QCD thermodynamics at low and high temperatures

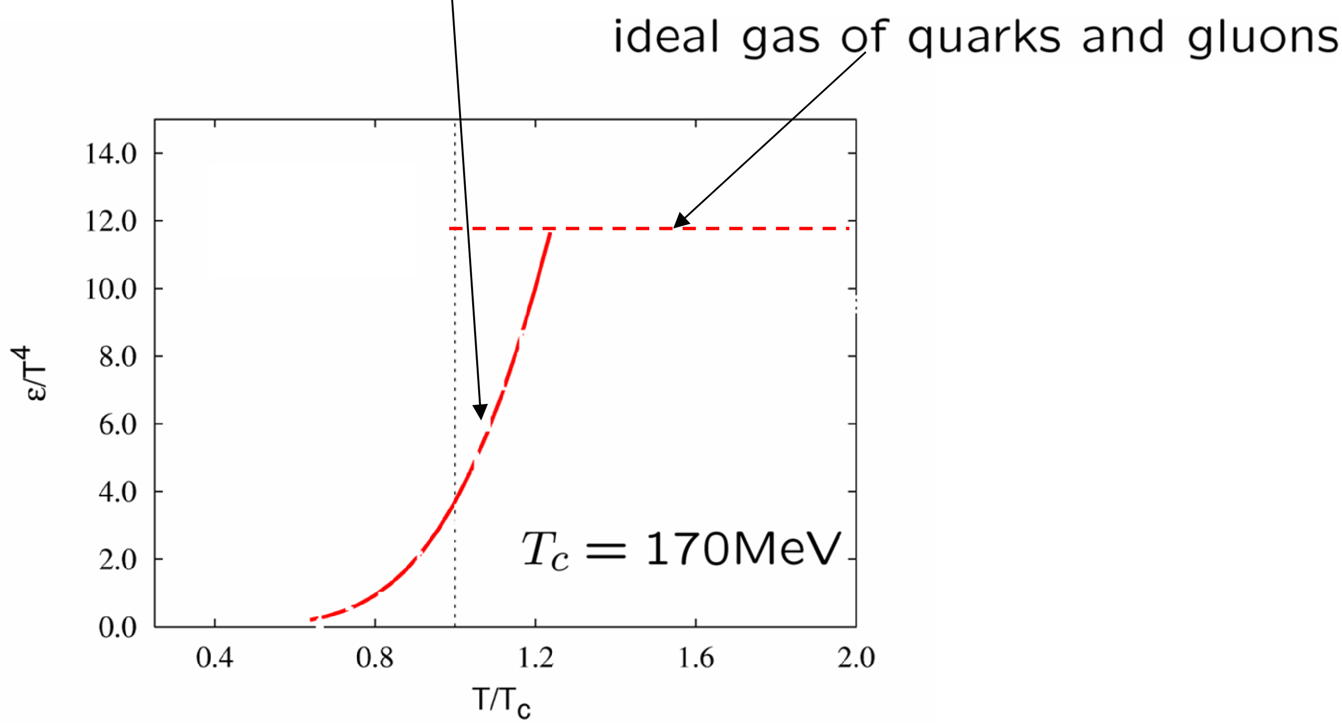
high-T (T>>Λ), perturbation theory should work, however:
expansion in g shows poor convergence
(reorganization of the perturbative series)
g^6 -order contribution is not calculable in the loop expansion
⇓
non-perturbative (lattice) method is desirable

low-T : hadrons are "good" degrees of freedom and weakly interacting for T<<Λ
(use chPT, Gerber, Leutwyler, NPB 321 (89) 387)

The simplest approach : consider gas of non-interacting hadrons
too naïve ? Not necessarily many hadronic interactions dominated by
resonance exchange in the s-channel , e.g. $\pi\pi \rightarrow \rho$
⇓
interacting hardon gas ⟶ non-interacting resonance gas
Hagedorn, Nouvo Cim. 35 (65) 395
Chapline et al, PRD 8 (73) 4302
Karsch et al, Eur.Phys.J.C29 (03)549

$$\ln Z(T,V) = \sum_i \frac{V d_i}{2\pi^2} \int_0^\infty dk k^2 \eta \ln(1 + \eta e^{-\beta\sqrt{p^2+m_i^2}})$$

$\eta = -1$ -boson, $\eta = +1$ -fermion Calculate $\ln Z$ using the masses of about 1000 experimentally known non-strange resonances



Deconfinement transition : rapid increase of the pressure, energy denisty, entropy density (liberation of many new degrees of freedom ?) Cabbibo, Parisi, PLB 59 (75) 67

Is it a phase transition ? What is the order parameter ?

Lattice Monte-Carlo simulations :
Kuti et al, PLB 98 (81) 199
McLerran, Svetitsky, PRD 24 (81) 450

→ free energy of static quarks

Engels et al, PLB 101 (81) 89 → rapid rise in the energy density

Lattice set-up:

$U_\mu(\tau,x) = e^{igA_\mu(\tau,x)}$, $N_\sigma^3 \times N_\tau$, $T = 1/(N_\tau a)$

Thermodynamic limit: $N_\sigma/N_\tau \to \infty$; Continuum limit : $N_\tau \to \infty$, T-fixed Temperature is set by $a \leftrightarrow \beta = 2N_c/g^2$ allowable gauge transformations $U_\mu(x) \to \Omega(x+\mu)U_\mu(x)\Omega^\dagger(x)$

$$\Omega(0,\vec{x}) = \Omega(\beta,\vec{x})C, \quad C = e^{2\pi i n/N_c} I \quad \to Z(N) - \text{symmetry}$$

$$L(\vec{x}) = \frac{1}{N_c}\mathrm{tr}\prod_{\tau=1}^{N_\tau} U_0(\tau,\vec{x})$$

Polyakov loop is changed $L(\vec{x}) \to e^{2\pi n i/N_c} L(\vec{x})$

$< L > \neq 0 \;\; \to Z(N)$ spontaneously broken; $< L > = e^{-F_Q/T}$ -free energy of an isolated static quark is finite $\Rightarrow$ deconfinement

L is order parameter

$\psi_a^\dagger(\tau,x),\ \psi_a(\tau,x)$-creation annihilation operators for static quarks at time τ and position x

$\psi_a^{\dagger c}(\tau,x),\ \psi_a^c(\tau,x)$-creation annihilation operators for static anti-quarks at time τ and position x

$$[\psi_a(\tau,x),\psi_b^\dagger(\tau,y)]_+ = \delta(x-y)\delta_{ab}$$

$$(-i\partial_\tau - gA_0(\tau,x))\,\psi(\tau,x) = 0$$

formal solution $\psi(\tau,x) = \mathcal{P}\,\mathrm{exp}\left(ig\int_0^\tau d\tau' A_0(\tau',x)\right)\psi(0,x) = W(x)\psi(0,x)$

Free energy of static quark anti-quark pair

$$e^{-\beta F(x,y)} = \sum_s < s|e^{-\beta H}|s >$$

$|s>$ denotes any state with a static quark at position x and static anti-quark at position y; $|s'>$ states with no static quarks

$$e^{-\beta F(x,y)} = \sum_{s'} < s'|\psi_a(0,x)\psi_b^c(0,y)e^{-\beta H}\psi_{a'}^\dagger(0,x)\psi_{b'}^{\dagger c}(0,y)|s' >$$

$e^{-\beta H}e^{\beta H}$ $\qquad e^{-\beta H}O(\tau)e^{\beta H} = O(\tau+\beta)$

$$= \sum_{s'} < s'|e^{-\beta H}\psi_a(\beta,x)\psi_b^c(\beta,y)\psi_{a'}^\dagger(0,x)\psi_{b'}^{\dagger c}(0,y)|s' >$$

$$= Z(\beta) < W(x)W^\dagger(y) >$$

The order of the phase transition in pure gauge theories:

- $SU(2)$, center $Z(2)$ 2nd order transition with Ising universality class
  Engels et al,NPB 332 (90) 737

- $SU(3)$, weak 1st order transition
  Fukugita et al, PRL 63 (89) 1768

- $SU(N)$, $N > 3$ strong 1st order transition
  Lucini et al, JHEP 0401 (04) 061

- Sp(N), $N > 1$ Z(2) center but 1st order transition
  Holland et al, NPB 694 (04) 35

