QCD at finite temperature and density

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1) Introduction : quantum statistical mechanics, scalar field theory at T>0, high temperature QCD

2) The deconfinement transition and color screening

3) Chiral transition, cutoff effects and improved actions, bulk thermodynamic quantities, transition temperature in QCD

4) Correlation functions and spectral functions at finite T

5) QCD at finite baryon density

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Quantum Statistical mechanics

Transition amplitude in QM and its path integral represenation

$$F(q', t'; q, t) = \langle q'|e^{-i\hat{H}(t'-t)}|q\rangle$$

$t \to -i\tau, \ t' \to -i\tau$ (imaginary time)

$$F(q' - i\tau'; q, -i\tau) = \langle q'|e^{-\hat{H}(\tau'-\tau)}|q\rangle$$

$$\hat{H} = \frac{1}{2}p^2 + V(q)$$

$$F(q, -i\tau'; q, -i\tau) = \int \mathcal{D}q \exp\left[-\int_{\tau}^{\tau'} d\tau'' \left(\frac{1}{2}\dot{q}^2(\tau'') + V(q(\tau''))\right)\right]$$

$$q(\tau) = q, \ q(\tau') = q'$$

Partition function in statistical mechanics:

$$Z(\beta) = \mathrm{Tr} e^{-\beta\hat{H}}, \ \beta = 1/T$$

$$Z(\beta) = \sum e^{-\beta E_n}, \quad \hat{H}|n\rangle = E_n|n\rangle$$

$$Z(\beta) = \int dq \langle q | e^{-\beta \hat{H}} | q \rangle$$

$$Z(\beta) = \int dq F(q, -i\beta; q, 0)$$

$$\Downarrow$$

$$Z(\beta) = \int \mathcal{D}q(\tau) \exp \left[- \int_0^\beta d\tau \left(\frac{1}{2} \dot{q}^2(\tau) + V(q(\tau)) \right) \right],$$

$$q(\beta) = q(0)$$

Euclidean action $S_E(\beta) = \int_0^\beta d\tau \left(\frac{1}{2} \dot{q}^2(\tau) + V(q(\tau)) \right)$

We can also calculate the generating functional

$$Z(\beta; j) = \int \mathcal{D}q \exp \left[-S_E(\beta) + \int_0^\beta j(\tau) q(\tau) d\tau \right]$$

$$\Delta(\tau_1, \tau_2) = \frac{1}{Z(\beta)} \frac{\delta^2 Z(\beta; j)}{\delta j(\tau_1) \delta j(\tau_2)} |_{j=0} = \frac{1}{Z(\beta)} \int \mathcal{D}q q(\tau_1) q(\tau_2) e^{-S_E(\beta)}$$

Correlation function in real and imaginary time in the operator formalism:

$$\hat{q}(-i\tau) = e^{\hat{H}\tau}\hat{q}e^{-\hat{H}\tau}$$
$$\hat{q}(t) = e^{i\hat{H}t}\hat{q}e^{-i\hat{H}t}$$

$$\Delta(\tau_1,\tau_2) = \langle T\hat{q}(-i\tau_1)\hat{q}(-i\tau_2)\rangle_\beta = \frac{1}{Z(\beta)}\mathrm{Tr}\left[T\hat{q}(-i\tau_1)q(-i\tau_2)\right]$$

$$\Delta(\tau) = \Delta(\tau,0) = \Delta(\tau-\beta)$$

$$D^>(t,t') = \langle\hat{q}(t)\hat{q}(t')\rangle_\beta$$
$$D^<(t,t') = \langle\hat{q}(t')\hat{q}(t)\rangle_\beta$$
$$D_R(t,t') = \langle\theta(t-t')[\hat{q}(t),\hat{q}(t')]\rangle_\beta$$

$$e^{-\beta\hat{H}}\hat{q}(t)e^{\beta\hat{H}} = \hat{q}(t+i\beta) \Rightarrow D^>(t,t') = D^<(t+i\beta,t')$$

