Bare Perturbation Theory, MOM schemes, finite volume schemes (lecture II)

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A couple of exercises

1. Check that Λ and M are indeed solutions of the Callan-Symanzik equation
2. Minimal subtraction of logarithms:
   in perturbation theory we may introduce a renormalised coupling $g_{\rm lat}(\mu)$ such that

$$g_{\rm lat}(\mu = 1/a) = g_0$$

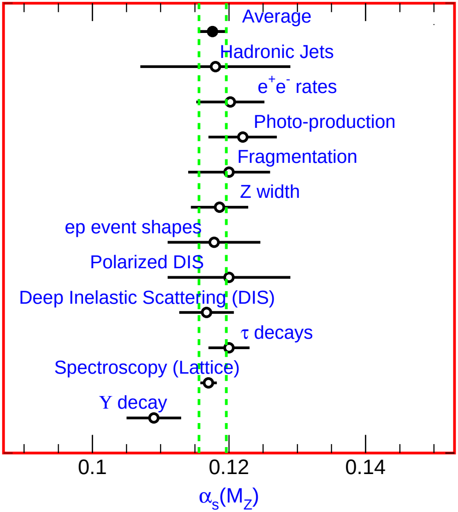
The couplings can be related in perturbation theory

$$g^2_{\rm lat}(\mu) = g_0^2 + c_1(a\mu)g_0^4 + \mathrm{O}(g_0^6)$$

The l.h.s. is renormalised and has a continuum limit. Compute $c_1(a\mu)$ and derive the behaviour of $g_0(a)$ for $a \to 0$, which follows from assuming that $g_{\rm lat}(\mu)$ is independent of a.

Contents

World average for $\alpha_s(m_Z)$



[PDG 2005]

N.B. Lattice result claims the smallest error!

Staggering results from the HPQCD coll. ("High Precision QCD"):

[HPQCD coll., Q. Mason et al. '05] Determination of QCD parameters in the $\overline{\mathrm{MS}}$ scheme with very small errors:

$$\begin{aligned} \alpha_s(m_Z) &= 0.1170(12) \\ \hat{m}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) &= 3.2(0)(2)(2)(0)\,\mathrm{MeV} \quad \text{errors:} \\ m_s^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) &= 87(0)(4)(4)(0)\,\mathrm{MeV} \quad \text{(stat.)(syst.) (pert.)(e.m. isospin)} \end{aligned}$$

- $N_\mathrm{f} = 2 + 1$ rooted staggered quarks (MILC configurations), staggered chiral perturbation theory (cf. Claude Bernard's lecture)
- perturbation theory at 2-loop order (impressive!)
- various versions of bare perturbation theory, some internal consistency checks

Variants of the bare coupling

bare coupling: defined at the cutoff scale, vanishes in continuum limit

Example:

- expand the plaquette expectation value P in powers of α_0

$$P = 1 - p_1\alpha_0 - p_2\alpha_0^2 + \ldots$$

- define modified bare couplings:

$$\alpha_P \stackrel{\mathrm{def}}{=} (1-P)/p_1, \qquad \tilde{\alpha}_P = -\ln(P)/p_1,$$

  where P is measured in the numerical simulation.

- Motivation: the perturbative series may behave differently for different bare couplings;
- In principle any short distance quantity on the lattice can be chosen: $m \times n$ Wilson loops with $m, n = 1, 2, 3$, or expectation of the link variable in a fixed gauge

Perturbation theory in the bare coupling

A shortcut method: use bare perturbation theory to relate to the renormalised coupling and quark masses (e.g. $\overline{\mathrm{MS}}$); Allowing for a constant $d = \mathrm{O}(1)$ one sets

$$\begin{aligned} \alpha_{\overline{\mathrm{MS}}}(d/a) &= \alpha_0(a) + c_1\alpha_0^2(a) + c_2\alpha_0^3(a) + \ldots, \qquad \alpha_0 = \frac{g_0^2}{4\pi} \\ \overline{m}_{\overline{\mathrm{MS}}}(d/a) &= m(a)\left(1 + Z_m^{(1)}\alpha_0(a) + Z_m^{(2)}\alpha_0^2(a) + \ldots\right\} \end{aligned}$$

