QCD at finite baryon density

The naive continuum prescription of introducing chemical potential by adding a term $\mu \int d^4x_E \bar{\psi}\gamma_0\psi$ does not work !

$$S = a^3 \sum_x \left[ma\bar{\psi}_x\psi_x + \mu a\bar{\psi}_x\gamma_0\psi_x + \frac{1}{2}\sum_\mu (\bar{\psi}_x\gamma_\mu\psi_{x+\mu} - \bar{\psi}_{x-\mu}\gamma_\mu\psi_x) \right]$$

$$\Downarrow$$

$\epsilon(\mu) \sim \mu^2/a^2$ instead of $\epsilon(\mu) \sim \mu^4$

The correct prescription is

$$U_0(x) \to e^{\mu a} U_0(x), \quad U_0^\dagger(x) \to e^{-\mu a} U_0^\dagger(x)$$

Hasenfratz, Karsch, PLB 125 (83) 308

$$S = (\overline{\psi}_x e^{\mu a} U_0(x)\psi_{x+0} - \overline{\psi}_x e^{-\mu a} U_0^+(x)\psi_{x-0}) + \sum_{x,i} \eta_i(x)\, (\overline{\psi}_x U_i(x)\psi_{x+i} - \overline{\psi}_x U_i^+(x)\psi_{x-i}) + am\sum_x \overline{\psi}_x\psi_x$$

$$\Downarrow$$

det M is complex => sign problem *det M exp(-S)* cannot be a probability

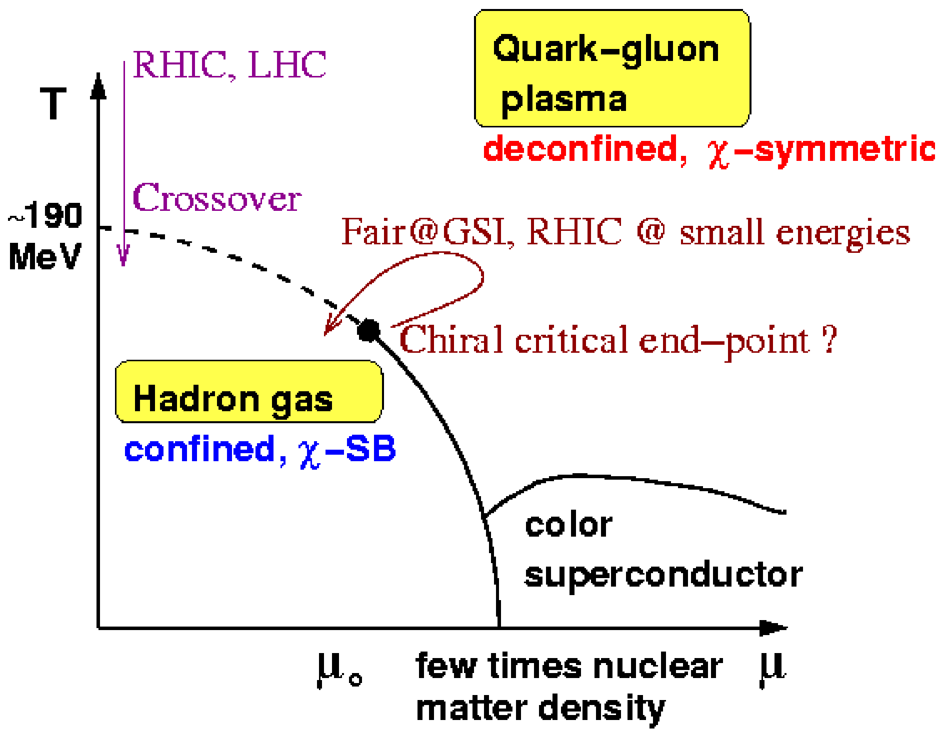
QCD Phase diagram

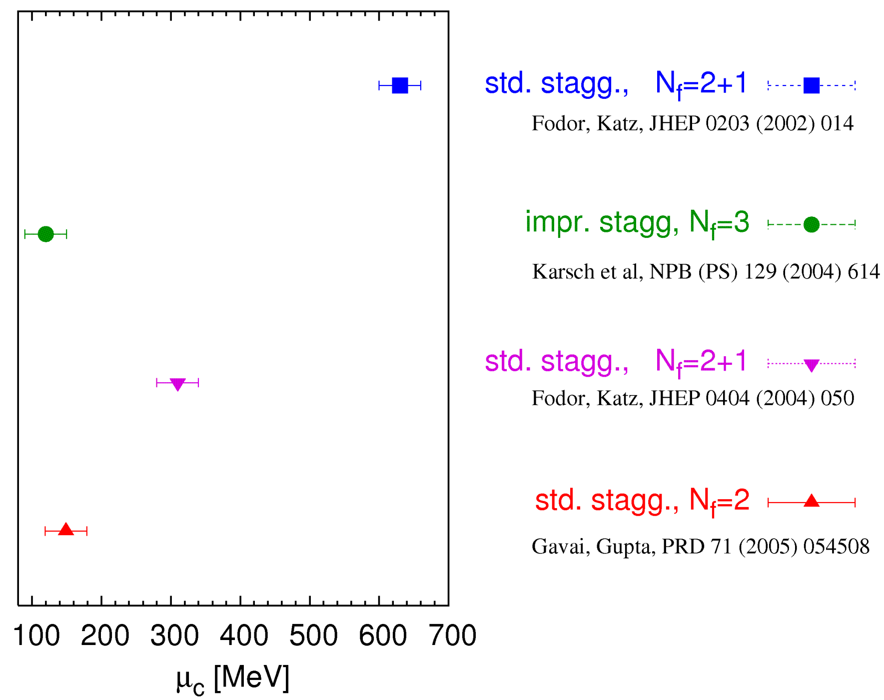
At physical quark masses the transition is likely a rapid crossover for $\mu = 0$

