Non-perturbative Renormalisation of Lattice QCD

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Some (more or less) pedagogical references

1. R. Sommer, "Non-perturbative renormalisation of QCD", Schladming Winter School lectures 1997, hep-ph/9711243v1; "Non-perturbative QCD: Renormalization, O(a) improvement and matching to heavy quark effective theory" Lectures at Nara, November 2005 hep-lat/0611020
2. M. Lüscher: "Advanced lattice QCD", Les Houches Summer School lectures 1997 hep-lat/9802029
3. S. Capitani, "Lattice perturbation theory" Phys. Rept. 382 (2003) 113-302 hep-lat/0211036
4. S. Sint "Nonperturbative renormalization in lattice field theory" Nucl. Phys. (Proc. Suppl.) 94 (2001) 79-94, hep-lat/0011081

Contents

Non-perturbative definition of QCD (1)

To define QCD as a QFT it is not enough to write down its classical Lagrangian:

$$\mathcal{L}_{\mathrm{QCD}}(x) = \frac{1}{2g^2}\,\mathrm{tr}\,\{F_{\mu\nu}(x)F_{\mu\nu}(x)\} + \sum_{i=1}^{N_{\mathrm{f}}} \overline{\psi}_i(x)\,(\not{D} + m_i)\,\psi_i(x)$$

One needs to define the functional integral:

- Introduce a Euclidean space-time lattice and discretise the continuum action such that the doubling problem is solved
- Consider a finite space-time volume $\Rightarrow$ the functional integral becomes a finite dimensional ordinary or Grassmann integral, i.e. mathematically well defined!
- Take the infinite volume limit $L \rightarrow \infty$
- Take the continuum limit $a \rightarrow 0$

Non-perturbative definition of QCD (2)

- The infinite volume limit is reached with exponential corrections $\Rightarrow$ no major problem.
- Continuum limit: existence only established order by order in perturbation theory; only for selected lattice regularisations:
  - lattice QCD with Wilson quarks [Reisz '89]
  - lattice QCD with overlap/Neuberger quarks [Reisz, Rothe '99]
  - not (yet ?) for lattice QCD with staggered quarks [cf. Giedt '06]
- From asymptotic freedom expect

$$g_0^2 = g_0^2(a) \quad \overset{a \to 0}{\sim} \quad \frac{-1}{2b_0 \ln a}, \qquad b_0 = \tfrac{11N}{3} - \tfrac{2}{3} N_\mathrm{f}$$

Non-perturbative definition of QCD (3)

Working hypothesis: the perturbative picture is essentially correct:

- The continuum limit of lattice QCD exists and is obtained by taking $g_0 \to 0$
- Hence, QCD is asymptotically free, naive dimensional analysis applies: Non-perturbative renormalisation of QCD is based on the very same counterterm structure as in perturbation theory!
- Absence of analytical methods: try to take the continuum limit numerically, i.e. by numerical simulations of lattice QCD at decreasing values of g_0.

WARNING:

Perturbation Theory might be misleading (cp. triviality of ϕ_4^4-theory)

Renormalisation of QCD

- The basic parameters of QCD are g_0 and m_i, $i = u, d, s, c, b, t$.
- To renormalise QCD one must impose a corresponding number of renormalisation conditions
- We only consider gauge invariant observables $\Rightarrow$ no need to consider field renormalisations for quark, gluon or ghost fields or the renormalisation of the gauge parameter.
- All physical information (particle masses and energies, particle interactions) is contained in the (Euclidean) correlation functions of gauge invariant composite, local fields $\phi_i(x)$

$$\langle\phi_1(x_1)\cdots\phi_n(x_n)\rangle$$

- a priori each ϕ_i requires renormalisation, and thus further renormalisation conditions.

Example: lattice QCD with Wilson quarks

The action $S = S_{\rm f} + S_{\rm g}$ is given by

$$S_{\rm f} = a^4 \sum_x \overline{\psi}(x)\,(D_W + m_0)\,\psi(x), \qquad S_{\rm g} = \frac{1}{g_0^2}\sum_{\mu,\nu} {\rm tr}\,\{1 - P_{\mu\nu}(x)\}$$

$$D_W = \frac{1}{2}\left\{\left(\nabla_\mu + \nabla^*_\mu\right)\gamma_\mu - a\nabla^*_\mu\nabla_\mu\right\}$$