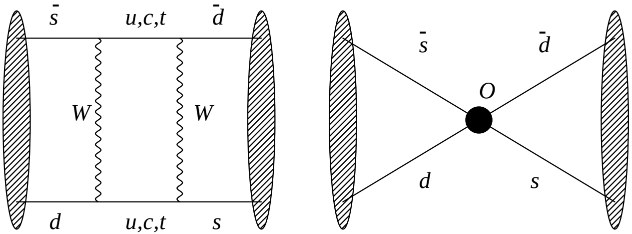
Main difficulties:

- The identification $\mu = da^{-1}$ means that cutoff effects and renormalisation effects cannot be disentangled; any change in the scale is at the same time a change in the cutoff.
- One needs to assume that the cutoff scale d/a is in the perturbative region
- One furthermore assumes that cutoff effects are negligible

$\Rightarrow$ how reliable are the error estimates?

QCD & composite operators (1)

Apart from the fundamental parameters of QCD one is interested in hadronic matrix elements of composite operators:
Example: $K^0 - \bar{K}^0$ mixing amplitude in the Standard Model:



A local interaction arises by integrating out W-bosons and t, b, c quarks, corresponding to a composite 4-quark operator

- The mixing amplitude reduces to the hadronic matrix element:

$$\langle \bar{K}^0 | O^{\Delta S=2} | K^0 \rangle = \frac{8}{3} m_K^2 F_K^2 B_K$$

$$O^{\Delta S=2} = \sum_\mu [\bar{s}\gamma_\mu(1-\gamma_5)d][\bar{s}\gamma_\mu(1-\gamma_5)d]$$

$O^{\Delta S=2}$ requires a multiplicative renormalization; it is initially defined in continuum scheme used for the Operator Product Expansion (OPE)

- Other composite operators arise by applying the OPE with respect to some hard scale, such as the photon momentum in Deep Inelastic Scattering (DIS)
- We thus need to discuss renormalisation of composite operators (cf. quark mass renormalisation for a first example)

RGI operators (1)

- Consider renormalized n-point function of multiplicatively renormalizable operators O_i:

$$G_{\mathrm{R}}(x_1, \cdots, x_n; m_{\mathrm{R}}, g_{\mathrm{R}}) = \prod_{i=1}^{n} Z_{O_i}(g_0, a\mu) G(x_1, \cdots, x_n; m_0, g_0)$$

- Callan-Symanzik equation:

$$\left\{ \mu \frac{\partial}{\partial \mu} + \beta(\bar{g}) \frac{\partial}{\partial \bar{g}} + \tau(\bar{g}) \overline{m} \frac{\partial}{\partial \overline{m}} + \sum_{i=1}^{n} \gamma_{O_i}(\bar{g}) \right\} G_R = 0$$

where

$$\gamma_{O_i}(\bar{g}(\mu)) = \left. \frac{\partial \ln Z_O(g_0, a\mu)}{\partial \ln(a\mu)} \right|_{\bar{g}(\mu)}$$

- Asymptotic behaviour for small couplings:

$$\gamma_O(g) \sim -g^2 \gamma_O^{(0)} - g^4 \gamma_O^{(1)} + \ldots$$

RGI operators (2)

RGI operators can be defined as solutions to the CS equation:

$$\left(\beta(\bar{g})\frac{\partial}{\partial\bar{g}} + \gamma_O\right) O_{\mathrm{RGI}} = 0$$

where

$$O_{\mathrm{RGI}} = O_{\mathrm{R}}(\mu)\left(\frac{\bar{g}^2(\mu)}{4\pi}\right)^{-\gamma_O^{(0)}/2b_0} \exp\left\{-\int_0^{\bar{g}} \mathrm{d}x \left[\frac{\gamma_O(x)}{\beta(x)} - \frac{\gamma_O^{(0)}}{b_0 x}\right]\right\}$$

- Its name derives from the fact that O_{RGI} is renormalisation scheme independent (analogous to M_i, verify it!)!
- Beware: the overall normalisation for O_{RGI} here follows the standard convention used for B_K, which differs from the one used for M.