Kubo-Martin-Schwinger (KMS) condition

$$\boxed{\Delta(\tau) = D^>(-i\tau,0)}$$

Different correlation functions $\Delta(\tau)$, $D^>(t)$, $D^<(t)$ and $D_R(t)$ are related to the spectral function $\sigma(k_0)$

$$D^>(k_0) = \int_{-\infty}^{\infty} dt e^{ik_0 t} D^>(t)$$

$$D^<(k_0) = \int_{-\infty}^{\infty} dt e^{ik_0 t} D^<(t) = \int_{-\infty}^{\infty} dt e^{ik_0 t} D^>(t-i\beta) = e^{-\beta k_0} D^>(k_0)$$

$$\boxed{\sigma(k_0) = \frac{D^>(k_0) - D^<(k_0)}{2\pi} = \frac{1}{\pi} Im D_R(k_0)}$$

$$\Downarrow$$

$$D^>(k_0) = (1 + f(k_0))\sigma(k_0),\ \ f(k_0) = (e^{-\beta k_0} - 1)^{-1}$$

$$\Downarrow$$

$$\boxed{\Delta(\tau) = \int_0^{\infty} dk_0 \sigma(k_0) \frac{\cosh(k_0 \cdot (\tau - \beta/2))}{\sinh(\beta k_0/2)}}$$

$$\sigma(k_0) = \frac{1}{Z(\beta)} \sum_{n,m} e^{-\beta E_n} \left[\delta(k_0 + E_n - E_m) - \delta(k_0 + E_m - E_n)\right] |\langle n|\hat{q}|m\rangle|^2$$

$$\sigma(k_0) = -\sigma(-k_0), \quad sgn(k_0)\sigma(k_0) > 0$$

Thermodynamics of scalar field theory

Straightforward generalization to infinite number of degrees of freedom $q(t) \rightarrow \phi_x(t) \equiv \phi(t,x)$

$$L = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

$$\Downarrow$$

$$S_E(\beta) = \int_0^\beta d\tau \int d^3x \left(\frac{1}{2}(\partial_\tau \phi)^2 + \frac{1}{2}(\partial_i \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right)$$

$$Z(\beta; j) = \int \mathcal{D}\phi \exp(-S_E(\beta) + \int_0^\beta d\tau \int d^3x j(\tau,x)\phi(\tau,x))$$

$$\phi(0,x) = \phi(\beta,x)$$

Free field limit ($\lambda = 0$):

$$Z(\beta; j) = \int \mathcal{D}\phi \exp\left[-\int d^4x_E \frac{1}{2}\phi(-\partial_\tau^2 - \nabla^2 + m^2)\phi + \int_0^\beta d^4x_E j(x_E)\phi(x_E)\right]$$

$$x_E = (\tau, x)$$

Gaussian integration:

$$Z_0(\beta; j) = Z(\beta) \exp\left[\int_0^\beta d^4x_E dy_E \; j(x_E)\Delta_0(x_E - y_E)j(y_E)\right]$$

$$Z(\beta) = (det\Delta_0)^{1/2} = \mathrm{Tr}\ln\Delta_0$$

$$\left[-\partial_\tau^2 - \nabla^2 + m^2\right]\Delta_0(x_E - y_E) = \delta(\tau_x - \tau_y)\delta(x - y)$$

$$\Downarrow$$

$$(\omega_n^2 + k^2 + m^2)\Delta_0(i\omega_n, k) = (\omega_n^2 + \omega_k^2)\Delta_0(i\omega_n, k) = 1$$

$$\omega_n = 2\pi T n, \;\; \omega_k^2 = k^2 + m^2$$

$$\Downarrow$$

$$\Delta_0(i\omega_n, k) = \frac{1}{\omega_n^2 + \omega_k^2}$$ -Matsubara propagator

