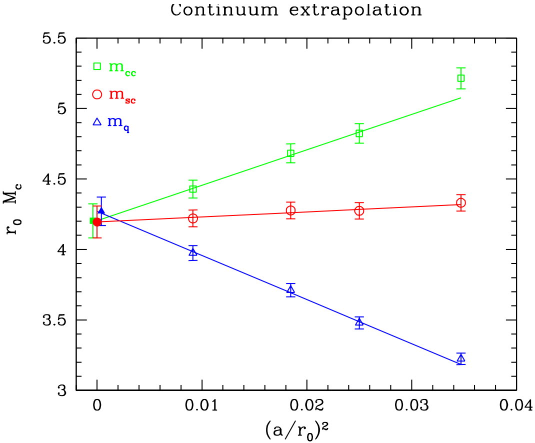
- N.B.: all O(a) counterterms are known non-perturbatively in the quenched case!

Continuum extrapolation of the quenched RGI charm quark mass

Continuum extrapolation:

$$r_0 M_c = A + B(a^2/r_0^2)$$
$$r_0 = 0.5\,\mathrm{fm}$$

$$M_c = 1.654(45)\,\mathrm{GeV}$$
$$\overline{m}_c^{\overline{\mathrm{MS}}}(\overline{m}_c) = 1.301(34)\,\mathrm{GeV}$$



Summary On-shell O(a) improvement

After O(a) improvement:

- The ambiguity in $m_{\rm cr}$ is reduced to O(a^2)
- Axial current normalisation can be defined up to O(a^2)
- Results exist for $c_{\rm sw}, c_{\rm A}$ for quenched and $N_{\rm f} = 2, 3$ and different gauge actions
- On-shell O(a) improvement seems to work; rather economical for spectral quantities (e.g. hadron masses): just need $c_{\rm sw}$!
- Quark bilinear operators are still tractable
- Four-quark operators are probably impractical
- Non-degenerate quark masses: rather complicated, proliferation of counterterms [Bhattacharya et al '99]; Not all can determined by chiral symmetry, due to violation of on-shell condition in Ward identities at finite mass
- However: for small quark masses and fine lattices $am_{\rm q}$ is small (a few percent at most) and perturbative estimates of improvement coefficients may be good enough!

Twisted mass QCD, continuum considerations (1)

Consider the continuum action of a doublet of massless quarks

$$S_{\rm f} = \int {\rm d}^4 x\ \overline{\psi}(x)\partial_\mu\gamma_\mu\psi(x)$$

The massless action is symmetric under chiral transformations

$$\begin{aligned} \psi \rightarrow \psi' &= \exp(i\omega_{\rm A}^a\gamma_5\tau^a/2)\psi \\ \overline{\psi} \rightarrow \overline{\psi}' &= \overline{\psi}\exp(i\omega_{\rm A}^a\gamma_5\tau^a/2) \end{aligned}$$

When introducing a quark mass term the choices $\overline{\psi}\psi$ or

$$\overline{\psi}'\psi' = \overline{\psi}\exp(i\omega_{\rm A}^a\gamma_5\tau^a)\psi = \cos(\omega_{\rm A})\ \overline{\psi}\psi + i\sin(\omega_{\rm A})u_{\rm A}^a\ \overline{\psi}\gamma_5\tau^a\psi$$

are equivalent!

($\omega_{\rm A}$ is the modulus of $(\omega_{\rm A}^1, \omega_{\rm A}^2, \omega_{\rm A}^3)$ and $u^a = \omega_{\rm A}^a/\omega_{\rm A}$ a unit vector)

- The choice of a mass term $\overline{\psi}\psi$ is a mere convention; in general one may pick any other direction in chiral flavour space
- The form of symmetry transformations depends on this choice:

Twisted mass QCD, continuum considerations (2)

- by definition,the flavour (isospin) symmetry leaves the mass term invariant:

$$\begin{aligned}\psi &\rightarrow \exp(-i\omega_{\rm A}^a\gamma_5\tau^a/2)\exp(i\omega_{\rm V}^b\tau^b/2)\exp(i\omega_{\rm A}^c\gamma_5\tau^c/2)\,\psi \\ \overline{\psi} &\rightarrow \overline{\psi}\exp(i\omega_{\rm A}^a\gamma_5\tau^a/2)\exp(-i\omega_{\rm V}^b\tau^b/2)\exp(-i\omega_{\rm A}^c\gamma_5\tau^c/2)\end{aligned}$$

- similarly for parity:

$$\begin{aligned}\psi(x) &\rightarrow \gamma_0\exp(i\omega_{\rm A}^a\gamma_5\tau^a)\,\psi(x_0,-\mathbf{x}), \\ \overline{\psi}(x) &\rightarrow \overline{\psi}(x_0,-\mathbf{x})\,\exp(i\omega_{\rm A}^a\gamma_5\tau^a)\gamma_0\end{aligned}$$

Question: why should one deviate from the standard convention for the quark mass term?