- Symmetries: $U(N_{\rm f})_{\rm V}$ (mass degenerate quarks), P, C, T and $O(4,\mathbb{Z})$

$\Rightarrow$ Renormalized parameters:

$$g_{\rm R}^2 = Z_g g_0^2, \qquad m_{\rm R} = Z_m\,(m_0 - m_{\rm cr})\,, \qquad am_{\rm cr} = am_{\rm cr}(g_0).$$

- In general: $Z = Z(g_0, a\mu, am_0)$;
- Quark mass independent renormalisation schemes: $Z = Z(g_0, a\mu)$
- Simple non-singlet composite fields, e.g. $P^a = \overline{\psi}\gamma_5\tau^a\psi$ renormalise multiplicatively, $P_{\rm R}^a = Z_{\rm P}(g_0, a\mu, am_0)P^a$

Approach to the continuum limit (1)

Suppose we have succeded to renormalise the theory non-perturbatively; for a numerical approach it is crucial to know how the continuum limit is reached. An essential tool is Symanzik's effective continuum theory [Symanzik '79]:

- purpose: render the a-dependence of lattice correlation functions explicit. $\Rightarrow$ structural insight into the nature of cutoff effects
- at scales far below the cutoff a^{-1}, the lattice theory is effectively continuum like; the influence of cutoff effects is expanded in powers of a:

$$S_{\rm eff} = S_0 + aS_1 + a^2 S_2 + \ldots, \qquad S_0 = S_{\rm QCD}^{\rm cont}$$
$$S_k = \int \mathrm{d}^4x\, \mathcal{L}_{\rm k}(x)$$

$\mathcal{L}_{\rm k}(x)$: linear combination of fields

- with canonical dimension $4+k$
- which share all the symmetries with the lattice action

Approach to the continuum limit (2)

A complete set of fields for $\mathcal{L}_1$ is given by:

$$\overline{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi, \qquad \overline{\psi}D_\mu D_\mu\psi, \qquad m\,\overline{\psi}\not{D}\psi, \quad m^2\overline{\psi}\psi, \qquad m\,\mathrm{tr}\,\{F_{\mu\nu}F_{\mu\nu}\}$$

The same procedure applies to composite fields:

$$\phi_{\rm eff}(x) = \phi_0 + a\phi_1 + a^2\phi_2 \ldots$$

for instance: $\phi(x) = P^a(x)$, basis for ϕ_1:

$$m\,\overline{\psi}\gamma_5\tfrac{1}{2}\tau^a\psi, \qquad \overline{\psi}\overleftarrow{\not{D}}\gamma_5\tfrac{1}{2}\tau^a\psi - \overline{\psi}\gamma_5\tfrac{1}{2}\tau^a\not{D}\psi$$

Consider renormalised, connected lattice n-point functions of a multiplicatively renormalisable field ϕ

$$G_n(x_1,\ldots,x_n) = Z_\phi^n\langle\phi(x_1)\cdots\phi(x_n)\rangle_{\rm con}$$

Approach to the continuum limit (3)

Effective field theory description:

$$\begin{aligned} G_n(x_1,\ldots,x_n) &= \langle\phi_0(x_1)\ldots\phi_0(x_n)\rangle_{\rm con} \\ &+ a\int {\rm d}^4y\; \langle\phi_0(x_1)\ldots\phi_0(x_n)\mathcal{L}_1(y)\rangle_{\rm con} \\ &+ a\sum_{k=1}^{n}\langle\phi_0(x_1)\ldots\phi_1(x_k)\ldots\phi_0(x_n)\rangle_{\rm con} + {\rm O}(a^2) \end{aligned}$$

- $\langle\cdots\rangle$ is defined w.r.t. continuum theory with S_0
- the a-dependence is now explicit, up to logarithms, which are hidden in the coefficients.
- In perturbation theory one expects at l-loop order:

$$P(a) \sim P(0) + \sum_{n=1}^{\infty}\sum_{k=1}^{l} c_{nk}\, a^n (\ln a)^k$$

where e.g. $P(a) = G_n$ at fixed arguments.