Mixed (Saclay) representation:

$$\Delta_0(\tau, k) = T\sum_n e^{-i\omega_n\tau}\Delta_0(i\omega_n, k)$$

$$[-\partial_\tau^2 + \omega_k^2]\Delta_0(\tau, k) = \delta(\tau_x - \tau_y), \;\; \Delta_0(\tau - \beta) = \Delta(\tau)$$

$$\rightarrow \Delta_0(\tau) = \tfrac{1}{2\omega_k}((1 + f(\omega_k))e^{-\omega_k\tau} + f(\omega_k)e^{\omega_k\tau}), f(\omega_k) = (e^{\beta\omega_k} + 1)^{-1}$$

$$\ln Z(\beta) = \frac{1}{2}\mathrm{Tr}\ln\Delta_0 = \frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_k \ln\beta^2\Delta(i\omega_n, k) =$$

$$-\frac{1}{2}\sum_{n=-\infty}^{\infty} V\int\frac{d^3k}{(2\pi)^3}\ln\beta^2[\omega_n^2+\omega_k^2] =$$

$$\sum_n \frac{d}{d\omega_k^2}\ln[\omega_n^2+\omega_k^2] = \sum_n\frac{1}{\omega_n^2+\omega_k^2} = \beta\Delta_0(\tau=0,k) = \frac{\beta}{2\omega_k}(1+2f(\omega_k))$$

$$\Downarrow$$

$$\sum_n \ln\beta^2(\omega_n^2+\omega_k^2) = \beta\omega + 2\ln(1+e^{-\beta\omega_k}) + const$$

$$\ln Z(\beta) = -V\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{2}\beta\omega_k + \ln(1-e^{-\beta\omega_k})\right]$$

$$F(T,V) = T\ln Z(\beta),\ p = -\partial F(T,V)/\partial V,\ S = -\frac{\partial F(T,V)}{\partial T}, U = F - TS$$

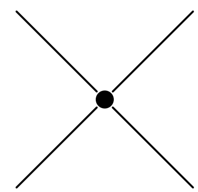
Massless case ($m = 0 \to \omega_k = k$):

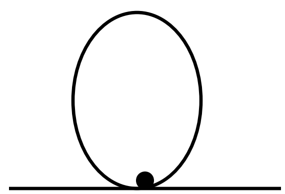
$$p = \int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{2}k + T\ln(1-e^{-\beta k}))\right] = \frac{\pi^2T^4}{90}$$

$$\epsilon(T) = U(T,V)/V = 3p,\ s(T) = S(T,V)/V = 4/3\epsilon(T)$$

Perturbative expansion:

$$e^{-S_E(\beta)} \simeq e^{-S_E^0(\beta)} \left(1 - \frac{\lambda}{4} \int d^4 x_E \phi^4(x_E)\right)$$



$\Pi =$ 

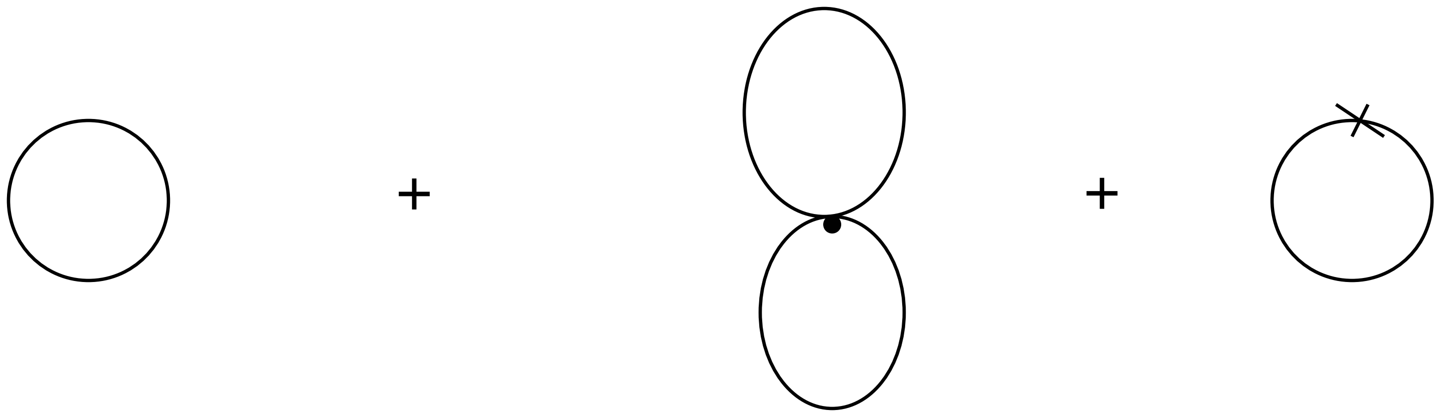
$$= \frac{\lambda}{2} T \sum_n \int \frac{d^3k}{(2\pi)^3} \Delta_0(i\omega_n, k)$$