Twisted Mass Lattice QCD (1)

Lattice action for a doublet ψ of mass degenerate light Wilson quarks quarks [Aoki '84]:

$$S_f = a^4 \sum_x \overline{\psi}(x) \left(D_{\rm W} + m_0 + i\mu_q\gamma_5\tau^3\right)\psi(x)$$

$D_{\rm W}$: Wilson-Dirac operator with/without Sheikholeslami-Wohlert (clover)

μ_q : bare twisted mass parameter

Properties:

- regularisation of QCD with $N_{\rm f} = 2$ mass degenerate quark flavours (see below)
- $\mu_q \neq 0 \Rightarrow$ no unphysical zero modes:

$$\det\Bigl(D_{\rm W} + m_0 + i\mu_q\gamma_5\tau^3\Bigr)$$
$$= \det\begin{pmatrix} \gamma_5(D_{\rm W}+m_0)+i\mu_q & 0 \\ 0 & \gamma_5(D_{\rm W}+m_0)-i\mu_q \end{pmatrix}$$
$$= \det\Bigl([D_{\rm W}+m_0)]^\dagger[D_{\rm W}+m_0] + \mu_q^2\Bigr) > 0$$

Twisted Mass Lattice QCD (2)

- positive and selfadjoint transfer matrix provided μ_q is real and $|\kappa| < 1/6$,
  $\kappa = (2am_0 + 8)^{-1} \Rightarrow$ unitarity
- The flavour symmetry is reduced to U(1) with generator $\tau^3/2$
- Discrete symmetries: C, axis permutations, reflections with flavour exchange, e.g.

$$\psi(x) \rightarrow \gamma_0\tau^1\psi(x_0, -\mathbf{x}), \qquad \overline{\psi}(x) \rightarrow \overline{\psi}(x_0, -\mathbf{x})\gamma_0\tau^1$$

Equivalence between tmQCD and QCD (1)

Classical continuum limit of twisted mass lattice QCD:

$$S_f = \int \mathrm{d}x \, \overline{\psi}(x)\left(\not{D} + m + i\mu_q\gamma_5\tau^3\right)\psi(x).$$

Perform a global chiral (non-singlet) rotation of the fields:

$$\psi' = R(\alpha)\psi, \quad \overline{\psi}' = \overline{\psi}R(\alpha), \quad R(\alpha) = \exp\left(i\alpha\gamma_5\frac{\tau^3}{2}\right).$$

For $\tan\alpha = \mu_q/m$ the action reads:

$$\begin{aligned} S_f' &= \int \mathrm{d}x \, \overline{\psi}'(x)(\not{D} + M)\psi'(x), \qquad M = \sqrt{m^2 + \mu_q^2} \\ \overline{\psi}'\psi' &= \overline{\psi}\exp(i\alpha\gamma_5\tau^3)\psi = \cos(\alpha)\overline{\psi}\psi + i\sin(\alpha)\overline{\psi}\gamma_5\tau^3\psi \end{aligned}$$

corresponds to $\omega_{\mathrm{A}}^a = \alpha\delta^{3a}$ in the previous discussion.

Equivalence between tmQCD and QCD (2)

Introduce polar mass coordinates $m = M\cos(\alpha)$, $\mu_q = M\sin(\alpha)$, and consider the formal functional integral

$$\langle O[\psi,\overline{\psi}]\rangle_{(M,\alpha)} = \mathcal{Z}^{-1}\int D[U,\psi,\overline{\psi}]\,O[\psi,\overline{\psi}]\,\mathrm{e}^{-S[m,\mu_q]}$$

The change of variables leads to the identity:

$$\left\langle O[\psi,\overline{\psi}]\right\rangle_{(M,0)} = \left\langle O[R(\alpha)\psi,\overline{\psi}R(\alpha)]\right\rangle_{(M,\alpha)}$$

For a member $\phi_A^{(r)}$ of a chiral multiplet in the representation r,

$$\phi_A^{(r)}[R(\alpha)\psi,\overline{\psi}R(\alpha)] = R_{AB}^{(r)}(\alpha)\phi_B^{(r)}[\psi,\overline{\psi}]$$