Perturbative vs. non-perturbative renormalisation

Distinguish 3 cases:

1. finite renormalisations: e.g. axial current normalisation for Wilson quarks $Z_A(g_0)$ (cf. lecture 4)

$\Rightarrow$ perturbation theory to high orders in g_0^2 might be an option [Di Renzo et al. '2006]

2. multiplicative scale dependent renormalisations, e,g, $O^{\Delta S=2}$:

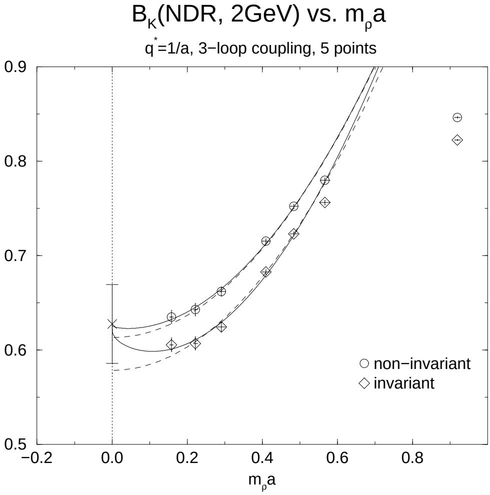
$\Rightarrow$ strong case for non-perturbative renormalisation (see below)

3. Power divergences: mixing with operators with lower dimensions, additive quark mass renormalisation with Wilson quarks:

$\Rightarrow$ total failure of perturbation theory (s. below)

Quenched B_K with staggered quarks [JLQCD, '98]

2 different discretised operators, perturbative 1-loop renormalisation



$\Rightarrow$ Continuum extrapolation difficult due to $\mathrm{O}(\alpha^2)$ terms.

Power divergences and perturbation theory

What problems arise if we just use perturbation theory?
In the case of power divergent subtraction PT is clearly insufficient:

additive mass subtraction with Wilson quarks

$$m_{\mathrm{R}} = Z_m(m_0 - m_{\mathrm{cr}}), \qquad m_{\mathrm{cr}} = \frac{1}{a} f(g_0^2)$$

Suppose one uses a perturbative expansion of f up to g_0^{2n}:

$$\Delta f(g_0^2) = \mathrm{O}(g_0^{2n}), \qquad g_0^{2n} \quad \sim \quad \frac{1}{(\ln a\Lambda)^n}$$

Remainder (after perturbative subtraction at finite order),

$$\frac{1}{a}\Delta f(g_0^2) \quad \sim \quad \{a(\ln a\Lambda)^n\}^{-1} \rightarrow \infty$$

is still divergent!

Momentum Subtraction Schemes (MOM)

Recall procedure in continuum perturbation theory:

- example: renormalisation of the pseudoscalar density $P^a(x) = \overline{\psi}(x)\gamma_5 \frac{1}{2}\tau^a \psi(x)$:
- Correlation functions in momentum space with external quark states:

$$
\begin{aligned}
\left\langle \tilde{\psi}(p)\tilde{\overline{\psi}}(q) \right\rangle &= (2\pi)^4 \delta(p+q) S(p) \qquad \text{quark propagator} \\
\left\langle \tilde{\psi}(p)\tilde{P}^a(q)\tilde{\overline{\psi}}(p') \right\rangle &= (2\pi)^4 \delta(p+q+p') S(p) \Gamma_P^a(p,q) S(p+q)\,,
\end{aligned}
$$

- At tree-level:

$$
\begin{aligned}
\Gamma_P^a(p,q)|_{\mathrm{tree}} &= \gamma_5 \tfrac{1}{2}\tau^a, \\
\Rightarrow \quad \frac{1}{4}\sum_{b=1}^{3} \mathrm{tr}\left\{\gamma_5 \tau^b \Gamma_P^a(p,q)|_{\mathrm{tree}}\right\} &= 1
\end{aligned}
$$

- Renormalised fields:

$$\psi_\mathrm{R} = Z_\psi \psi, \qquad \overline{\psi}_\mathrm{R} = Z_\psi \overline{\psi}, \qquad P^a_\mathrm{R} = Z_\mathrm{P} P^a$$