$$= \frac{\lambda}{2} T \int \frac{d^3k}{(2\pi)^3} \Delta_0(\tau = 0, k)$$

$+$ 

$$= \frac{\lambda}{2} T \int \frac{d^3k}{(2\pi)^3} (\frac{1}{2}\omega_k + n(\omega_k))$$

Massless case $(\omega_k = k)$: $\Pi = \dfrac{\lambda T^2}{24}$

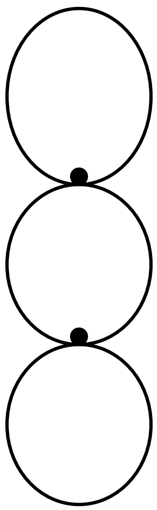


$$T\sum_n \int \frac{d^3k}{(2\pi)^3} \ln \Delta_0(i\omega_n, k) \quad + \quad \frac{\lambda}{8}\left(T\sum_n \int \frac{d^3k}{(2\pi)^3} \Delta_0(i\omega_n, k)\right)^2$$

Massless case:

$$P = \frac{\pi^2 T^4}{90}\left(1 - \frac{5\lambda}{64\pi^2}\right)$$

Infrared problems at finite temperature: the next-to-leading correction to the pressure is not of order λ^2 but $\lambda^{3/2}$ from $m = 0$!



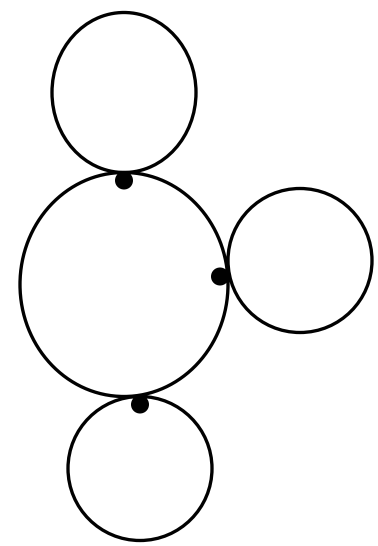
$$\lambda^2 \left(T \sum_l \int \frac{d^3p}{(2\pi)^3} \frac{1}{(p^2+\omega_l^2)^2} \right) \left(T \sum_n \int \frac{d^3k}{(2\pi)^3} \Delta_0(i\omega_n, k) \right)^2$$

the $l = 0$ term is IR divergent as $\int d^3p/p^4$

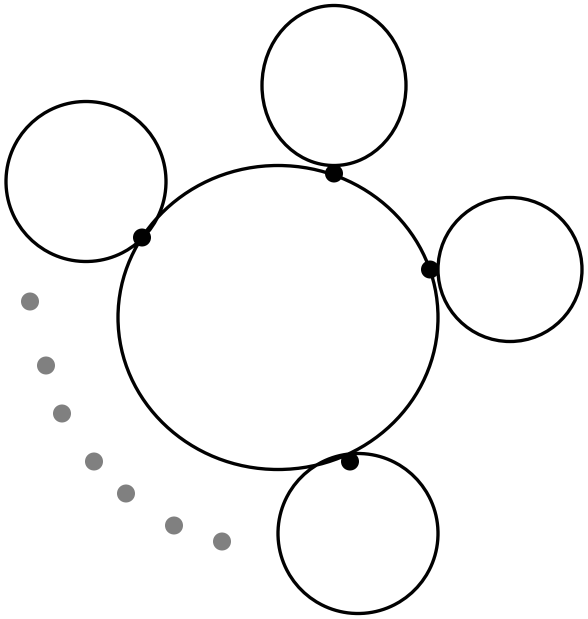
In the 4-loop diagram

$$\lambda^3 \left(T \sum_l \int \frac{d^3p}{(2\pi)^3} \frac{1}{(p^2+\omega_l^2)^3} \right) \left(T \sum_n \int \frac{d^3k}{(2\pi)^3} \Delta_0(i\omega_n, k) \right)^3$$