MILC, PRD 71 (04) 034504, RBC-Bielefeld, PRD 74 (06) 054507,
Aoki et al, Nature 443 (06) 675

There should a critical end-point at some $\mu = \mu_c$ where the transition turns from crossover to 1st odrer. Where it is located ?

Lattice QCD methods to estimate μ_c: multi-parameter re-weighting
(Fodor, Katz 2001, partial breakthrough)
Taylor expansion, imaginary mu





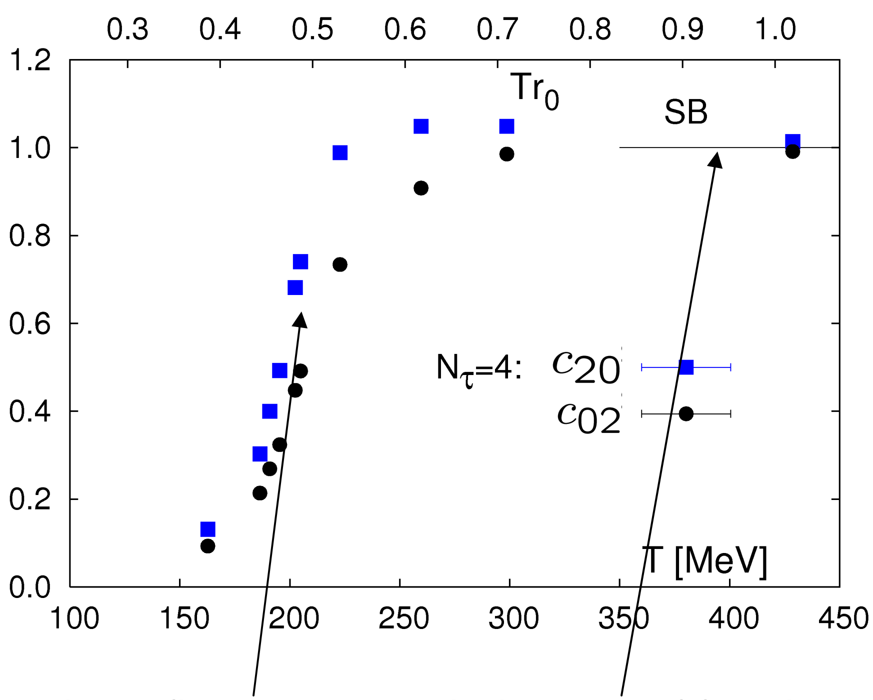
Quark number and strangeness fluctuations

$$\frac{p}{T^4} = \sum_{i,j} c_{i,j} \hat{\mu}_q^i \hat{\mu}_s^j, \quad c_{i,j} = \frac{1}{i!j!} \frac{\partial^i}{\partial \hat{\mu}_q^i} \frac{\partial^j}{\partial \hat{\mu}_s^j} \frac{1}{VT^3} \ln Z(T,V), \;\; \hat{\mu}_i = \mu_i/T$$

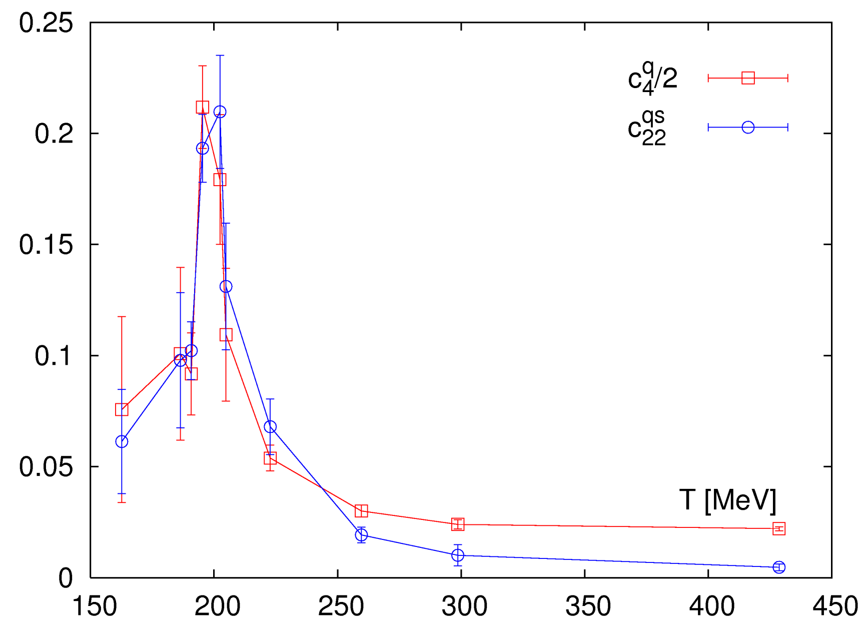
Quark number fluctuations: $c_{20} \sim \langle q^2 \rangle - \langle q \rangle^2$