$\Rightarrow$ renormalised vertex function:

$$\Gamma^a_{P,\mathrm{R}}(p,q) = Z_\mathrm{P} Z_\psi^{-2} \Gamma^a_P(p,q)$$

- typical MOM renormalisation condition (quark masses set to zero):

$$\Gamma^a_{P,\mathrm{R}}(p,0)|_{\mu^2=p^2} = \gamma_5 \tfrac{1}{2}\tau^a \qquad \Rightarrow \qquad Z_\mathrm{P} Z_\psi^{-2}$$

- equivalently using "projector":

$$\tfrac{1}{4}\sum_{b=1}^{3} \mathrm{tr}\left\{\gamma_5 \tau^b \, \Gamma^a_{P,\mathrm{R}}(p,0)|_{\mu^2=p^2}\right\} = 1$$

- Determine Z_ψ either from propagator or use MOM scheme for vertex function of a conserved current

$$\Gamma_{V,\mathrm{R}}(p,q) = Z_\psi^{-2} \Gamma_V(p,q)$$

Summary: MOM schemes in the continuum

- Renormalisation condions are imposed on vertex functions in the gauge fixed theory with external quark, gluon or ghost lines
- The vertex functions are taken in momentum space.
- A particular momentum configuration is chosen, such that the vertex function becomes a function of a single momentum p; quark masses are set to zero
- MOM condition: a renormalised vertex function at subtraction scale $\mu^2 = p^2$ equals its tree-level expression
- Can also be used to define a renormalised gauge coupling: take vertex function of either the 3-gluon vertex, the quark-gluon vertex or the ghost-gluon vertex.
- Renormalisation constants depend on the chosen gauge! Need wave function renormalisation for quark, gluon and ghost fields.

RI/MOM Schemes (RI = Regularisation Independent; MOM = Momentum Subtraction)

[Martinelli et al '95]: mimick the procedure in perturbation theory:

- choose Landau gauge

$$\partial_\mu A_\mu = 0$$

  can be implemented on the lattice by a minimisation procedure
- RI/MOM schemes are very popular: many major collaborations use it because
  - it is straightforward to implement on the lattice; many improvements over the years regarding algorithmic questions
  - it can be used on the very same gauge configurations which are produced for hadronic physics
- Regularisation Independence (RI) means: correlation functions of a renormalised operator do not depend on the regularisation used (up to cutoff effects).

RI/MOM schemes, discussion

- Suppose we have calculated a renormalised hadronic matrix element of the multiplicatively renormalisable operator O

$$\mathcal{M}_O(\mu) = \lim_{a\to 0} \langle h|O_{\mathrm{R}}(\mu)|h'\rangle$$

- Provided μ is in the perturbative regime, one may evaluate the MOM scheme in continuum perturbation theory and evolve to a different scale:

$$\begin{aligned} \mathcal{M}_O(\mu') &= U(\mu',\mu)\mathcal{M}_O(\mu), \\ U(\mu',\mu) &= \exp\left\{\int_{\bar{g}(\mu)}^{\bar{g}(\mu')} \frac{\gamma_O(g)}{\beta(g)}\mathrm{d}g\right\} \end{aligned}$$

- N.B. Continuum perturbation theory is available to 3-loops in some cases!

RI/MOM schemes, what could go wrong?

- The scale μ could be too low; need to hope for a "window"

$$\Lambda_{\mathrm{QCD}} \ll \mu \ll a^{-1}$$

  In practice scales are often too low: non-perturbative effects (e.g. pion poles, condensates) are then eliminated by fitting to expected functional form (from OPE in fixed gauge);

$\Rightarrow$ errors are difficult to quantify!

- Gribov copies: the (Landau) gauge condition does not have a unique solution on the full gauge orbit
- Perturbative calculations are made using
  - infinite volume
  - vanishing quark masses

$\Rightarrow$ inconvenient for numerical simulations especially in full QCD.

- Wilson quarks: a priori cutoff effects are O(a) even in on-shell O(a) improved theory.