the $l = 0$ term is IR divergent as $\int d^3p/p^6$



We need to resum all diagrams of the following type (ring diagrams)



$$= -\frac{1}{2} VT \sum_n \int \frac{d^3p}{(2\pi)^3} \sum_{N=2}^{\infty} \frac{1}{N} \left(\frac{(-\Pi)}{(\omega_n^2 + p^2)} \right)^N$$

$$= -\frac{1}{2} VT \sum_n \int \frac{d^3p}{(2\pi)^3} \left(\ln \left(1 + \frac{\Pi}{p^2 + \omega_n^2} \right) - \frac{\Pi}{p^2 + \omega_n^2} \right)$$

keeping only the contribution from (IR sensitive) $n = 0$ and using $\Pi = \lambda T^2/24$ we get

$$F_{ring} = \frac{VT^4}{12\pi} \left(\frac{\lambda}{24} \right)^{3/2}$$

Collective effects in the medium have to be taken into account at all orders !

$$P = \frac{\pi^2 T^4}{90} \left(1 - \frac{15}{8} \left(\frac{\lambda}{24\pi^2} \right) + \frac{15}{2} \left(\frac{\lambda}{24\pi^2} \right)^{3/2} + ... \right)$$

Dirac Fields at finite temperature

Free Dirac Hamiltonian

$$\hat{H} = \int d^3x \psi^\dagger \gamma_0 (-i\gamma \cdot \nabla + m)\psi(x)$$

$\hat{Q} = \int d^3x \psi^\dagger \gamma^0 \psi$ -conserved charge

Canonical and grand canonical partition functions

$$Z_{can} = \mathrm{Tr} e^{-\beta \hat{H}}, \quad Z = \mathrm{Tr} e^{-\beta \hat{H} + \mu \hat{Q}}$$

$$Z = \int \mathcal{D}(\psi_\alpha^*,\ \psi_\alpha) \exp\left(- \int_0^\beta d\tau \left[\psi_\alpha (\partial_\tau - \mu) \psi_\alpha + H(\psi_\alpha^*, \psi_\alpha) \right] \right)$$

fermion fields anticommute $\Rightarrow \psi_\alpha(\beta) = -\psi_\alpha(0)$

$\Rightarrow \quad \omega_n = (2n+1)\pi T, n = 0, \ \pm 1, \pm 2...$

$$Z = \mathrm{Tr} \ln \left[-i\beta((-i\omega_n + \mu) - \gamma^0 \gamma \cdot k - m\gamma_0) \right]$$

$$= 2 \sum_n \sum_k \ln \left[\beta^2 \left((\omega_n + i\mu)^2 + \omega_k \right) \right]$$

$$2V \int \frac{d^3k}{(2\pi)^3} \left[\beta\omega_k + \ln(1 + e^{-\beta(\omega_k - \mu)}) + \ln(1 + e^{-\beta(\omega_k + \mu)}) \right]$$

Gauge field at finite temperature

$$Z(\beta) = \int \mathcal{D}(A_\mu^a, \eta_b, \eta_c) \exp\left[-\int_0^\beta d^4x_E \mathcal{L}_{eff}(x)\right]$$

$$\mathcal{L}_{eff}(x) = \frac{1}{4} F_{\mu\nu}^a(x) F_{\mu\nu}^a(x) + \frac{1}{2\xi}(\partial_\mu A_\mu^a)^2 + \bar{\eta}_a(x)\left[\partial^2 \delta_{ab} + f_{abc} A_\mu^c \partial_\mu\right] \eta_b(x)$$

$$A_\mu(0, x) = A_\mu(\beta, x), \quad \eta_a(0, x) = \eta_a(\beta, x)$$

$$\ln Z(\beta) = -\frac{1}{2} \times 4(N_c^2 - 1) \sum_n \sum_k \ln[\beta^2(\omega_n^2 + k^2)] +$$