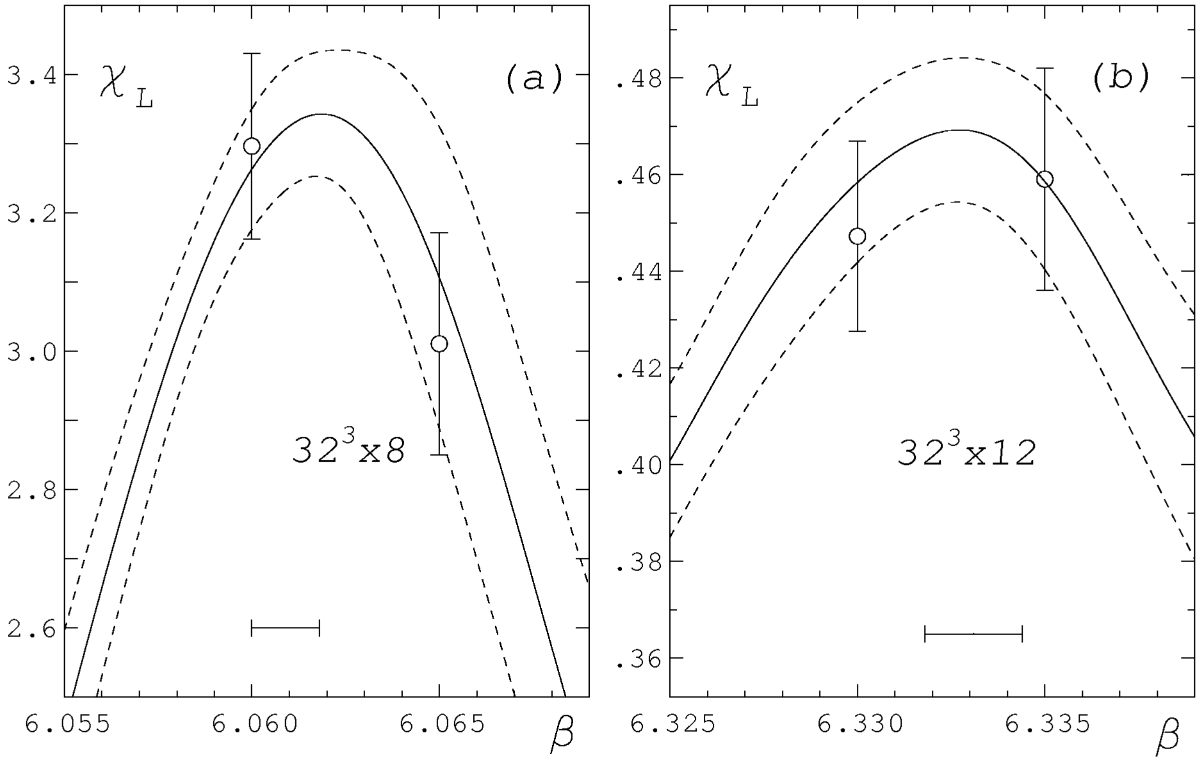
- $G(2)$ no center but 1st order transition
  Pepe, Wiese, NPB 768 (07) 21

The order of the phase transition is determined by mismatch in d.o.f : $\mathcal{O}(1)$ at low T and $\mathcal{O}(N_c^2)$ at high T

How to determine the transition temperature ?

$$\frac{\chi_L}{T^2} = N_\sigma^3 \left(\langle L^2 \rangle - \langle L \rangle^2 \right) = \langle (\delta L)^2 \rangle \text{ has a peak at } \beta_c$$

Boyd et al., Nucl. Phys. B496 (1996) 167



• Use different volumes and Ferrenberg-Swedsen re-weighting to combine information collected at different gauge couplings

Finite volume behavior can tell the order of the phase transition, e.g. for 1st order transition the peak height scales as spatial volume !

Determine the static potential at $T=0$,

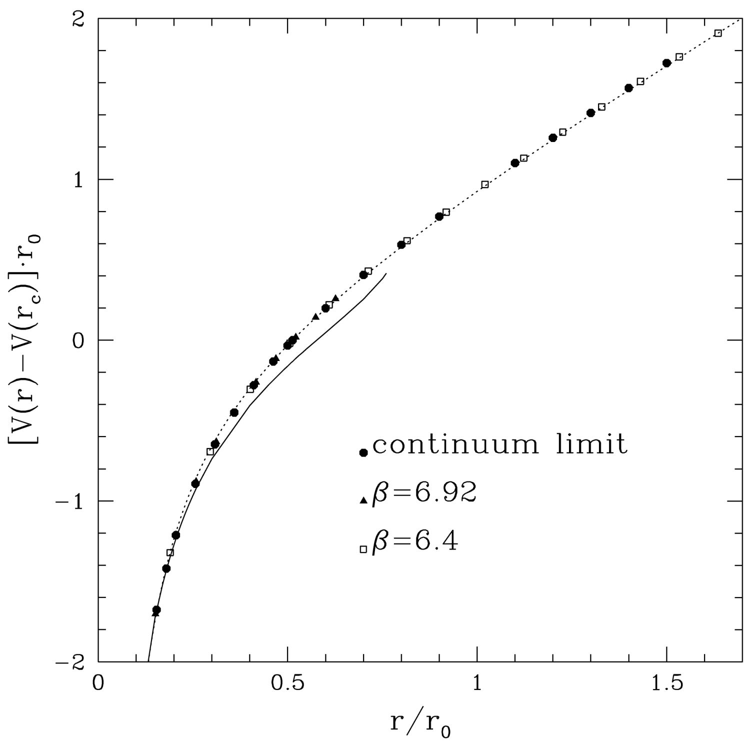
determine the lattice spacing $a(\beta_c)$ using the string tension σ or the Sommer scale:

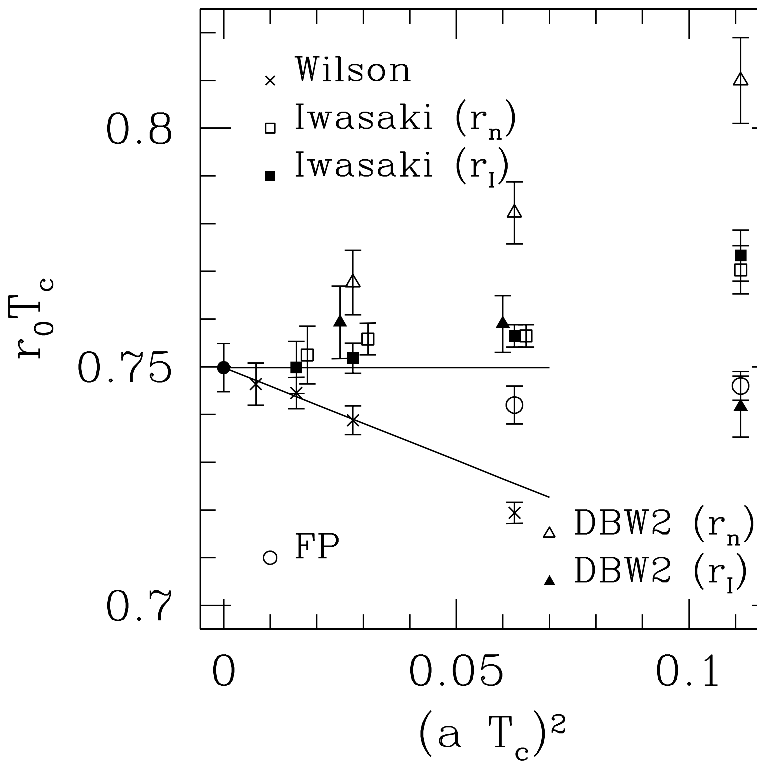
$$r^2 \frac{dV}{dr}|_{r=r_0} = 1.65$$

$$T_c = 1/(N_\tau a(\beta_c))$$

Necco, Nucl. Phys. B683 (2004) 167

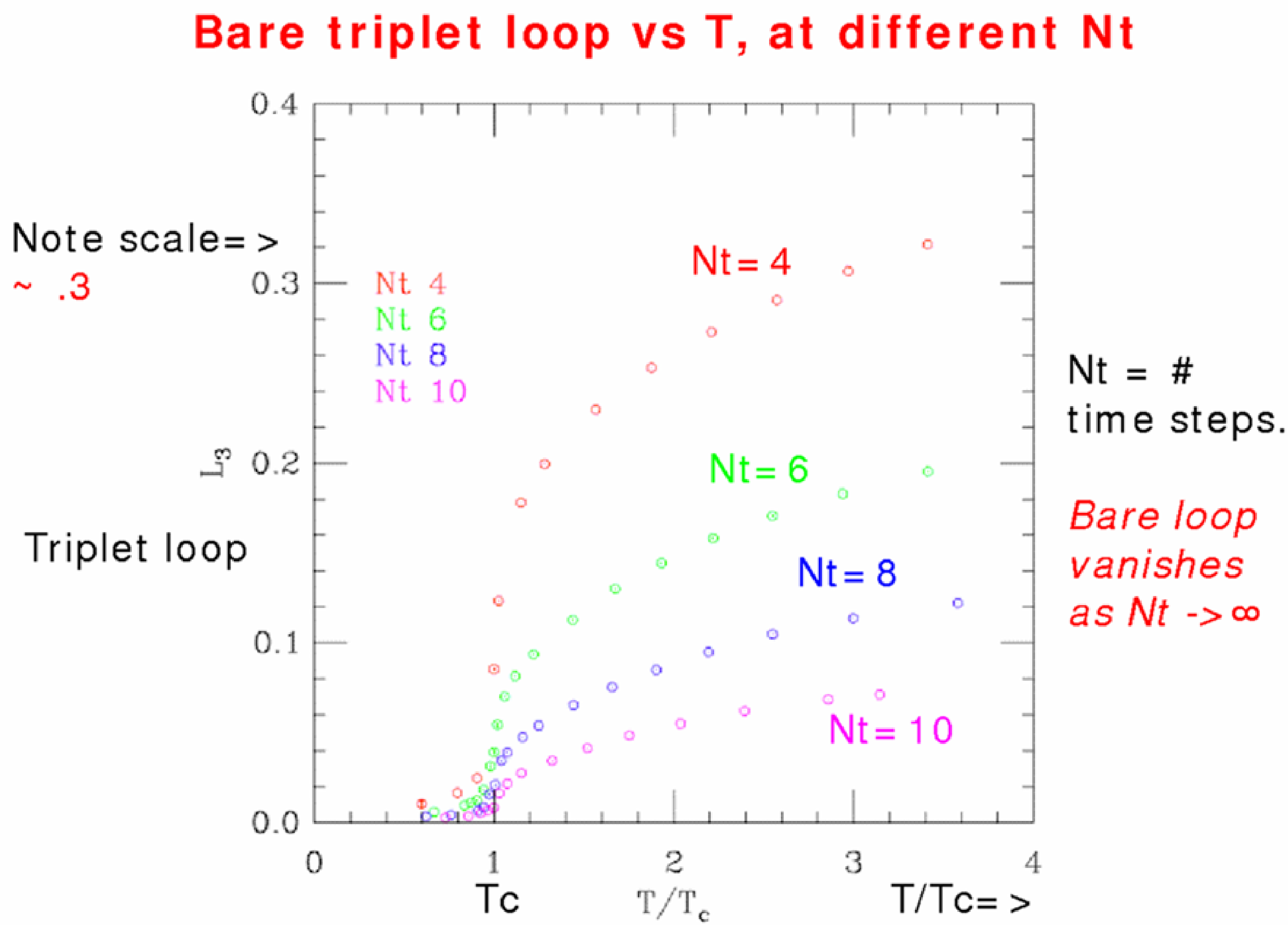
Necco, Sommer, NPB 623 (02) 271





Continuum limit for *L* ?