The identity for n-point functions of such fields becomes

$$\left\langle \phi_{A_1}^{(r_1)}(x_1)\cdots\phi_{A_n}^{(r_n)}(x_n)\right\rangle_{(M,0)} =$$

$$\left\{\prod_{i=1}^{n} R_{A_iB_i}^{(r_i)}(\alpha)\right\}\left\langle \phi_{B_1}^{(r_1)}(x_1)\cdots\phi_{B_n}^{(r_n)}(x_n)\right\rangle_{(M,\alpha)}$$

Equivalence between tmQCD and QCD (3)

Examples: chiral multiplets (A_μ^a, V_μ^a) and $(\frac{1}{2}S^0, P^a)$

$$A_\mu^a = \overline{\psi}\gamma_\mu\gamma_5\frac{\tau^a}{2}\psi, \qquad V_\mu^a = \overline{\psi}\gamma_\mu\frac{\tau^a}{2}\psi,$$
$$P^a = \overline{\psi}\gamma_5\frac{\tau^a}{2}\psi, \qquad S^0 = \overline{\psi}\psi.$$

With $\psi' = R(\alpha)\psi$, $\overline{\psi}' = \overline{\psi}R(\alpha)$, $O' \equiv O[\psi', \overline{\psi}']$, $c \equiv \cos(\alpha)$, $s \equiv \sin(\alpha)$:

$$A'^1_\mu = cA^1_\mu + sV^2_\mu, \qquad V'^1_\mu = cV^1_\mu + sA^2_\mu,$$
$$A'^2_\mu = cA^2_\mu - sV^1_\mu, \qquad V'^2_\mu = cV^2_\mu - sA^1_\mu,$$
$$A'^3_\mu = A^3_\mu, \qquad V'^3_\mu = V^3_\mu,$$
$$P'^a = P^a, \quad (a = 1, 2), \qquad P'^3 = cP^3 + is\,\tfrac{1}{2}\overline{\psi}\psi.$$

For instance:

$$\left\langle A^1_\mu(x)P^1(y)\right\rangle_{(M,0)} = \cos(\alpha)\left\langle A^1_\mu(x)P^1(y)\right\rangle_{(M,\alpha)} + \sin(\alpha)\left\langle V^2_\mu(x)P^1\right\rangle_{(M,\alpha)}$$

Equivalence between tmQCD and QCD (4)

The PCAC and PCVC relations,

$$\partial_\mu A_\mu^a = 2mP^a + \delta^{3a} i\mu_q S^0, \qquad \partial_\mu V_\mu^a = -2\mu_q \varepsilon^{3ab} P^b,$$

take their standard form in the primed basis

$$\partial_\mu A'^a_\mu = 2MP'^a, \qquad \partial_\mu V'^a_\mu = 0.$$

Remarks:

- We refer to the basis of primed fields as "physical" because the mass term takes its standard form in this basis
- We still need to explain how the relationship between QCD with a standard mass term and twisted mass QCD works out beyond the formal continuum theory.

Beyond the formal continuum theory

- If tmQCD is regularized with Ginsparg-Wilson quarks the same identities can be derived in the bare theory
- If the renormalization procedure respects the chiral multiplet structure and the multiplicative renormalization constants do not depend on α (e.g. mass independent renormalization schemes)
  $\Rightarrow$ the formal continuum relations hold between renormalized theories.
  N.B.: no reference to perturbation theory! Assuming universality the correspondence is established non-perturbatively. In PT it works out order by order in the loop expansion.
- The angle α is given by the ratio between renormalized PCVC and PCAC masses: $\tan\alpha = \mu_\mathrm{R}/m_\mathrm{R}$

Lattice tmQCD with Wilson quarks

1. restore the chiral multiplets in the massless bare theory by imposing the chiral flavour Ward identities, e.g. $(Z_{\rm A} A_\mu^a, \tilde{V}_\mu^a)$.
2. If necessary renormalize a given chiral multiplet by imposing a renormalization condition on one of its members. Choose a mass independent renormalization scheme!
3. Renormalization of the parameters:

$$g_{\rm R}^2 = Z_g g_0^2, \qquad m_{\rm R} = Z_m(m_0 - m_{\rm c}), \qquad \mu_{\rm R} = Z_\mu \mu_q,$$

From the exact PCVC relation

$$\partial_\mu^* \tilde{V}_\mu^2 = 2\mu_q P^1 = 2\mu_{\rm R}(P_{\rm R})^1 \quad \Rightarrow Z_\mu Z_{\rm P} = 1.$$

$\Rightarrow$ to define α measure a bare PCAC mass m

$$m = \frac{\langle \partial_\mu A_\mu^1(x) O \rangle}{\langle P^1(x) O \rangle} \qquad \Rightarrow \qquad \tan\alpha = \frac{\mu_{\rm R}}{m_{\rm R}} = \frac{Z_{\rm P}^{-1}\mu_q}{Z_{\rm P}^{-1} Z_{\rm A} m} = \frac{\mu_q}{Z_{\rm A} m}.$$

the definition of α requires $Z_{\rm A}$, except for $\alpha = \pi/2$, where $m = 0$.