Approach to the continuum limit (4)

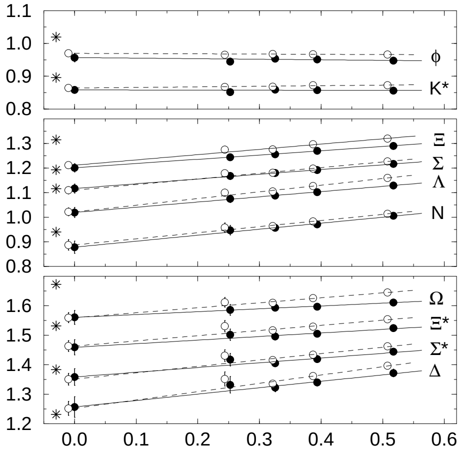
Conclusions from Symanzik's analysis:

- Asymptotically, cutoff effects are powers in a, modified by logarithms;
- In contrast to Wilson quarks, only even powers of a are expected for
  - bosonic theories (e.g. pure gauge theories, scalar field theories)
  - fermionic theories which retain a remnant axial symmetry (overlap, Domain Wall Quarks, staggered quarks, Wilson quarks with a twisted mass term, etc.)

  In QCD simulations a is typically varied by a factor 2

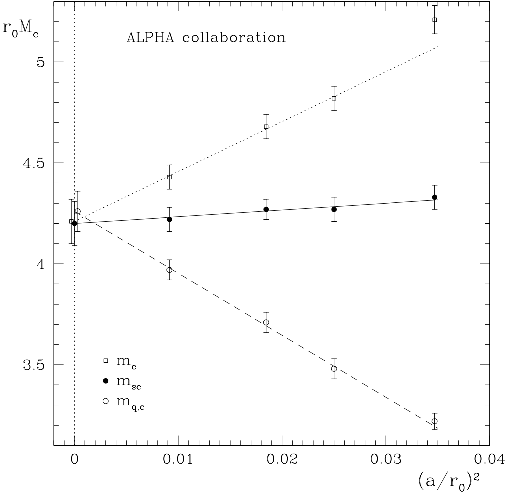
$\Rightarrow$ logarithms vary too slowly to be resolved; linear or quadratic fits (in a resp. a^2) are used in practice.

Linear continuum extrapolation of the quenched hadron spectrum; standard Wilson quarks with Wilson's plaquette action:[CP-PACS coll., Aoki et al. '02] $a = 0.05 - 0.1\,\mathrm{fm}$, experimental input: $m_K,\ m_\pi,\ m_\rho$



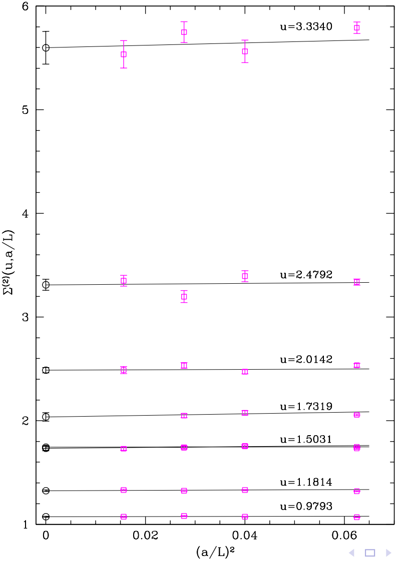
Example 2: O(a) improved charm quark mass (quenched)

[ALPHA coll. J. Rolf et al '02]



Example 3: Step Scaling Function for SF coupling ($N_f = 2$)

[ALPHA coll., Della Morte et al. 2005]



The 2d O(N) sigma model: a test laboratory for QCD?

$$S = \frac{N}{2\gamma} \sum_{x,\mu} (\partial_\mu \mathbf{s})^2, \qquad \mathbf{s} = (s_1, \ldots, s_N) \qquad \mathbf{s}^2 = 1$$

- like QCD the model has a mass gap and is asymptotically free
- many analytical tools: large N expansion, Bethe ansatz, form factor bootstrap, etc.
- efficient numerical simulations due to cluster algorithms.

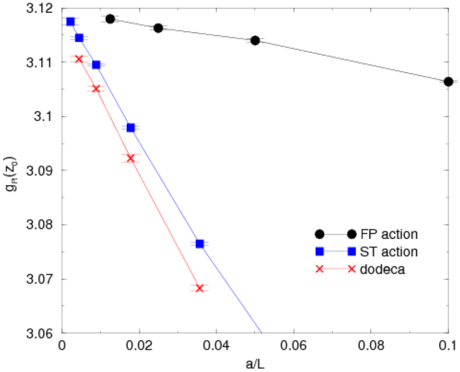
$\Rightarrow$ very precise data over a wide range of lattice spacing (a can be varied by 1-2 orders of magnitude).