Dumitru et al, hep-th/0311223



needs renormalization !

Free energy of static sources in QCD

- QCD partition function in the presence of a static $Q\bar{Q}$ pair (McLerran, Svetitsky PRD 24 (1981) 450)

$$\frac{Z_{q\bar{q}}(r,T)}{Z(T)} = \frac{1}{Z(T)}\int DA_\mu D\bar{\psi}D\psi e^{-\int_0^{1/T} d\tau \int d^3x L_{QCD}(\tau,\vec{x})} W(\vec{r})W^\dagger(0)$$
$$= \langle W(\vec{r})W^\dagger(0)\rangle,$$

$$Z(T) = \int DA_\mu D\bar{\psi}D\psi e^{-\int_0^{1/T} d\tau \int d^3x L_{QCD}(\tau,\vec{x})}$$

$W(\vec{x}) = Pe^{ig\int_0^{1/T} d\tau A_0(\tau,\vec{x})} = \prod_{\tau=0}^{N_\tau-1} U_0(\tau,\vec{x})$ temporal Wilson line, $L(\vec{x}) = \mathrm{Tr}\, W(\vec{x})$ Polyakov loop.

- Different color channels

$$Z_{q\bar{q}}(r,T) = Z_{q\bar{q}}^{(1)} P_1 + Z_{q\bar{q}}^{(8)} P_8$$

$P_1 = \frac{1}{9} 1 \otimes 1 - \frac{2}{3}\bar{t}^a \otimes t^a$, $P_8 = \frac{8}{9} 1 \otimes 1 + \frac{2}{3}\bar{t}^a \otimes t^a$, $\bar{t}^a = -t^{a*}$

Brown, Weisberger, PRD 20 (79) 3239; Nadkarni, PRD 34 (86) 3904

$$\frac{Z^{(1)}_{q\bar{q}}(r,T)}{Z(T)} = \frac{1}{Z(T)}\frac{\text{Tr }(P_1 Z_{q\bar{q}})}{\text{Tr }P_1} = \frac{1}{3}\text{Tr }\langle W(\vec{r})W^{\dagger}(0)\rangle$$

$$\frac{Z^{(8)}_{q\bar{q}}(r,T)}{Z(T)} = \frac{1}{Z(T)}\frac{\text{Tr }(P_8 Z_{q\bar{q}})}{\text{Tr }P_8} = \frac{1}{8}\langle\text{Tr }W(r)\text{Tr }W^{\dagger}(0)\rangle - \frac{1}{24}\text{Tr }\langle W(r)W^{\dagger}(0)\rangle$$

$$\frac{Z^{(av)}_{q\bar{q}}(r,T)}{Z(T)} = \frac{1}{Z(T)}\frac{\text{Tr }((P_1+P_8)Z_{q\bar{q}})}{\text{Tr }(P_1+P_8)} = \frac{1}{9}\langle\text{Tr }W(r)\text{Tr }W^{\dagger}(0)\rangle$$

- The free energy, internal energy energy and the entropy

$$F_i(r,T) = -T\ln\frac{Z^{(i)}_{q\bar{q}}(r,T)}{Z(T)} = V_i(r,T) - TS_i(r,T)$$

$$V_i(r,T) = T^2\frac{\partial}{\partial T}\ln\left(\frac{Z^{(i)}_{q\bar{q}}(r,T)}{Z(T)}\right) = -T^2\frac{\partial F_i(r,T)/T}{\partial T};\ S_i(r,T) = -\frac{\partial F_i(r,T)}{\partial T}$$

$$i = 1, 8, av$$

- Color average free energy is explicitely gauge invariant

$$\frac{1}{9} < L(R)L^\dagger(0) >= e^{-F_{av}(R,T)/T} = \frac{1}{9}e^{-F_1(R,T)/T} + \frac{8}{9}e^{-F_8(R,T)/T}$$

- $W(\vec{x})$ not gauge invariant $\Rightarrow$ gauge invariant version:

$$\tilde{W}(\vec{x}) = \Omega^\dagger(\vec{x})W(\vec{x})\Omega(\vec{x}),$$

($\Omega(\vec{x})$ is an SU(3) matrix, $\Omega = f_\alpha^n \; D_\mu^2 f_\alpha^{(n)} = \lambda_n f_\alpha^{(n)} \; \tau = 0$-ra)

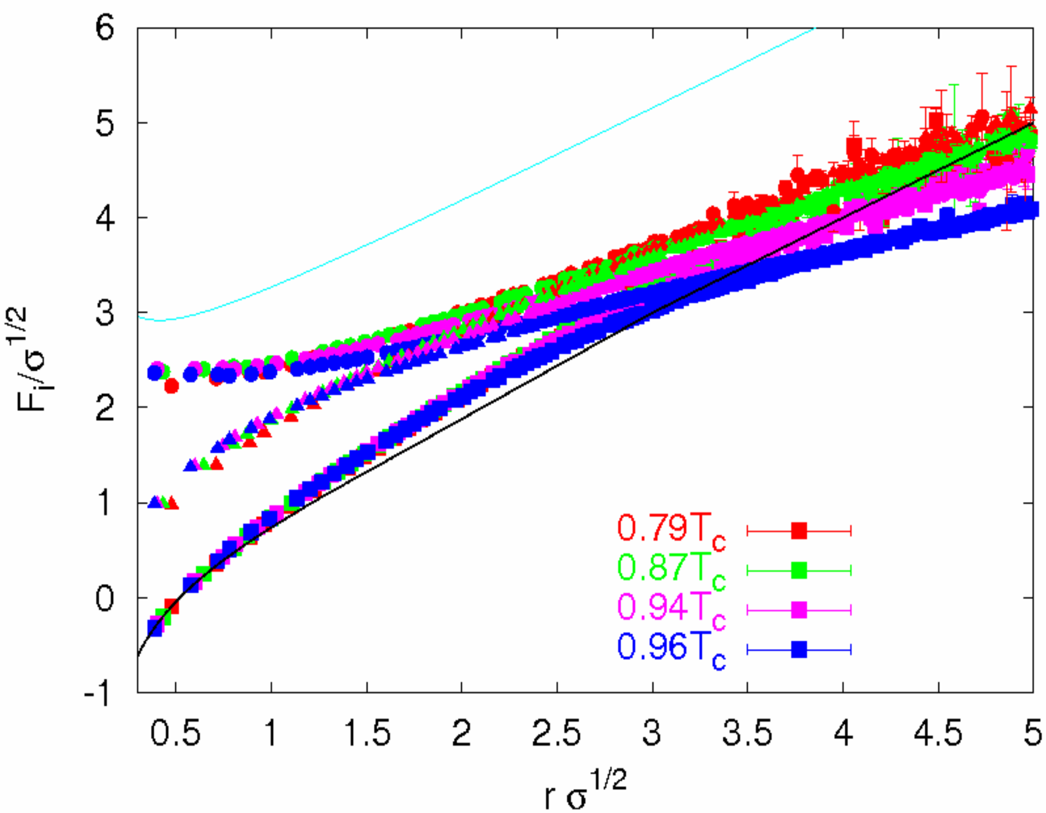
Philipsen, PLB 535 (02) 138

or fix to Coulomb gauge $\Downarrow$

in the $T = 0$ limit a transfer matrix can be defined and and both definitions are equivalent to the standard definition in terms of Wison loops .