Strangeness fluctuations: $c_{02} \sim \langle S^2 \rangle - \langle S \rangle^2$

2+1 flavor
RBC-Bielefeld Collaboration



deconfinement, quark degrees of freedom



precise method to locate the transition point scaling field: $t = \left| \frac{T - T_c}{T_c} \right| + A \left(\frac{\mu_q}{T_c} \right)^2$, $\mu_{crit} = 0$

singular part: $f_s(T, \mu_q) = b^{-1} f_s(t b^{1/(2-\alpha)}) \sim t^{2-\alpha}$

Y. Hatta, T. Ikeda, PRD67 (2003) 014028

$$c_2 \equiv \chi_q \sim \frac{\partial^2 \ln \mathcal{Z}}{\partial \mu_q^2} \sim t^{1-\alpha} \quad , \quad c_4 \sim \frac{\partial^4 \ln \mathcal{Z}}{\partial \mu_q^4} \sim t^{-\alpha} \quad (\mu = 0)$$

$$\epsilon \sim \frac{\partial \ln \mathcal{Z}}{\partial T} \sim t^{1-\alpha} \quad , \quad C_V \sim \frac{\partial^2 \ln \mathcal{Z}}{\partial T^2} \sim t^{-\alpha} \quad (\mu = 0)$$

$\Rightarrow 2^{nd}$ derivative w.r.t μ_q "looks like energy density"

$\Rightarrow 4^{th}$ derivative w.r.t μ_q "looks like specific heat"

Comparison with resonance gas at low T

$$P(T,\mu) = P_B(T,\mu) + P_M(T,\mu) \qquad \Delta P(T,\mu) = P(T,\mu) - P(T,\mu=0) = \Delta P_B$$

$$\frac{\Delta P_B}{T^4} \approx F(T)(\cosh(\frac{3\mu_q}{T})-1) \qquad F(T) = \frac{1}{2\pi^2}\int dm\,\rho(m)(\frac{m}{T})^2 K_2(\frac{m}{T})$$

Karsch, Redlich, Tawfik, PLB 571 (2003) 67

~1000 Exp. Know resonances

Compare with LGT results

(Bielefeld-Swansea Coll) :

$$\frac{\Delta P}{T^4} \approx F(T)\left[c_2(\frac{\mu_q}{T})^2 + c_4(\frac{\mu_q}{T})^4 + c_6(\frac{\mu_q}{T})^6\right]$$

$$\frac{n_q}{T^3} \approx F(T)\left[2c_2(\frac{\mu_q}{T}) + 4c_4(\frac{\mu_q}{T})^3 + 6c_6(\frac{\mu_q}{T})^5\right]$$

$$\frac{\chi_q}{T^2} \approx F(T)\left[2c_2 + 12c_4(\frac{\mu_q}{T})^2 + 30c_6(\frac{\mu_q}{T})^4\right]$$

Consequences:

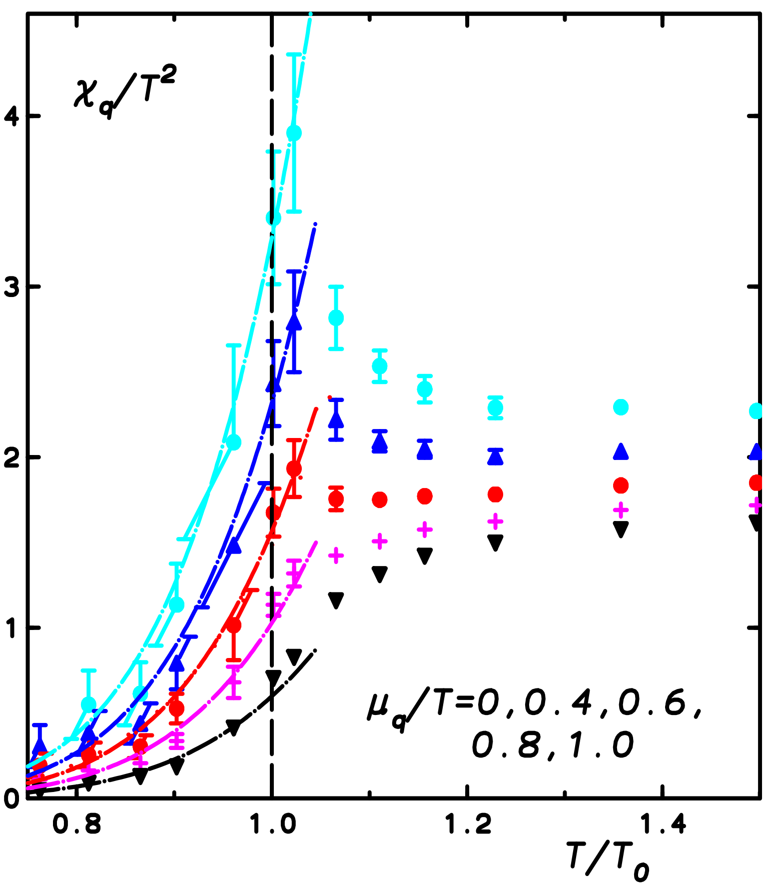
For fixed μ_q / T the ratio of These observable is T-independent