- Symanzik: expect O(a^2) effects, up to logarithms
- Large N, at leading [Caracciolo, Pelissetto '98] and next-to-leading [Knechtli, Leder, Wolff '05]:

$$P(a) \sim P(0) + \frac{a^2}{L^2}\left(c_1 + c_2 \ln(a/L)\right)$$

A sobering result:

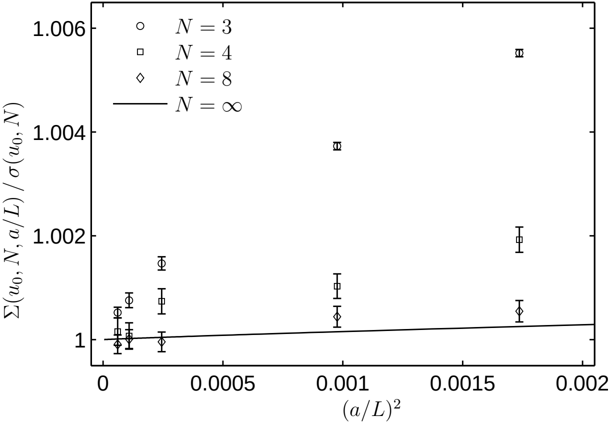
Numerical study of renormalised finite volume coupling to high precision ($N = 3$) [Hasenfratz, Niedermayer '00, Hasenbusch et al. '01]



- Cutoff effects (blue points) seem to be almost linear in a!
- Is this just an unfortunate case?

A closer look:

[Knechtli, Leder, Wolff '05], plot of cutoff effects vs. a^2/L^2:



Asymptotic behaviour seems to set in close to the continuum limit!

Tentative Conclusion

- Symanzik's analysis seems to be applicable beyond perturbation theory
- In quenched QCD numerical results seem to confirm expectations; still very few results in full QCD (expect more in the near future)
- However, The Symanzik expansion is only asymptotic, and powers of a are accompanied by (powers of) logarithms,
- Lesson from σ model: asymptotic behaviour may set in very late!
- It helps to combine results from different regularisations: renormalised quantities must agree in the continuum limit (assuming universality)

What would we like to achieve?

Natural question to ask:

What are the values of the fundamental parameters of QCD (and thus of the Standard Model!),

$$\alpha_s,\ m_u \approx m_d,\ m_s,\ m_c,\ m_b$$

if we renormalise QCD by using experimental low energy data as input. For instance, choose the same number of experimentally well-measured hadron properties:

$$F_\pi,\ m_\pi, m_K,\ m_D,\ m_B.$$

- QCD is regarded as a low energy approximation to the Standard Model: weak interactions are weak $(m_W,\ m_Z \gg m_p)$ and electromagnetic effects are small $(\alpha_{e.m.} = 1/137)$
- conceptually clean, natural question for lattice QCD
- alternative: combination of perturbation theory + additional assumptions ("quark hadron duality", sum rules, hadronisation Monte-Carlo, . . .).

From bare to renormalised parameters

- At fixed g_0:

$$F_\pi, m_\pi, m_K, m_D \quad \Rightarrow \quad a(g_0), am_{0,l}(g_0), am_{0,s}(g_0), am_{0,c}(g_0)$$

- These are bare quantities, the continuum limit cannot be taken!
- N.B.: due to quark confinement there is no natural definition of "physical" quark masses or the coupling constant from particle masses or interactions
- At high energy scales, $\mu \gg m_p$, one may use perturbative schemes to define renormalised parameters (e.g. dimensional regularisation and minimal subtraction)
- How can we relate the bare lattice parameters to the renormalised ones in, say, the $\overline{\mathrm{MS}}$ scheme?
- basic idea: introduce an intermediate renormalisation scheme which can be evaluated both perturbatively and non-perturbively

Non-perturbative renormalisation schemes

Example for a renormalised coupling

Consider the force $F(r)$ between static quarks at a distance r, and *define*

$$\alpha_{\rm qq}(r) = r^2 F(r)|_{m_{{\rm q},i}=0}$$

- at short distances:

$$\alpha_{\rm qq}(r) = \alpha_{\overline{\rm MS}}(\mu) + c_1(r\mu)\alpha^2_{\overline{\rm MS}}(\mu) + \ldots$$

- at large distances:

$$\lim_{r\to\infty} \alpha_{\rm qq}(r) = \begin{cases} \infty & \text{for } N_{\rm f} = 0 \\ 0 & \text{for } N_{\rm f} > 0 \end{cases}$$

- NB: renormalization condition is imposed in the chiral limit $\Rightarrow \alpha_{\rm qq}(r)$ and its β-function are quark mass independent.