Philipsen, PLB 535 (02) 138

Numerical results in SU(3) gauge theory at $T < T_c$



- 1) F_1 is T-ndependent for $r\sqrt{\sigma} < 1$ ($< 0.5fm$)

  2) the same $\sigma(T)r$ behavior of F_1, F_8, F_{av} for $r\sqrt{\sigma} > 3$ ($> 1.5fm$)

Free energy in the pertubative high temperature limit

$$\exp(-F_1(r,T)/T) = \frac{1}{3}\mathrm{Tr}\langle W(\vec{r})W^\dagger(0)\rangle, \quad W \simeq 1 + igA_0/T$$

$$\Rightarrow \quad F_1(r,T) = -g^2 C_F \frac{e^{-m_D r}}{4\pi r} = U_1(r,T), \quad C_F = \frac{N^2-1}{2N} \qquad m_D = gT$$

At leading order: $F_8(r,T)/F_1(r,T) = -1/8$ $\quad$ $S_1(r,T) = 0$

At next to leading order: $F_1(r,T) = -g^2 C_F \frac{e^{-m_D r}}{4\pi r} - \frac{C_F m_D g^2}{4\pi}$

$$F_1(r=\infty,T) = F_\infty(T) \neq 0$$

and the entropy appears: $S_1(r,T) = \frac{C_F g^2 m_{D0}}{4\pi T}(1 - e^{-m_{D0} r}) \sim \mathcal{O}(g^3)$

The internal energy is different from the free energy

$$U_1(r,T) = F_1(r,T) + TS_1(r,T) = -g^2 C_F \frac{e^{-m_{D0} r}}{4\pi r} - \frac{C_F g^2 m_{D0}}{4\pi} e^{-m_{D0} r}$$

$$U_1(r=\infty,T) = U_\infty(T) = 0$$

The work to be done in order to bring the static quark anti-quark pair separated by distance r_1 to distance r_2 is determined by color averaged free energy:

$$A = F_{av}(r_2) - F_{av}(r_1)$$

In leading order perturbation theory:

$$F_{av}(r,T) = -\frac{1}{9}\frac{\alpha_s^2}{r^2 T}\exp(-2m_D r)$$

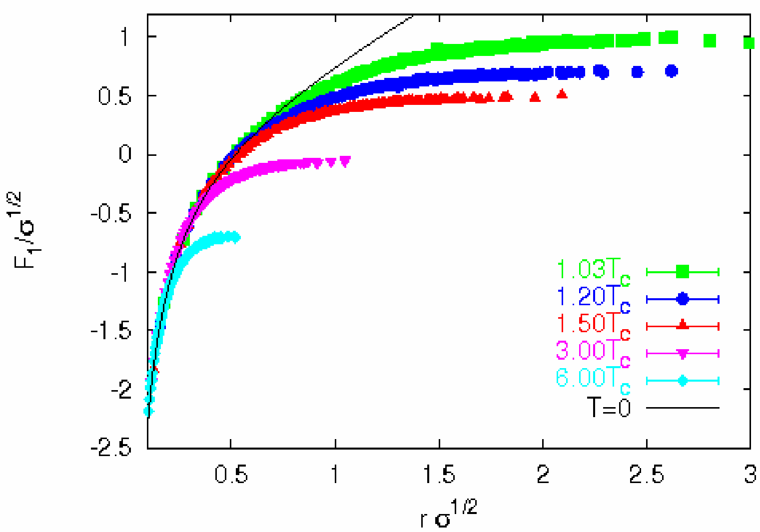
In QED

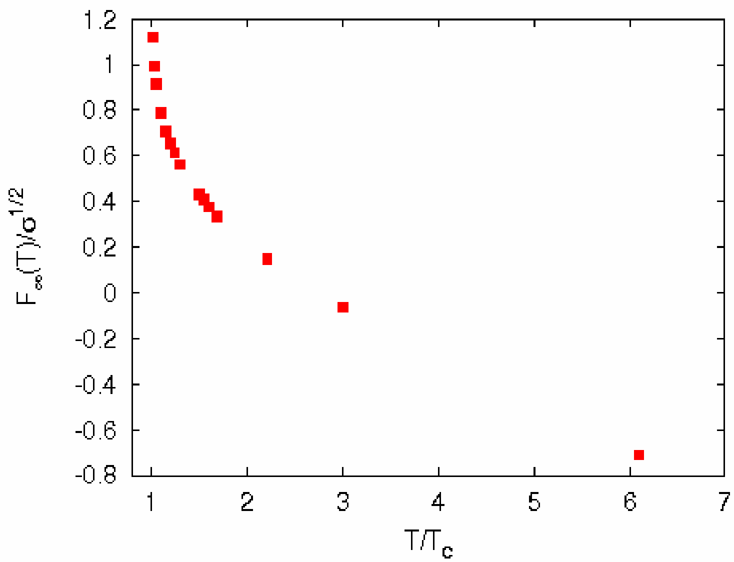
$$F_{av}(r,T) = -\frac{\alpha}{r}\exp(-m_D r)$$

In QCD the work is reduced due to cancelation between color singlet and octet contribution

Numeerical results for $T > T_c$

- The free energy and the entropy contribution





$$F_1(r\to\infty,T) = F_8(r\to\infty,T) = F_{av}(r\to\infty,T) = F_\infty(T)$$

$F_\infty(T)$ is decreasing with $T \Rightarrow$

$$-\frac{\partial F_\infty(T)}{\partial T} \equiv S_\infty(T) > 0$$

- Long vs. short distance physics, screening

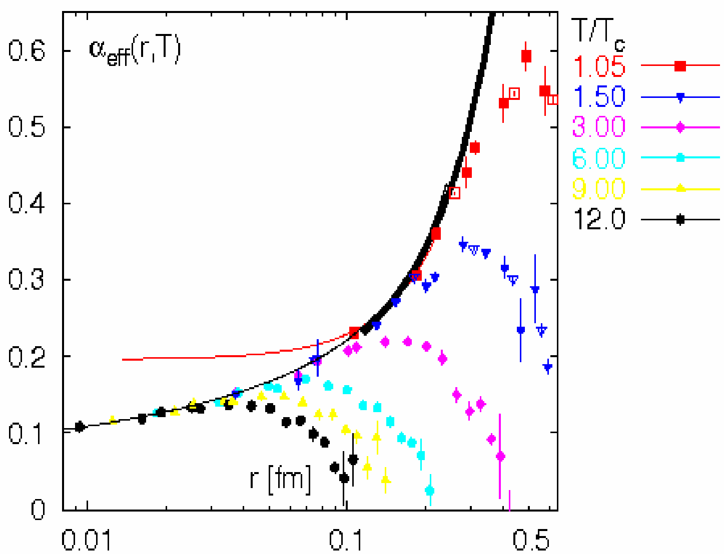
No temperature dependence in

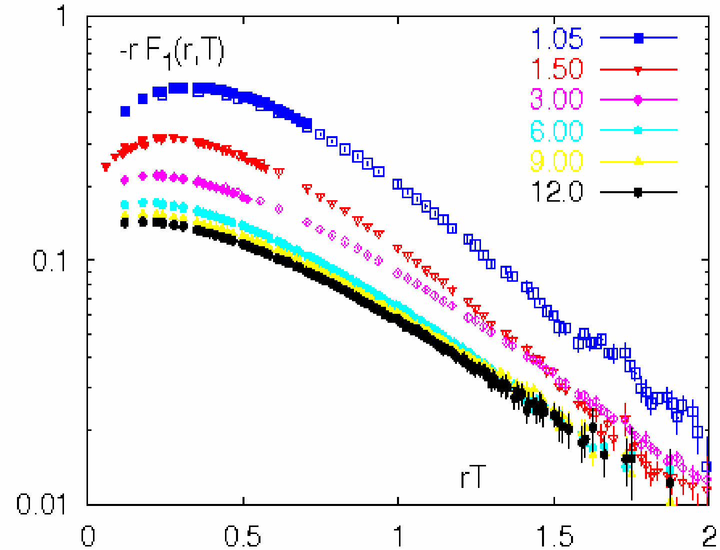
$$\alpha_{eff}(r,T) = \frac{3}{4}\frac{dF_1(r,T)}{dr}$$

$\alpha_{eff}(r_{med},T) = \alpha_{eff}^{max} \rightarrow r_{med}(T) = 0.5 fm \cdot T_c/T$ which separates the short and long distance physics.

$r < r_{med}(T)$ exponential screening for $r < r_{med}(T)$

remnants of $T = 0$ non-perturbative physics in the deconfined phase close to T_c



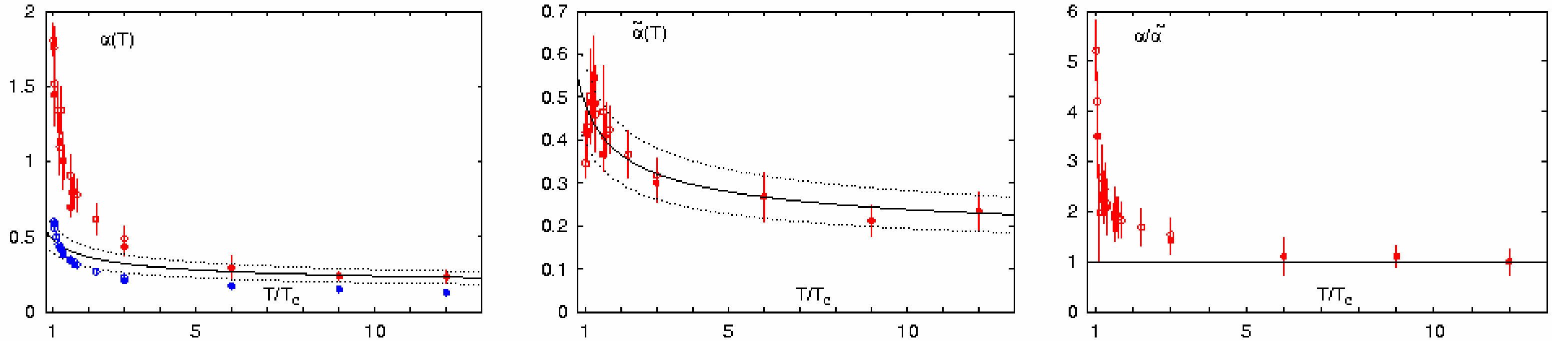


- Screening at coupling at large distances

$$F_1(r,T) = -\frac{4}{3}\frac{\alpha(T)}{r}\exp(-\sqrt{4\pi\tilde{\alpha}(T)}r)$$