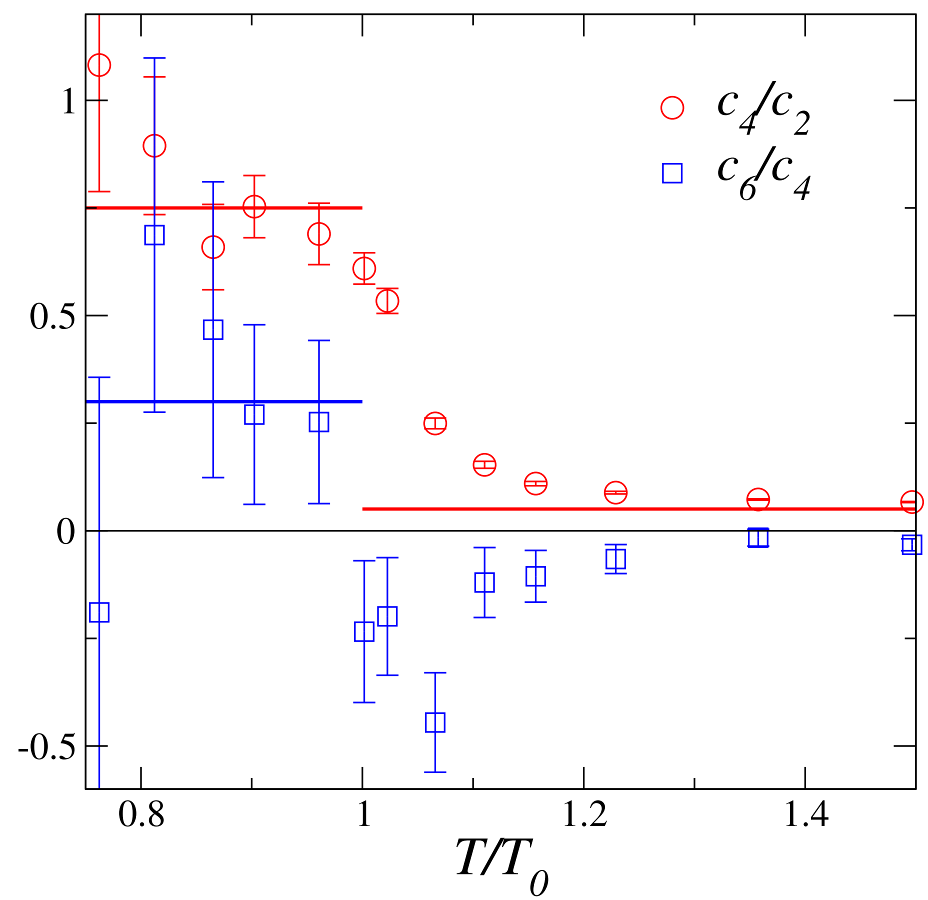
The ratios of the expansion coeffiecients are

$$\frac{c_4}{c_2} = \frac{3}{4}, \ldots, \frac{c_6}{c_4} = 0.3$$

2-falvor, Bielefeld-Swansea Collaboration

Resonance gas model :Karsch, Redlich, Tawfik, EPJC 29 (2003) 549, PLB 571 (2003) 67