A prominent non-perturbative effect: the pion pole

[Martinelli et al. '95]

- Consider the 3-point correlation function for P^a:

$$\int \mathrm{d}^4x \int \mathrm{d}^4y\, \mathrm{e}^{-ipx} \langle\overline{\psi}(0)\gamma_5 \tfrac{1}{2}\tau^b \psi(x)\overline{\psi}(0)P^a(y)\rangle$$

- For large p^2 it is dominated by short distance contributions either at $x \approx 0$ or $x \approx y$. The contribution for $x \approx 0$ is proportional to the pion propagator

$$\int \mathrm{d}^4y \langle P^b(0)P^a(y)\rangle \propto \frac{1}{m_\pi^2}$$

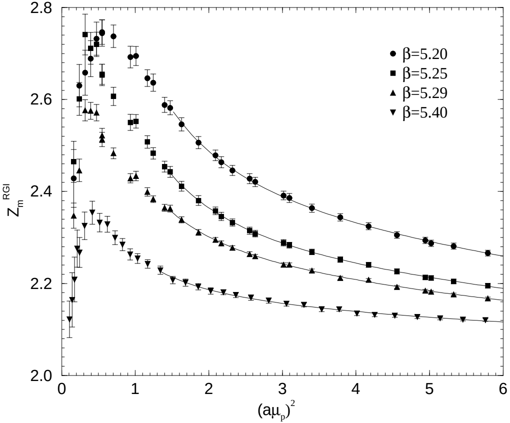
- Dimensional counting: suppression by $1/p^2$ relative to the perturbative term at $x \approx y$:

$$Z_{\mathrm{P}}^{\mathrm{MOM,non-pert}} \sim \frac{A}{\mu^2 m_{\mathrm{q}}} + \ldots$$

$\Rightarrow$ the chiral limit is ill-defined!

RI/MOM scheme, example 1

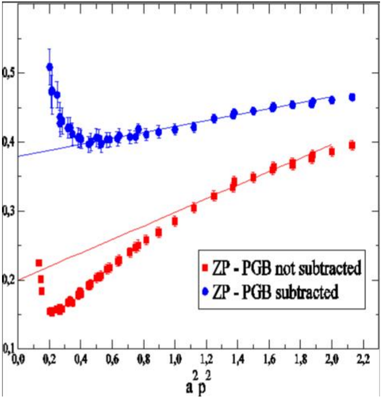
[QCDSF-UKQCD collaboration, Göckeler et al. '06]

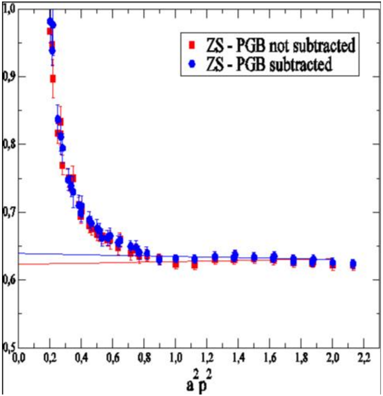


- Z_P^{-1} for the RGI operator after subtraction of the pion pole through a fit. While there is no plateau at fixed β, the situation seems to improve towards higher β, as μ gets larger in physical units.

RI/MOM scheme, example 2

[ETMC collaboration, talk by P. Dimopoulos at Lattice '07]
twisted mass QCD with $N_{\mathrm{f}} = 2$, subtraction of pion pole à la [Giusti, Vladikas '00]

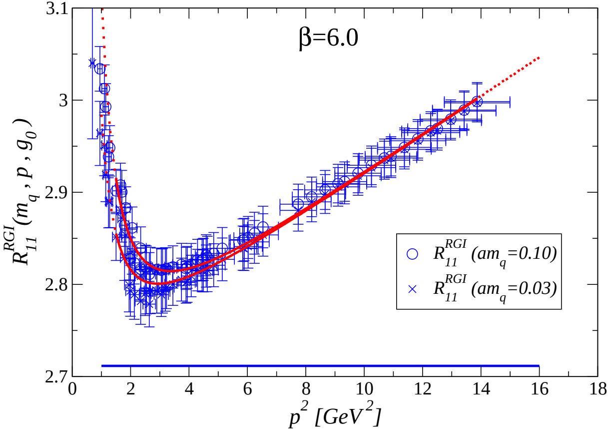




While Z_S shows the expected plateau, Z_{P} shows some slope even after subtraction of the pion pole (cutoff effects?)