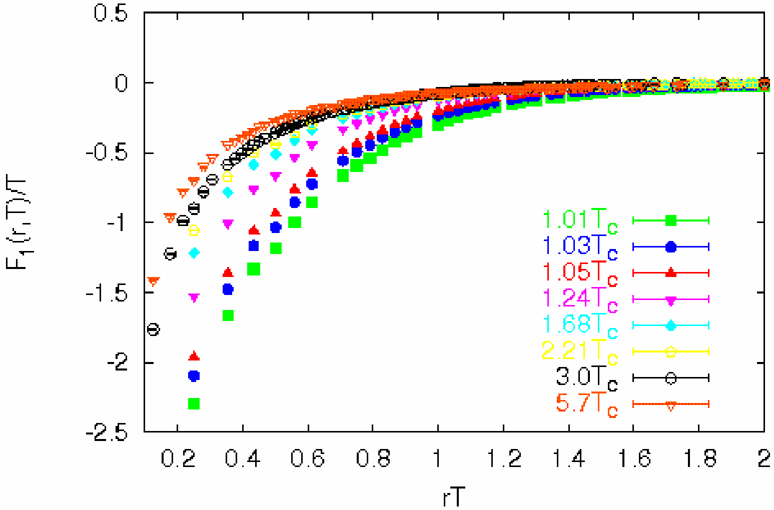
LO expectations $\alpha(T) = \tilde{\alpha}(T)$ holds for $T \geq 6T_c$

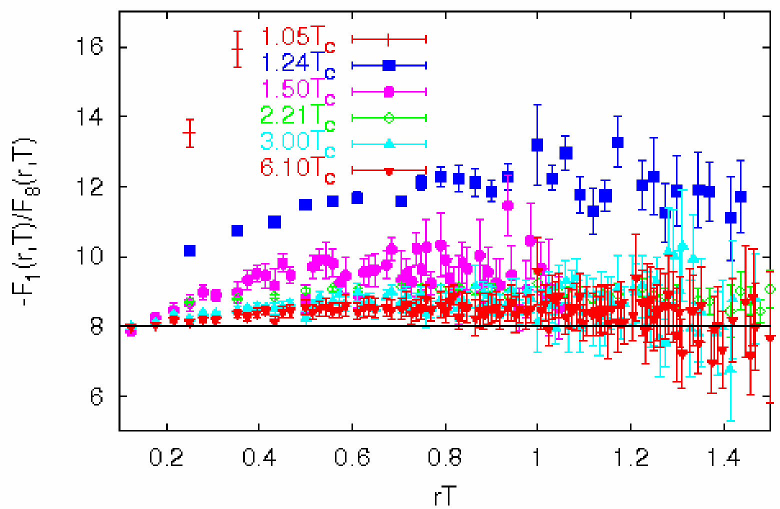
$\alpha(T) = 2.17(7)\alpha_{\bar{MS}}(\mu = 2\pi T)$



- High temperature limit:

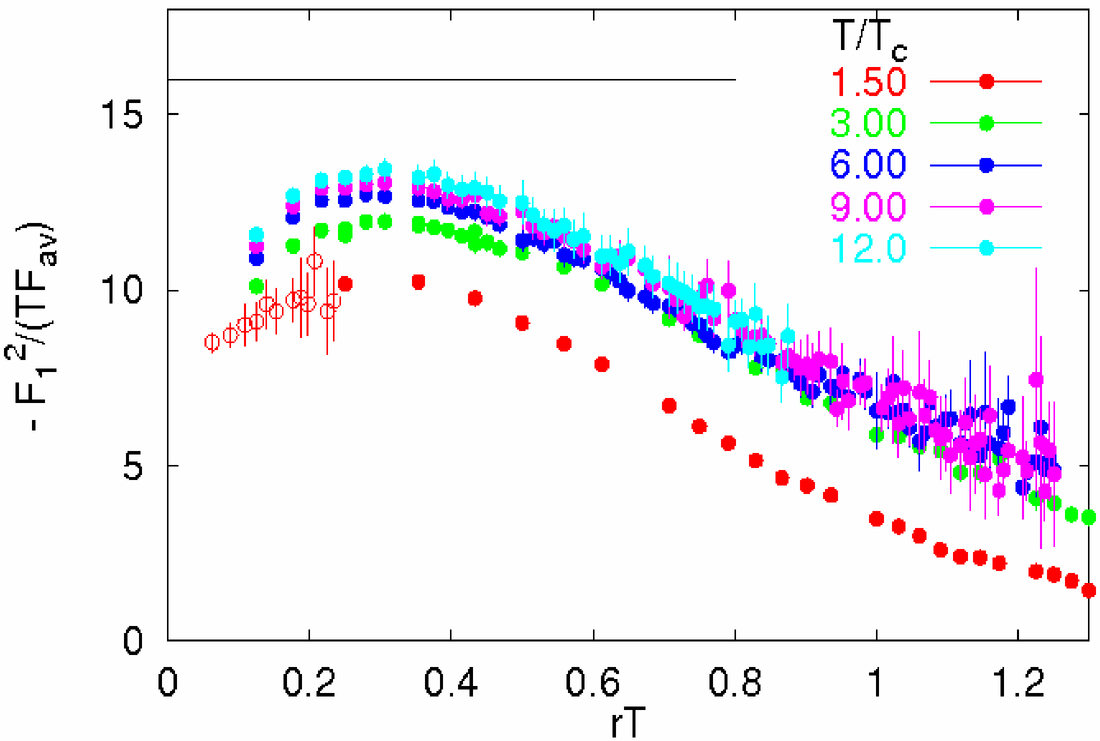
conventional normalization : $F_1 = F_8 = F_{av} = 0, r = \infty$





Leading order expansion ($rT \gg 1$, $g \ll 1$): $-F_1(r,T)/F_8(r,T) = 8$, $-F_1^2/(TF_{av}) = 16$

$$F_1 = -\frac{4}{3}\frac{\alpha_s(T)}{r}\exp(-m_{D0}r), F_{av} = -\frac{1}{9}\frac{\alpha_s^2}{r^2 T}\exp(-m_{D0}r)$$



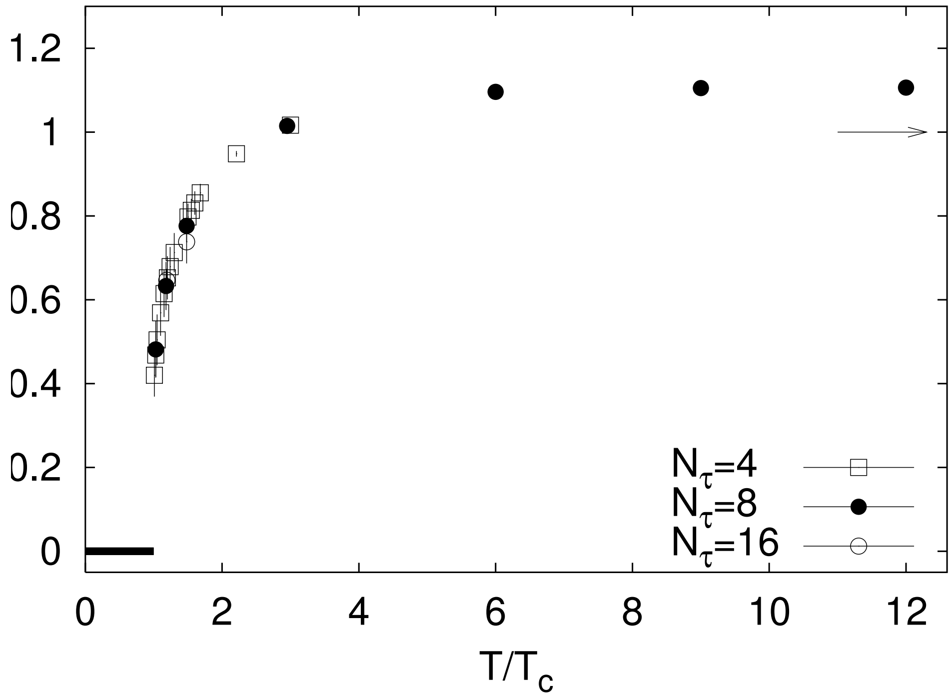
$-F_1^2/(TF_{av}) \sim const \neq 16$, $rT < 0.6$

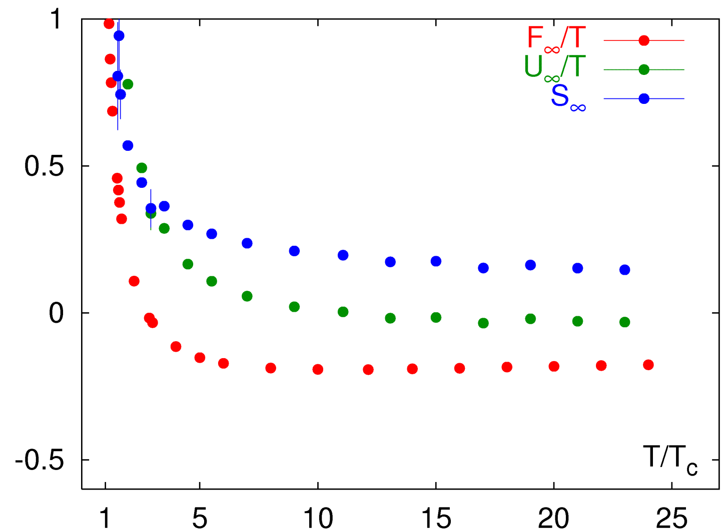
$rT > 0.6$ non-perturbative behavior

The renormalized Polyakov loop

Kaczmarek et al, PLB 543 (02) 41, PRD 70 (04) 074505, hep-lat/0309121

$$L_{ren} = \exp(-F_\infty(T)/(2T))$$





LO: $F_\infty \simeq -TS_\infty, \quad S_\infty = -\frac{4}{3}\alpha_s \frac{m_D}{T}$