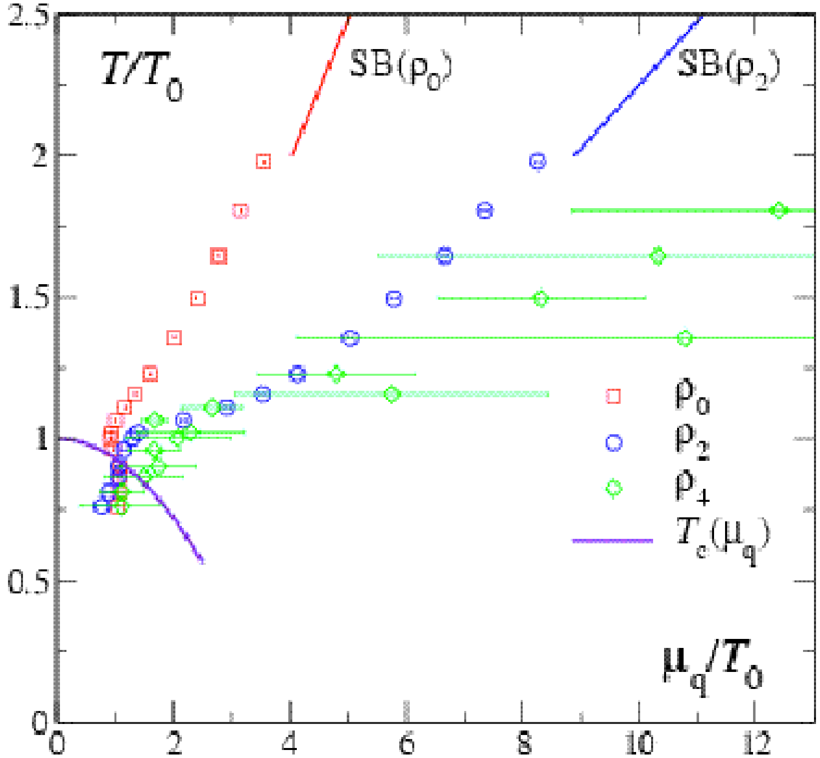


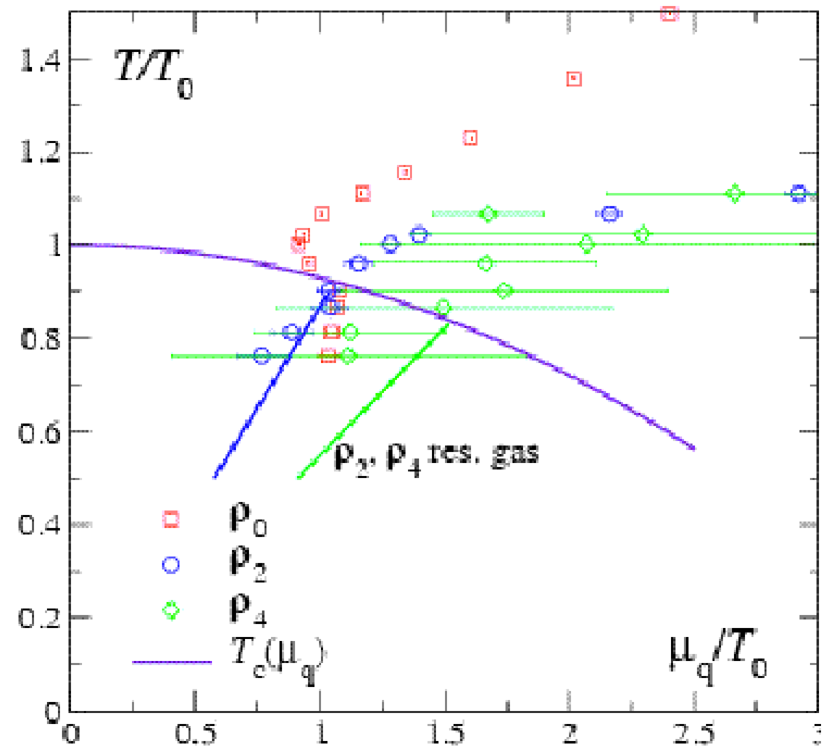
High temperatures :

$$c_2 \simeq \frac{1}{2},\ c_4 \simeq \frac{1}{4\pi^2},\ c_6 \simeq -\frac{g^3}{768\pi^7}(1+N_f/6)^{-3/2} \ll c_4$$

Radius of convergence : $\rho = \lim_{n\to\infty} \rho_{2n} = \lim_{n\to\infty} \sqrt{\left|\frac{c_{2n}}{c_{2n+2}}\right|}$

gives the position of the nearest singularity, if all coefficients are positive the singularity is on the real axis





➡ need higher order terms in the Taylor expansion

Locating the transition point with Lee-Yang zeros

$Z(\beta_c) = 0$ for the true phase transition ($V = \infty$) For finite volume there is no singularity but $Z(\beta^*) = 0,\ \mathrm{Im}\beta^* > 0$ On the lattice one calculates

$$Z_{norm}(\mathrm{Re}\beta^*, \mathrm{Im}\beta^*) = \left| \frac{Z(\mathrm{Re}\beta^*, \mathrm{Im}\beta^*)}{Z(\mathrm{Re}\beta^*, 0)} \right|$$

$$= \left| \langle e^{-i\mathrm{Im}\beta^* S_G} \rangle_{(\mathrm{Re}\beta^*, 0)} \right|$$