Example for a renormalised quark mass

Use PCAC relation as starting point:

$$\partial_\mu (A_\mathrm{R})^a_\mu = 2m_\mathrm{R}(P_\mathrm{R})^a$$

- A^a_μ, P^a: isotriplet axial current & density
- The normalization of the axial current is fixed by current algebra (i.e. axial Ward identities) and scale independent!

$\Rightarrow$ Quark mass renormalization is inverse to the renormalization of the axial density:

$$(P_\mathrm{R})^a = Z_\mathrm{P} P^a, \qquad m_\mathrm{R} = Z_\mathrm{P}^{-1} m_\mathrm{q}.$$

$\Rightarrow$ Impose renormalization condition for the axial density rather than for the quark mass

Renormalization condition for axial density

Define $\langle P_{\rm R}^a(x) P_{\rm R}^b(y)\rangle = \delta^{ab} G_{\rm PP}(x-y)$, and impose the condition

$$G_{\rm PP}(x)\Big|_{\mu^2x^2=1,\, m_{{\rm q},i}=0} = -\frac{1}{2\pi^4(x^2)^3}$$

$G_{\rm PP}(x)$ is defined at all distances:

$$G_{\rm PP}(x) \stackrel{x^2\rightarrow 0}{\sim} -\frac{1}{2\pi^4(x^2)^3} + {\rm O}(g^2), \qquad G_{\rm PP}(x) \stackrel{x^2\rightarrow \infty}{\sim} -\frac{1}{4\pi^2 x^2} G_\pi^2 + \ldots$$

$\Rightarrow Z_{\rm P}$ is defined at all scales μ:

- at large μ (but $\mu \ll 1/a$):

$$Z_P(g_0, a\mu) = 1 + g_0^2 d_0 \ln(a\mu) + \ldots,$$

- at low scales μ:

$$Z_{\rm P}(g_0, a\mu) \propto \mu^2$$

Renormalization group functions

The renormalized coupling and quark mass are defined non-perturbatively at all scales

$\Rightarrow$ Renormalization group functions are defined non-perturbatively, too:

- β-function

$$\beta(\bar{g}) = \mu \frac{\partial \bar{g}(\mu)}{\partial \mu}, \qquad \bar{g}^2(\mu) = 4\pi\alpha_{\mathrm{qq}}(1/\mu)$$

- quark mass anomalous dimension:

$$\tau(\bar{g}) = \frac{\partial \ln \overline{m}(\mu)}{\partial \ln \mu} = -\lim_{a\rightarrow 0} \frac{\partial \ln Z_{\mathrm{P}}(g_0, a\mu)}{\partial \ln a\mu}\bigg|_{\bar{g}(\mu)}$$

Asymptotic expansion for weak couplings:

$$\begin{aligned} \beta(g) &\sim -g^3 b_0 - g^5 b_1 \ldots, \qquad & b_0 &= \{\tfrac{11}{3}N - \tfrac{2}{3}N_{\mathrm{f}}\}(4\pi)^{-2}, \ldots \\ \tau(g) &\sim -g^2 d_0 - g^4 d_1 \ldots, \qquad & d_0 &= 3(N - N^{-1})(4\pi)^{-2}, \ldots \end{aligned}$$

The Callan-Symanzik equation

Physical quantities P are independent of μ, and thus satisfy the CS-equation:

$$\left\{\mu\frac{\partial}{\partial\mu}+\beta(\bar{g})\frac{\partial}{\partial\bar{g}}+\tau(\bar{g})\overline{m}\frac{\partial}{\partial\overline{m}}\right\}P=0$$

Λ and M_i are special solutions:

$$\begin{aligned}\Lambda &= \mu\,(b_0\bar{g}^2)^{-b_1/2b_0^2}\,\exp\left\{-\frac{1}{2b_0\bar{g}^2}\right\}\\ &\quad\times\exp\left\{-\int_0^{\bar{g}}\mathrm{d}x\left[\frac{1}{\beta(x)}+\frac{1}{b_0x^3}-\frac{b_1}{b_0^2x}\right]\right\}\\ M_i &= \overline{m}_i\,(2b_0\bar{g}^2)^{-d_0/2b_0}\exp\left\{-\int_0^{\bar{g}}\mathrm{d}x\left[\frac{\tau(x)}{\beta(x)}-\frac{d_0}{b_0x}\right]\right\}\end{aligned}$$

N.B. no approximations involved!