Correlation length near the transition

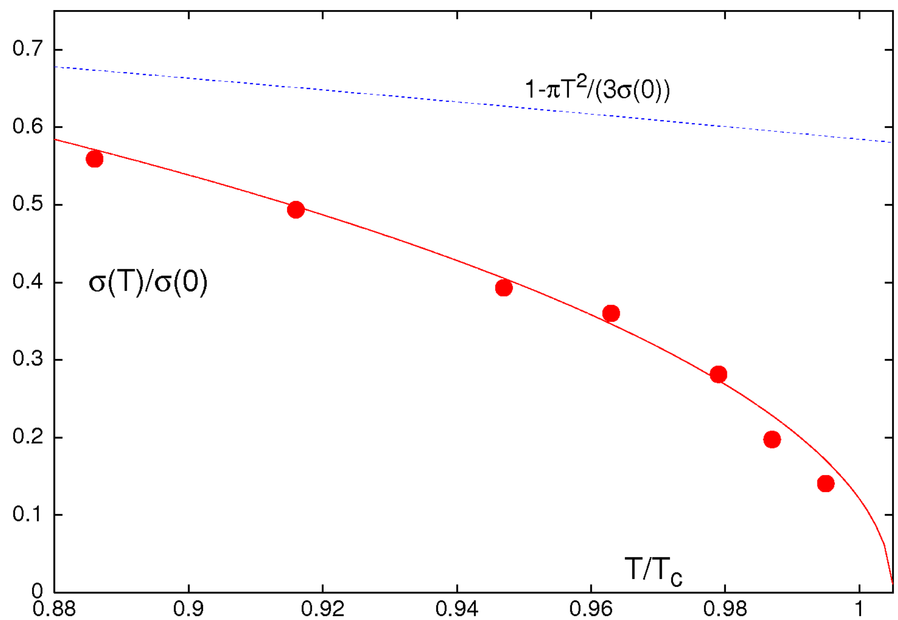
$T < T_c :$

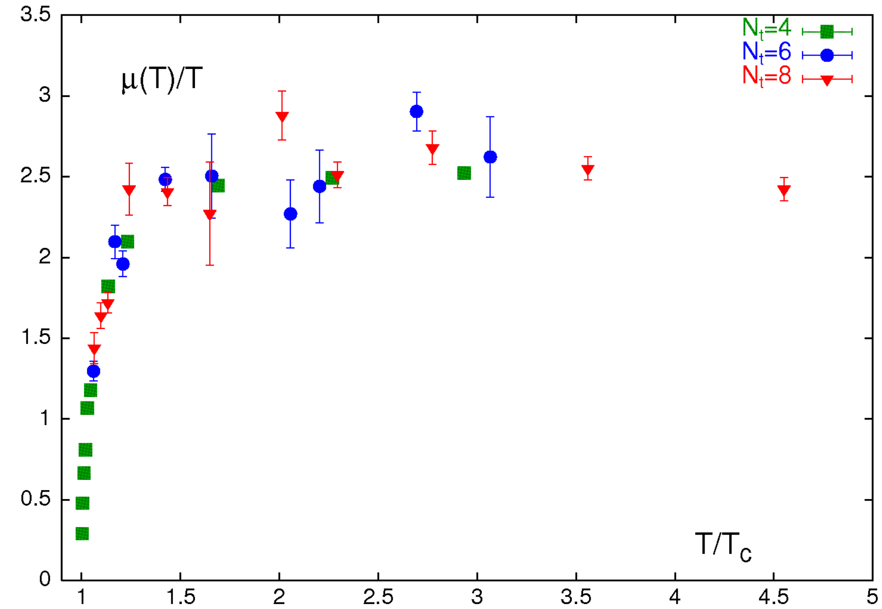
$$< L(r)L^\dagger(0) > \sim e^{-\sigma(T)r/T}$$

$T > T_c :$

$$\ln\left(\frac{< L(r)L^\dagger(0) >}{|< L >|^2}\right) \sim e^{-\mu(T)r}$$

Kaczmarek, Phys.Rev.D62 (00) 034021



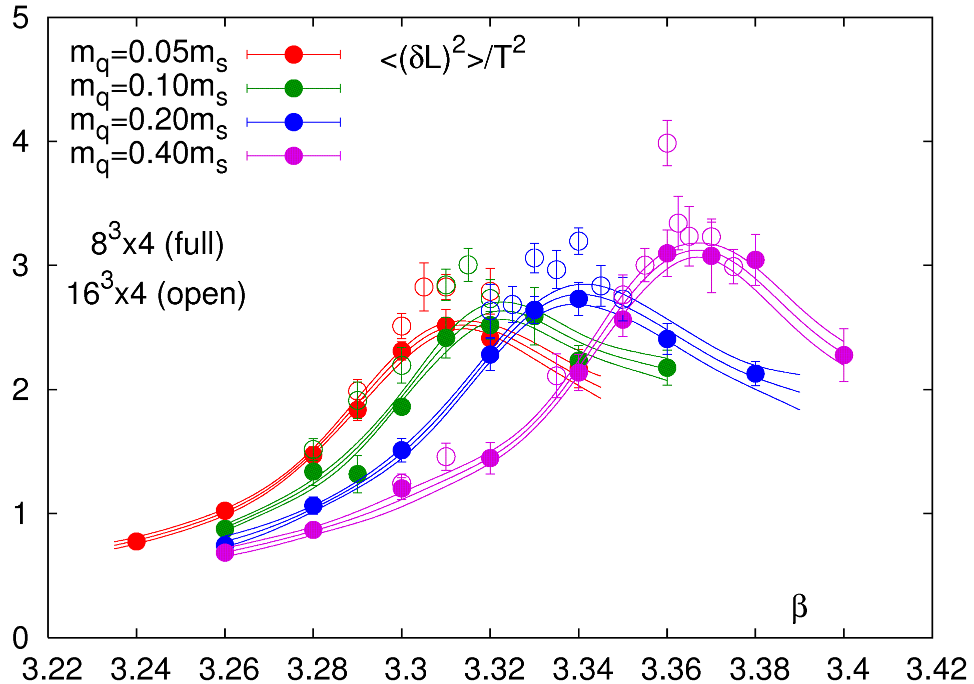


small inverse correlation length => weak 1st order phase transition
QCD is far from the large N-limit !

Deconfinement transition in QCD

Dynamical quarks break Z(3) symmetry => no phase transition

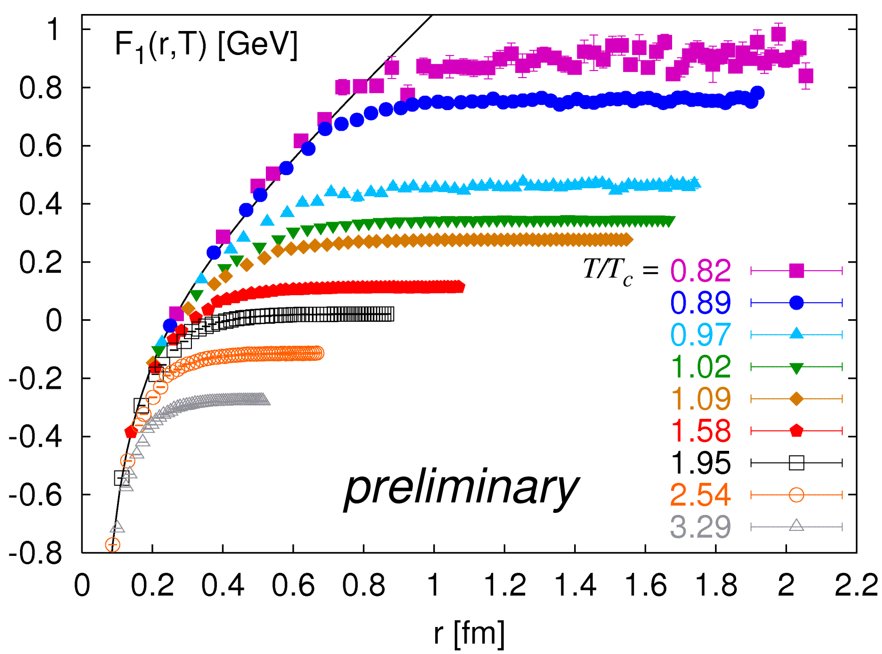
$$\frac{\chi_L}{T^2} = N_\sigma^3 \left(\langle L^2 \rangle - \langle L \rangle^2 \right) = \langle (\delta L)^2 \rangle$$