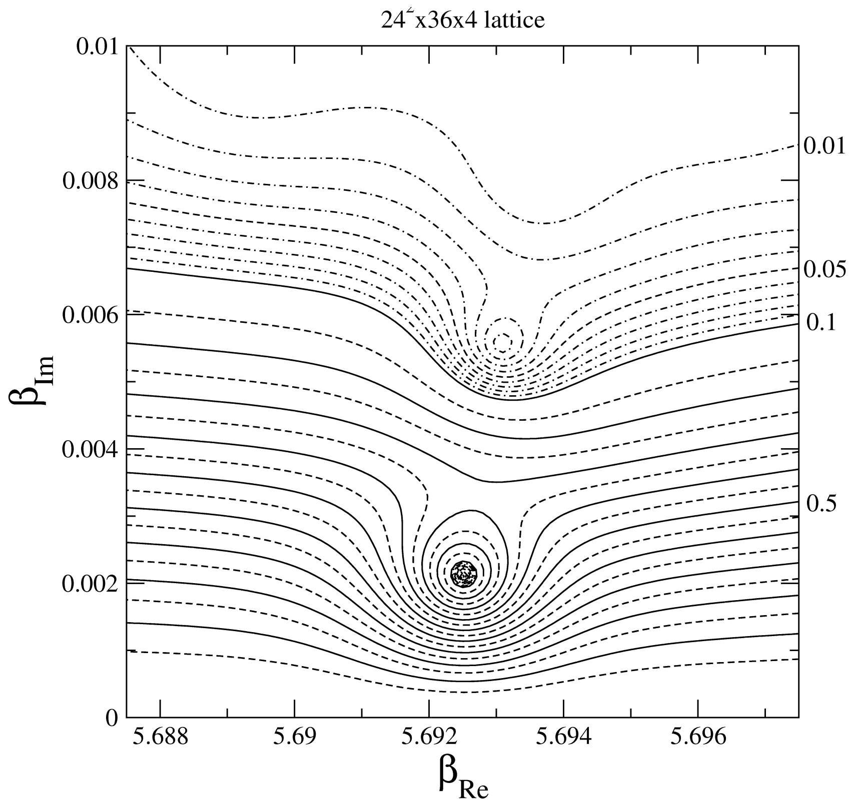
$$\Downarrow$$

Z_{norm} could be zero only if fluctuations of S_G are large (phase $> \pi/2$), i.e when the gauge action susceptibility $\langle S_G^2 \rangle - \langle S_G \rangle^2$ has its maximum

$$\Downarrow$$

$\mathrm{Re}\beta^*$ for the β^* closest to the real axis provides an estimate of the pseudo-critical coupling

for a 1st order tramsition $\mathrm{Im}\beta^* \sim 1/V$



Ejiri, hep-lat/0506023

Locating the critical end-point with re-weighting

- Multi-parameter re-weghting:

$$Z(\beta,\mu) = \int \mathcal{D}U e^{-S_g(\beta_0,\mu)} [detM(\mu=0,U)]^{n_f/4} \left\{ e^{-S_g(\beta,U)+S_g(\beta_0,U)} \left[\frac{detM(\mu,U}{detM(\mu=0,U)} \right]^{n_f/4} \right\}$$

- Lee-Yang zeroes: $Z(\beta^*,\mu) = 0$

$\mathrm{Im}\beta^*(L_s \to \infty) \neq 0$ ➡ Crossover

$\mathrm{Im}\beta^*(L_s \to \infty) = 0$ ➡ Phase transition

re-weighting in mu only (Glasgow method) has poor overlap

2001: Fodor, Katz, JHEP 0203 (2002) 014

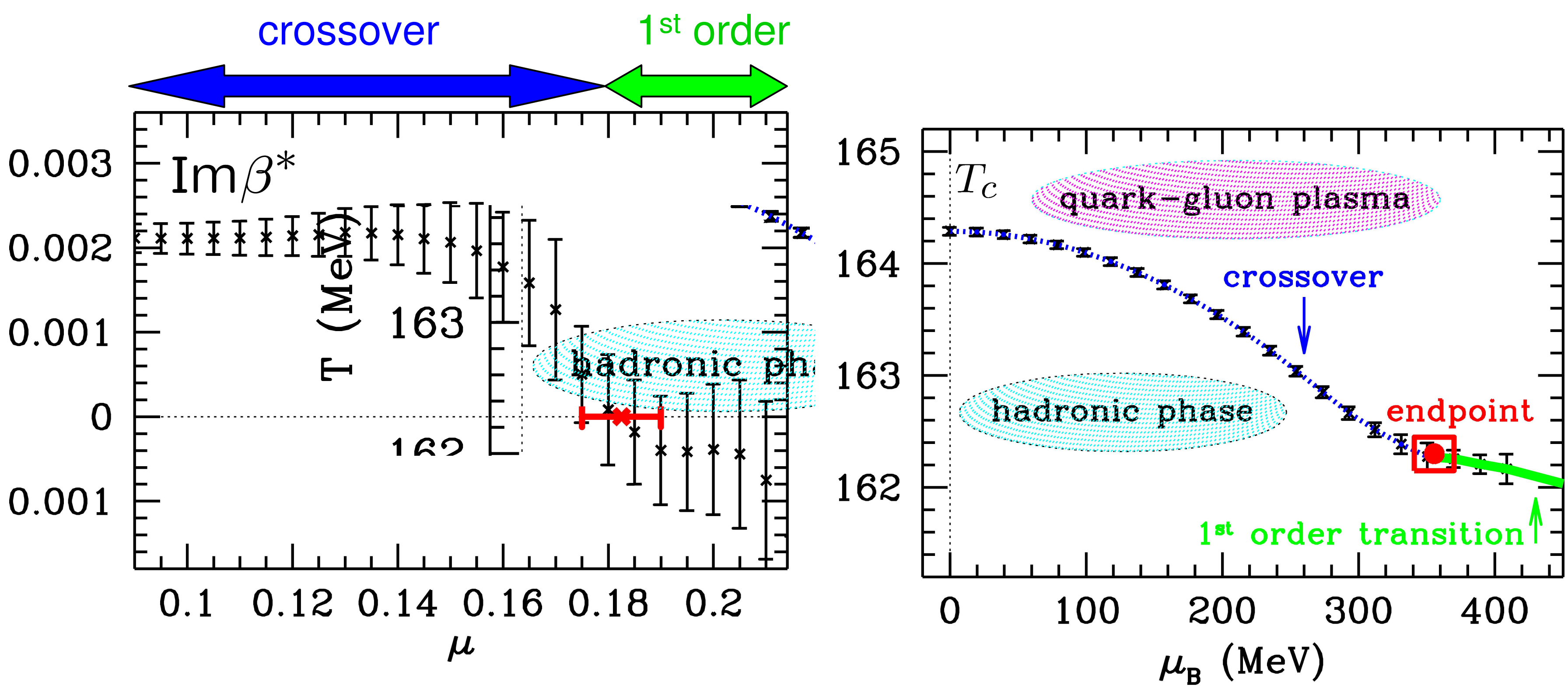
Lattices: $4^4,\ 6^3 \times 4,\ 8^3 \times 4$ $\quad m_\pi/m_\rho \simeq 0.3$