Λ and M_i as fundamental parameters of QCD

- defined beyond perturbation theory
- scale independent
- scheme dependence? Consider finite renormalization:

$$g'_{\mathrm{R}} = g_{\mathrm{R}} c_g(g_{\mathrm{R}}), \qquad m'_{\mathrm{R},i} = m_{\mathrm{R},i} c_m(g_{\mathrm{R}})$$

with asymptotic behaviour $c(g) \sim 1 + c^{(1)} g^2 + \ldots$
$\Rightarrow$ find the exact relations

$$M'_i = M_i, \qquad \Lambda' = \Lambda \exp(c_g^{(1)}/b_0).$$

$\Rightarrow \Lambda_{\overline{\mathrm{MS}}}$ can be defined indirectly beyond PT; to obtain Λ in any other scheme requires the one-loop matching of the respective coupling constants.

Strategy to compute Λ and M_i

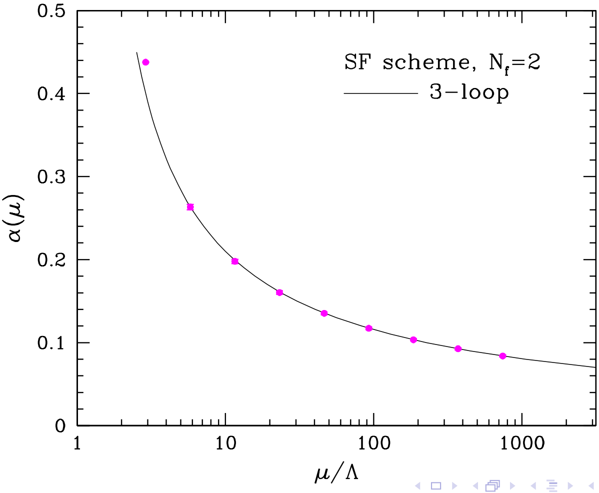
- At fixed g_0 determine the bare parameters corresponding to the experimental input.
- Determine $\alpha_{\rm qq}(1/\mu)$ and $Z_{\rm P}(g_0, a\mu)$ at the same g_0 in the chiral limit
- use $Z_{\rm P}$ to pass from bare to renormalised quark masses
- do this for a range of μ-values
- repeat the same for a range of g_0-values and take the continuum limit

$$\lim_{a\to 0} Z_{\rm P}^{-1}(g_0, a\mu) m_i(g_0), \qquad \lim_{a\to 0} \alpha_{\rm qq}(1/\mu)$$

- check wether perturbative scales μ have been reached
- if this is the case, use the perturbative β- and τ-function to extrapolate to $\mu = \infty$; extract Λ and M_i (equivalently convert to $\overline{\rm MS}$ scheme deep in perturbative region).

Example: running of the coupling (SF scheme, $N_f = 2$)

[ALPHA, M. Della Morte et al. 2005]



The problem of large scale differences

Λ and M_i refer to the high energy limit of QCD

- The scale μ must reach the perturbative regime: $\mu \gg \Lambda_{\mathrm{QCD}}$
- The lattice cutoff must still be larger: $\mu \ll a^{-1}$
- The volume must be large enough to contain pions: $L \gg 1/m_\pi$
- Taken together a naive estimate gives

$$L/a \gg \mu L \gg m_\pi L \gg 1 \quad \Rightarrow L/a \simeq \mathrm{O}(10^3)$$

$\Rightarrow$ widely different scales cannot be resolved simultaneously on a finite lattice!

In practice ...

This estimate may be a little too pessimistic:

- $Lm_\pi \approx 3-4$ often sufficient
- if cutoff effects are quadratic one only needs $a^2\mu^2 \ll 1$.
- when working in momentum space one may argue that the cutoff really is π/a;
- in any case, one must satisfy the requirement $\mu \gg \Lambda_{\rm QCD}$

Heavy quark thresholds

Λ and M_i implicitly depend on $N_{\rm f}$ the number of active flavours! If computed for $N_{\rm f} = 2, 3$ one needs to perform a matching across the charm and bottom thresholds to match the real world at high energies.