The freedom of introducing more general mass terms can be used to avoid lattice renormalization problems:

1. F_π can be obtained from the 2-point function

$$
\begin{aligned}
\left\langle (A_\mathrm{R})^1_0(x)(P_\mathrm{R})^1(y)\right\rangle_{(M_\mathrm{R},0)} &= \cos(\alpha)\left\langle (A_\mathrm{R})^1_0(x)(P_\mathrm{R})^1(y)\right\rangle_{(M_\mathrm{R},\alpha)} \\
&\quad + \sin(\alpha)\left\langle \widetilde{V}^2_0(x)(P_\mathrm{R})^1(y)\right\rangle_{(M_\mathrm{R},\alpha)}.
\end{aligned}
$$

   At $\alpha = \pi/2$ one has $\cos(\alpha) = 0$ and F_π is obtained from the vector current. The determination of Z_A is avoided!

2. The chiral condensate:

$$
\left\langle (S_\mathrm{R})^0(x)\right\rangle_{(M_\mathrm{R},0)} = \cos(\alpha)\left\langle (S_\mathrm{R})^0(x)\right\rangle_{(M_\mathrm{R},\alpha)} + 2i\sin(\alpha)\left\langle (P_\mathrm{R})^3(x)\right\rangle_{(M_\mathrm{R},\alpha)}
$$

   At $\alpha = \pi/2$ the chiral condensate is represented by P^3 which only renormalizes multiplicatively in the chiral limit!

Application to B_K: The B_K parameter is defined in QCD with dynamical u, d, s quarks:

$$\langle \bar{K}^0 | O^{\Delta S=2}_{\rm (V-A)(V-A)} | K^0 \rangle = \frac{8}{3} F_K^2 m_K^2 B_K$$

The local operator

$$O^{\Delta S=2}_{\rm (V-A)(V-A)} = \sum_\mu [\bar{s}\gamma_\mu(1-\gamma_5)d]^2$$

is the effective local interaction induced by integrating out the massive gauge bosons and t, b, c quarks in the Standard Model.

- only the parity-even part contributes to B_K

$$O_{\rm (V-A)(V-A)} = \underbrace{O_{\rm VV+AA}}_{\rm parity-even} - \underbrace{O_{\rm VA+AV}}_{\rm parity-odd}$$

- Operator mixing problem with Wilson quarks [Bernard et al.,'88]:

$$[O_{\rm VV+AA}]_{\rm R} = Z_{\rm VV+AA}\left\{O_{\rm VV+AA} + \sum_{i=1}^{4} z_i \; O_i^{d=6}\right\}$$

$$[O_{\rm VA+AV}]_{\rm R} = Z_{\rm VA+AV} O_{\rm VA+AV}$$

$\Rightarrow$ parity-odd component renormalizes multiplicatively!

Question: Can we avoid the mixing problem by using the multiplicatively renormalized operator $O_{\mathrm{VA+AV}}$ to compute B_K?

- consider continuum theory for a light quark doublet ψ and the s-quark:

$$
\begin{aligned}
\mathcal{L}_f &= \overline{\psi}\left(\not{D} + m + i\mu_q\gamma_5\tau^3\right)\psi + \bar{s}\left(\not{D} + m_{\mathrm{s}}\right)s \\
\Rightarrow \quad O'_{\mathrm{VV+AA}} &= \cos(\alpha)O_{\mathrm{VV+AA}} - i\sin(\alpha)O_{\mathrm{VA+AV}} \\
&= -iO_{\mathrm{VA+AV}} \quad (\alpha = \pi/2)
\end{aligned}
$$

Conclusions

- Wilson quarks break all chiral/axial symmetries which leads to additive quark mass renormalisation, non-trivial axial current normalisation and O(a) effects; can be "cured" by imposing chiral continuum Ward Identities
- Twisted mass QCD with Wilson type quarks is a regularisation which
  - is equivalent to standard QCD with $N_\mathrm{f} = 2, 4, \ldots$
  - has an additional unphysical parameter, the twist angle α. This angle determines the physical interpretation (flavour vs. chiral symmetries, parity) and can be used to circumvent certain lattice specific renormalization problems: F_π without Z_A, the chiral order parameter without cubic divergence, B_K without mixing ….
  - enjoys automatic O(a) improvement at $\alpha = \pi/2$ (cf. lecture V)
  - breaks flavour and parity symmetries; expect that these are restored in the continuum limit (just as axial symmetry with standard Wilson quarks).