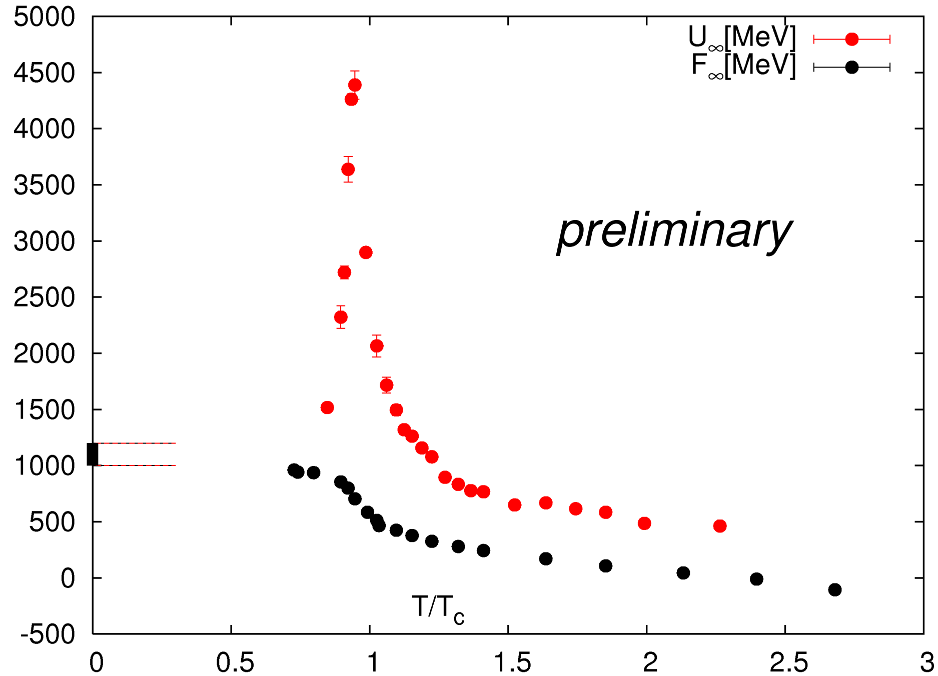


Static quark anti-quark free energy in 2+1f QCD

RBC-Bielefeld Collaboration:
M. Cheng, N.H. Christ, S. Eijiri, K. Hübner, C. Jung, F., O. Kaczmarek, F. Karsch,
E. Laermann, J. Liddle, R. Mawhinney, C. Miao, P. Petreczky, K. Petrov, C. Schmidt,
W. Söldner, J. Van der Heide

$16^3 \times 4$, $24^3 \times 6$ lattices, $m_\pi \simeq 200$ MeV





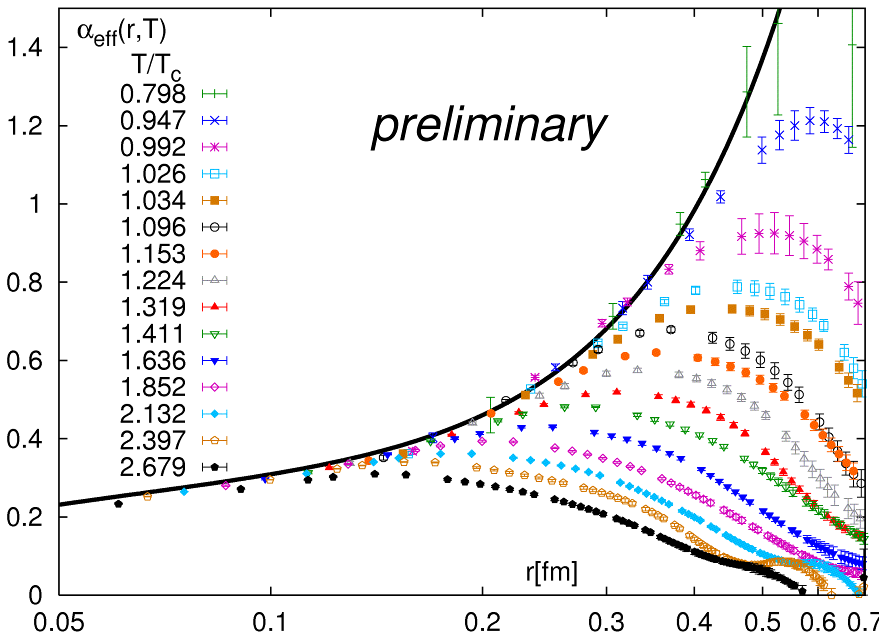
Strong temperature dependence of the static quark anti-quark correlators

Both $F_\infty(T)$ and $U_\infty(T)$ decrease for $T > 1.1 T_c$

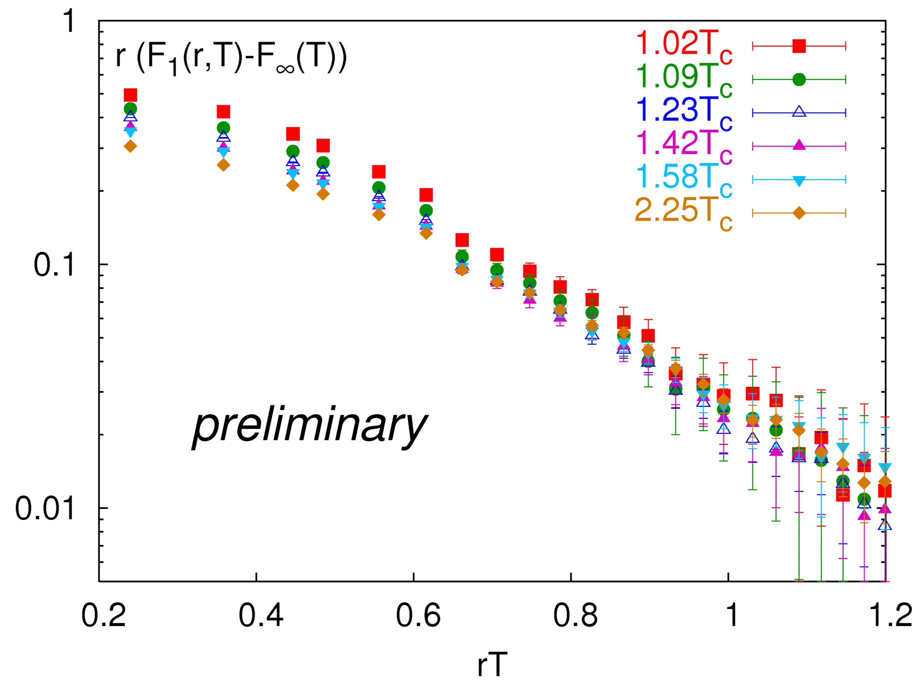
Static quark anti-quark free energy in 2+1f QCD

$$\alpha_{eff}(r,T) = \frac{3r^2}{4}\frac{dF_1(r,T)}{dr}$$

$$F_1(r \gg 1/T) = -\frac{4}{3}\alpha_s \frac{e^{-m_D r}}{r}$$

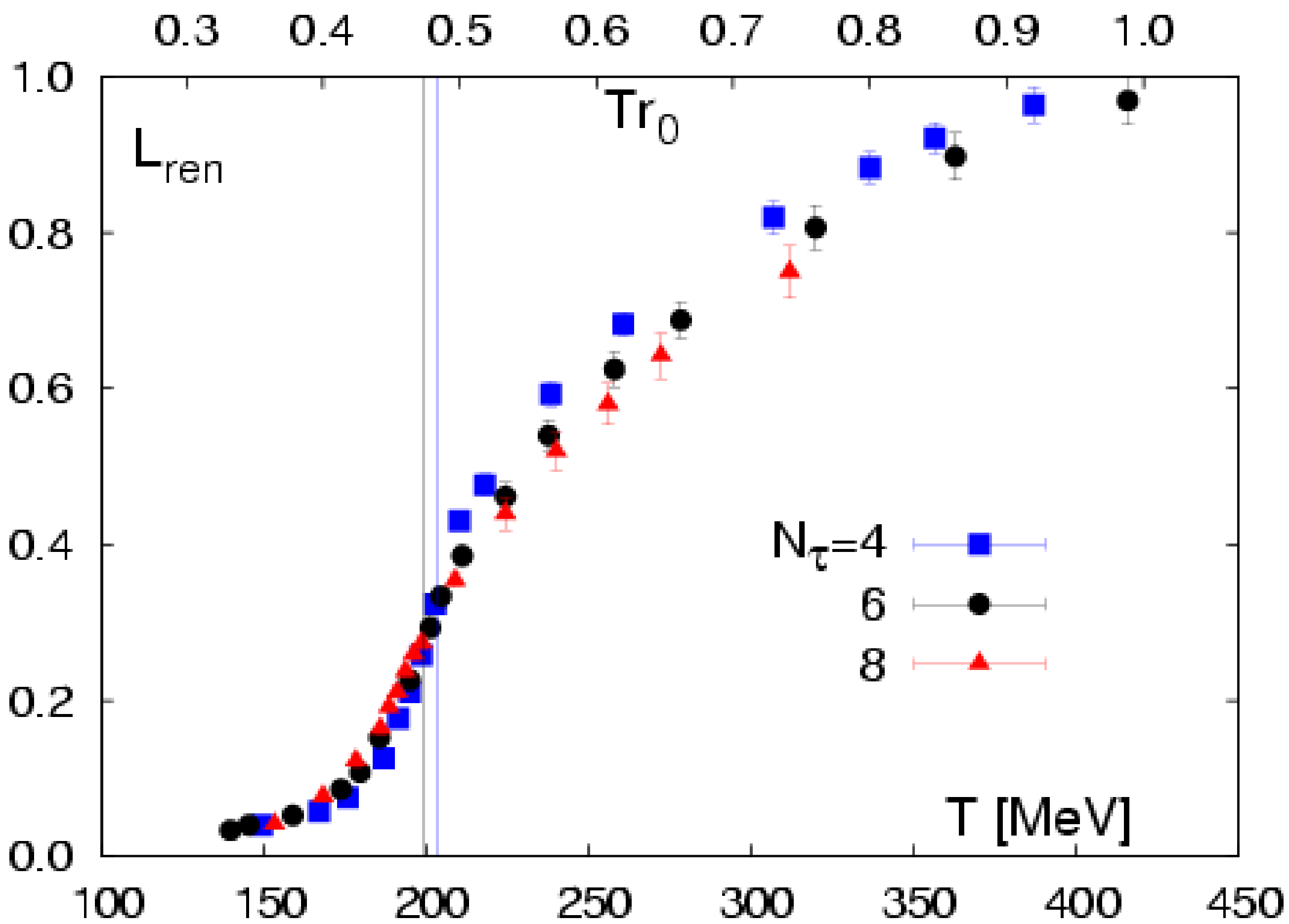


$\alpha_{eff}(r,T)$ and $F_1(r,T)$ are temperature independent for $r < \frac{0.4\text{fm}}{T/T_c}$