$$T_c = 172(3)\mathrm{MeV},\ T_E = 160(4)\mathrm{MeV},\ \mu_E = 725(35) MeV$$

2004: Fodor, Katz, JHEP 0404 (2004) 050

$$m_\pi/m_\rho = 0.188(2) \qquad 6^3\times 4,\ 8^3\times 4,\ 10^3\times 4,\ 12^3\times 4$$

$$\mathrm{Im}\beta^*(L_s) = \mathrm{Im}\beta^*(\infty) + \zeta/L_s^3$$

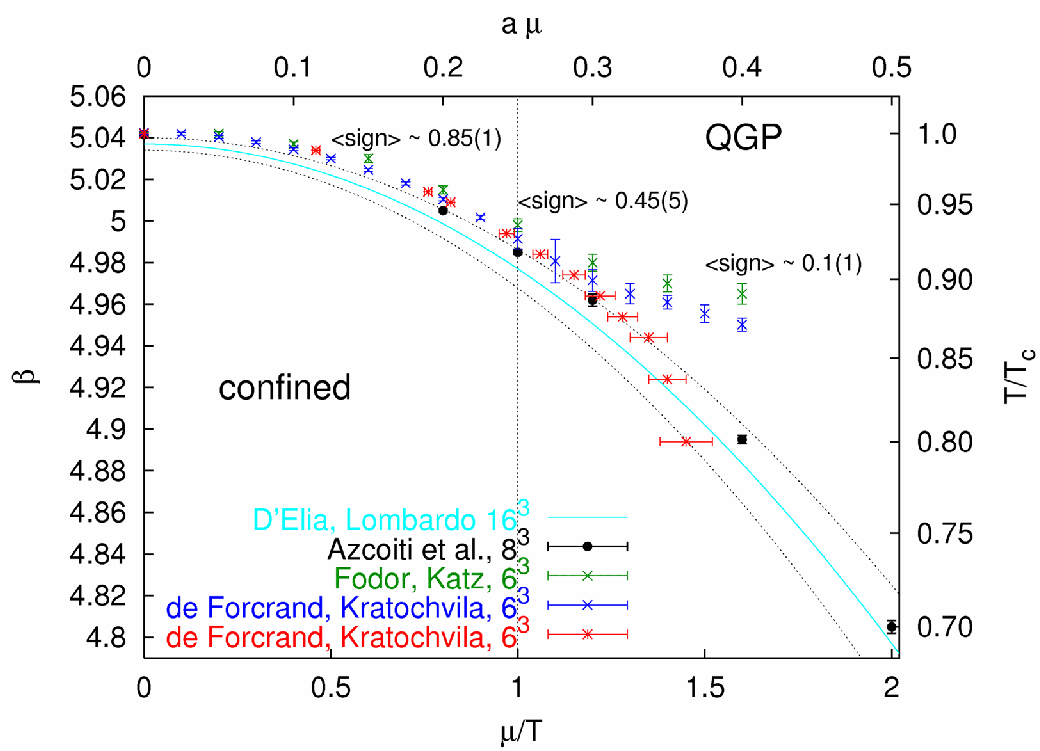


$$r_0 = 0.5\mathrm{fm}: \quad T_c = 164(2)\mathrm{MeV},\ T_E = 162(2)\mathrm{MeV},\ \mu_E = 360(40)\mathrm{MeV}$$

Imaginary chemical potential

det M no sign problem => direct simulations are possible
no singularity at finite volume => continue to real chemical potential
=> alternative way to do the Taylor expansion => canonical partition function

$N_f = 4$ de Forcrand, Kratochvila, PoS LAT2005 (06) 167



$$T_c(\mu) = T(0) - t_2\mu^2$$

different methods
agree quite well !