RI/MOM scheme, example 3

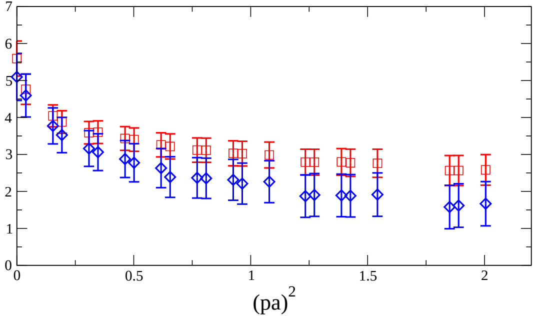
[R. Babich et al. 06] four-quark operator for B_K with overlap quarks (quenched QCD at $\beta = 6.0$):



- non-perturbative effects are eliminated through fit function from OPE including logarithmic terms

RI/MOM scheme, example 4

[Huey-Wen Lin '06] study of quark gluon vertex:



- Comparison of Landau gauge fixed results obtained from 2 gauge equivalent configurations
- Influence of Gribov copies can be sizable!

RI/MOM schemes; tentative conclusions

- There are examples where the method seems to work fine
- Non-perturbative effects like the pion pole are either subtracted or taken into account by fits to the expected p^2-behaviour; error estimates seem difficult!
- A warning from the quark-gluon vertex: the effect of Gribov copies while often found to be small should be monitored!
- finite volume and quark mass effects seem to be small
- Since the method can be applied at little cost on the existing configurations it should always be tried!
- However it seems difficult to get reliable errors down to the desired level (say 1-2 percent for Z-factors)

Possible improvements of RI/MOM schemes

- use gauge invariant states $\Rightarrow$ no trouble with Gribov copies, but more demanding in perturbation theory; expect larger cutoff effects
- use non-exceptional momentum configurations (P. Boyle, Lattice 2007): could reduce the problem with Goldstone poles; Perturbation theory needs to be re-done!
- reach higher scales: Promote to finite volume scheme: fix μL
- need gauge fixing on the torus (complicated)
- twisted gauge field boundary conditions; link between N_c and $N_{\rm f}$
- in any case perturbation theory needs to be re-done from scratch and may be complicated
- dimensional argument: gauge invariant fermionic correlation functions typically suffer from larger cutoff effects

Main idea of finite volume schemes

[Lüscher et al. 91-94, Jansen et al '95] Main idea:

- define a finite volume renormalisation scheme, where

$$\mu = L^{-1}$$

- $\Rightarrow$ possibility to construct the RG evolution recursively, by going in steps

$$L \rightarrow 2L \rightarrow 4L \rightarrow \cdots \rightarrow 2^n L$$

- in practice e.g. $n = 8$, can bridge 2 orders of magnitude in scale
- The problem of large scale differences is solved by NOT having all scales on a single lattice.

Sketch of the recursive procedure:

- suppose we have a non-perturbative definition of the running coupling $\bar{g}(L)$

$$\sigma(u) = \bar{g}^2(2L)\big|_{u=\bar{g}^2(L)}$$

At fixed $u = \bar{g}^2(L)$ the function $\sigma(u)$ can be obtained from a sequence of pairs of lattices with sizes L/a and $2L/a$:

$$\sigma(u) = \lim_{a\to 0} \Sigma(u, a/L)$$

- repeat the procedure for a range of u-values in $\left[\bar{g}^2(L_{\rm min}), \bar{g}^2(L_{\rm max})\right]$.

$$u_0 = \bar{g}^2(L_{\rm min}), \qquad u_k = \sigma(u_{k-1}) \quad \left[= \bar{g}^2\left(2^k L_{\rm min}\right)\right], \quad k = 1, 2, \ldots$$

$\Rightarrow$ after 7-8 steps scale differences of O(100) are bridged!

- need to compute $F_\pi L_{\rm max}$ and $\bar{g}^2(L_{\rm min}) = g^2 + k_1 g^4 + \ldots$