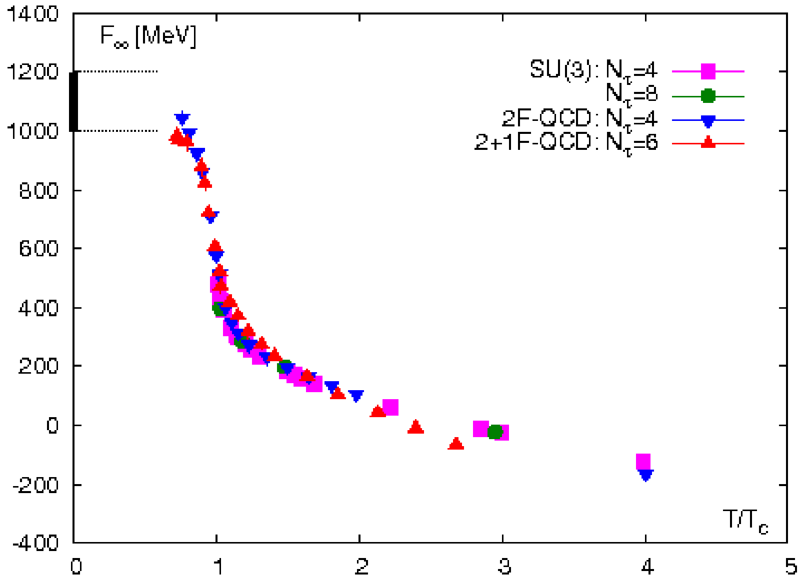
$F_1(r,T)$ scales with T and is exponentially screened for $r > 0.8/T$

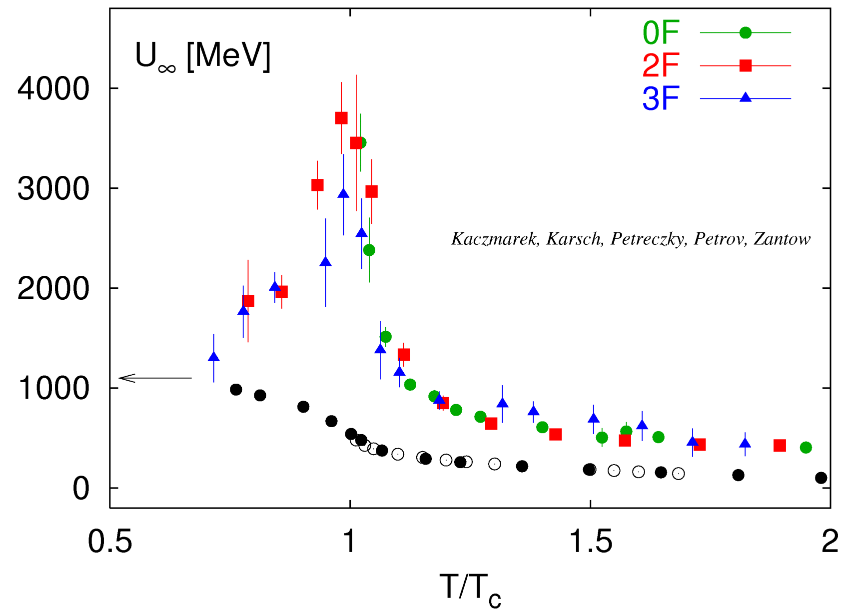
Renormalized Polyakov loop in 2+1F QCD

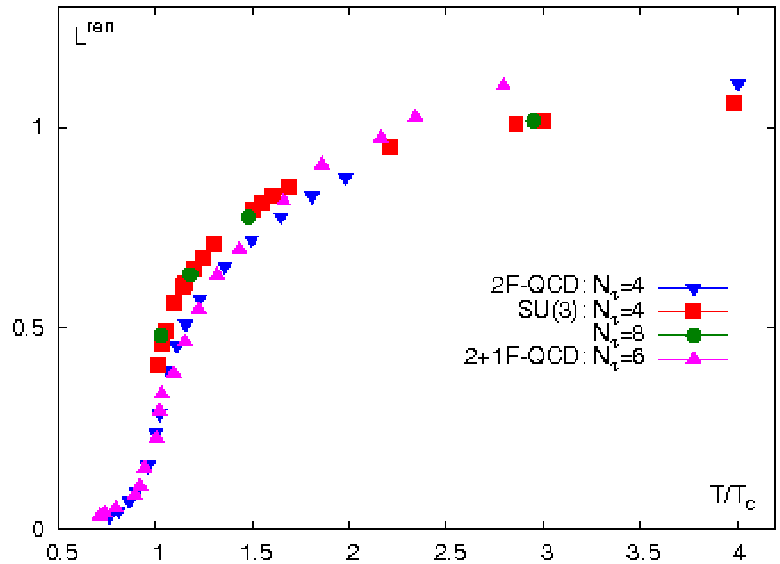


see talk by Karsch at Lattice 2007,
data for temporal extent=8 are from HotQCD Collaboration

How different is full QCD from SU(3) gauge theory ?

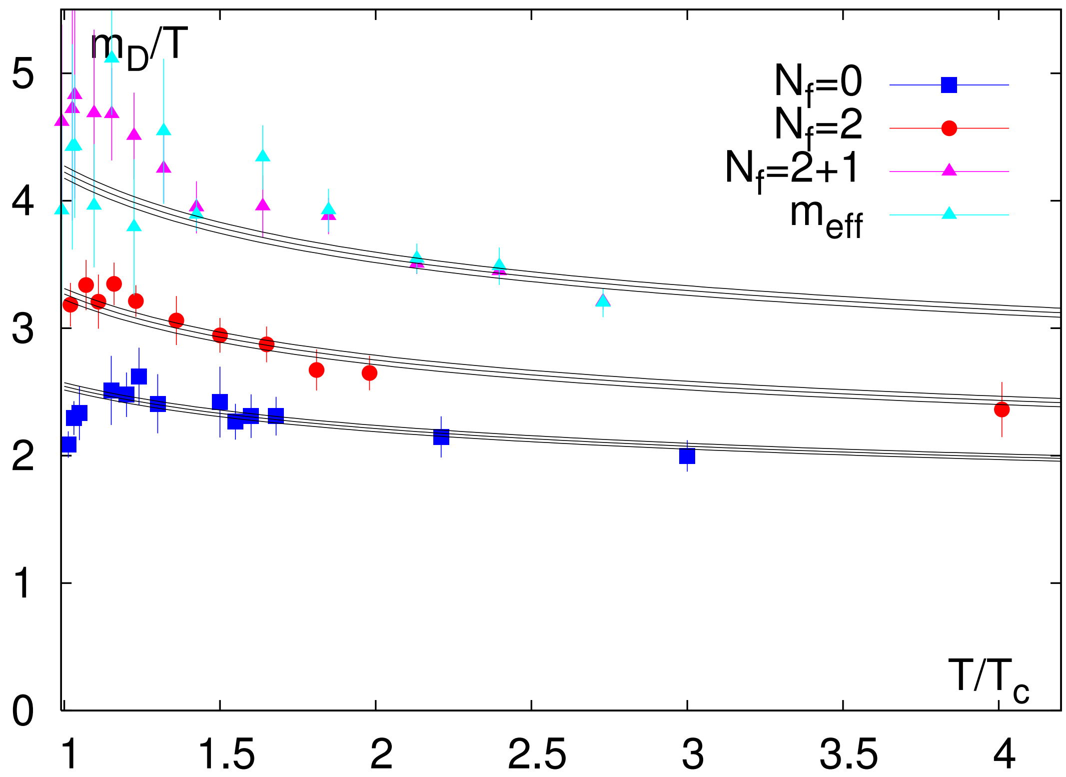






Quark flavor dependence in $F_\infty(T)$, $U_\infty(T)$ and $L_{ren}(T)$ is small when ploted as function of T/T_c

Screening mass in 2+1F QCD



$$m_D(T) = A\sqrt{1+\frac{N_f}{6}}g^{\text{2-loop}}(T)T$$

$$m_D \simeq 1.4 m_D^{LO}$$

quarks modify the perturbative pre-factors, non-perturbative effects are in the soft gluon sector

Homework:

Using the definition

$$\exp(-F_1(r,T)/T) = \frac{1}{3}\mathrm{Tr}\langle W(\vec{r})W^\dagger(0)\rangle, \quad \text{and} \quad W \simeq 1 + igA_0/T$$

show that $F_1 = -\frac{4}{3}\frac{\alpha_s}{r}\exp(-m_D r)$ at leading order (hint : use a gauge where A_0 is time independent and the re-summed gluon propagator $D_{00}^{ab}(k) = \delta_{ab}/(\vec{k}^2 + m_D^2)$)

Using the above result for F_1 and the leading order relation $F_8/F_1 = -1/8$ as well as the relation

$$\exp(-F_{av}/T) = \frac{1}{9}\exp(-F_1/T) + \frac{8}{9}\exp(-F_8/T)$$

show that $F_{av} = -\frac{1}{9}\frac{\alpha_s^2}{r^2 T}\exp(-2m_D r)$ (hint : expand the exponent to second order)

Arrive at the above result for F_{av} starting from its definition $\exp(-F_{av}/T) = \frac{1}{9} < \mathrm{Tr}W(r)\mathrm{Tr}W^\dagger(0) >$ and using the perturbative expansion for the Wilson line W. (hint : use the same steps as in the redivation of F_1 above)