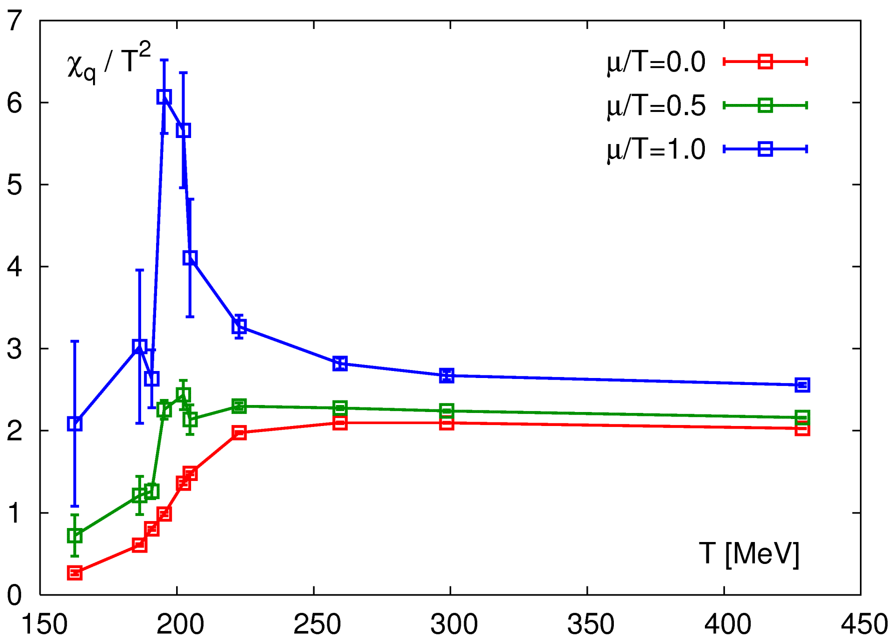
Quark number and strangeness fluctuations at finite density

Net strangeness density: $n_s(\hat{\mu}_q, \hat{\mu}_s) = \sum_{i,j} (j+1) c_{i(j+1)} \hat{\mu}_q^i \hat{\mu}_s^j$

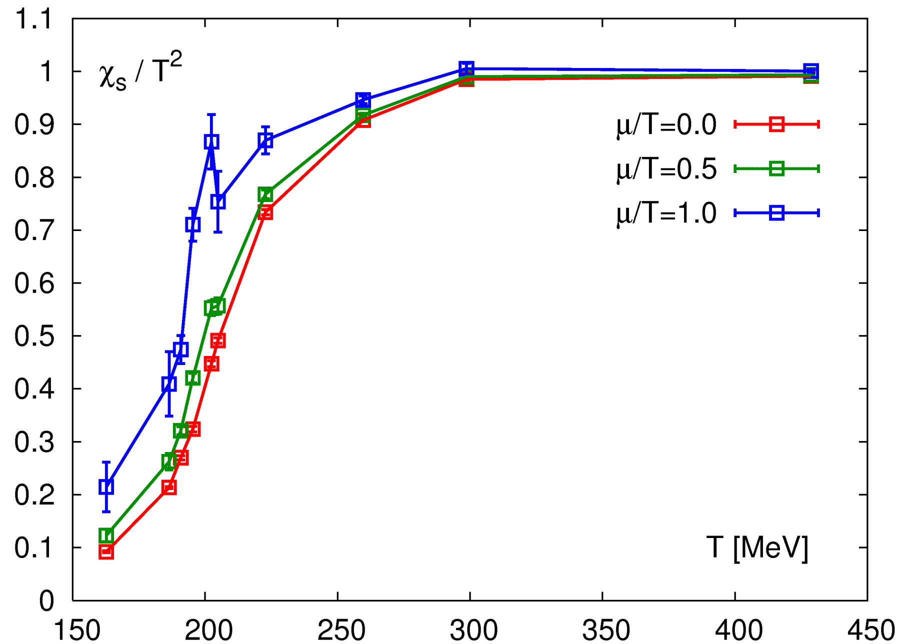
$$n_s = 0 \Rightarrow \hat{\mu}_s = \sum_i d_i \hat{\mu}_q^i = -\frac{c_{11}}{c_{20}} \hat{\mu}_q + \mathcal{O}(\hat{\mu}_q^3)$$

$$\frac{\chi_q}{T^2} = 2c_{20} + 6c_{31}\hat{\mu}_q\hat{\mu}_s + +2c_{22}\hat{\mu}_s^2 + 12c_{40}\hat{\mu}_q^4 + \mathcal{O}(\mu_{q,s}^4)$$

$$\frac{\chi_s}{T^2} = 2c_{02} + 6c_{13}\hat{\mu}_q\hat{\mu}_s + +2c_{22}\hat{\mu}_q^2 + 12c_{04}\hat{\mu}_s^4 + \mathcal{O}(\mu_{q,s}^4)$$



Large enhancement of quark number fluctuations at finite μ_q



Moderate enhancement of strangeness fluctuations at finite μ_q