4 gluons

$$\frac{1}{2} \times 2(N_c^2 - 1) \sum_n \sum_k \ln[\beta^2(\omega_n^2 + k^2)]$$

ghosts

$$p(T) = 2(N_c^2 - 1) \frac{\pi^2 T^4}{90}$$

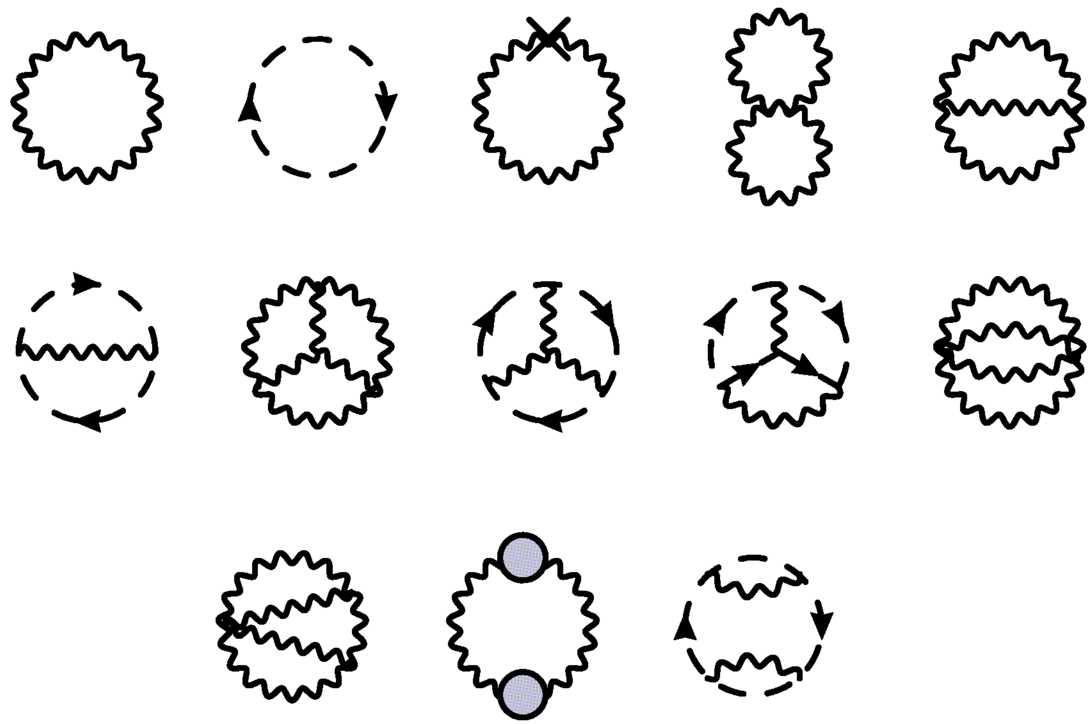
QCD at finite temperature

Because of asymptotic freedom thermodynamics quantities can be calculated in perturbation theory if $T \gg \Lambda_{QCD}$, at least in principle

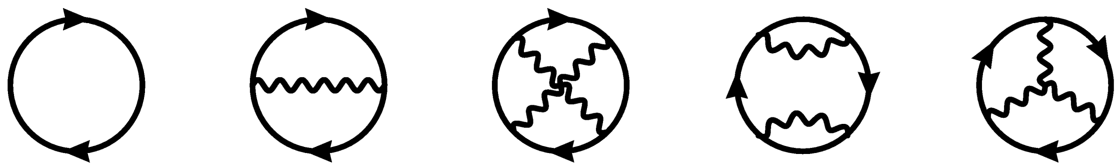
Pressure has been calculated to 3-loop order

Arnold, Zhai, Phys.Rev. D51 (1995) 1906, Kastening, Zhai, Phys.Rev. D52 (1995) 7232

Bosonic contribution:



Fermionic contribution:



Gluon self energy in the static limit:

$$\Pi_{00}(\omega_n = 0, k \to 0) = m_D^2 = (\frac{N_c}{3} + \frac{N_f}{6}) g^2 T^2$$

$$\Pi_{ii}(\omega_n = 0, k \to 0) = 0$$

$$\Downarrow$$

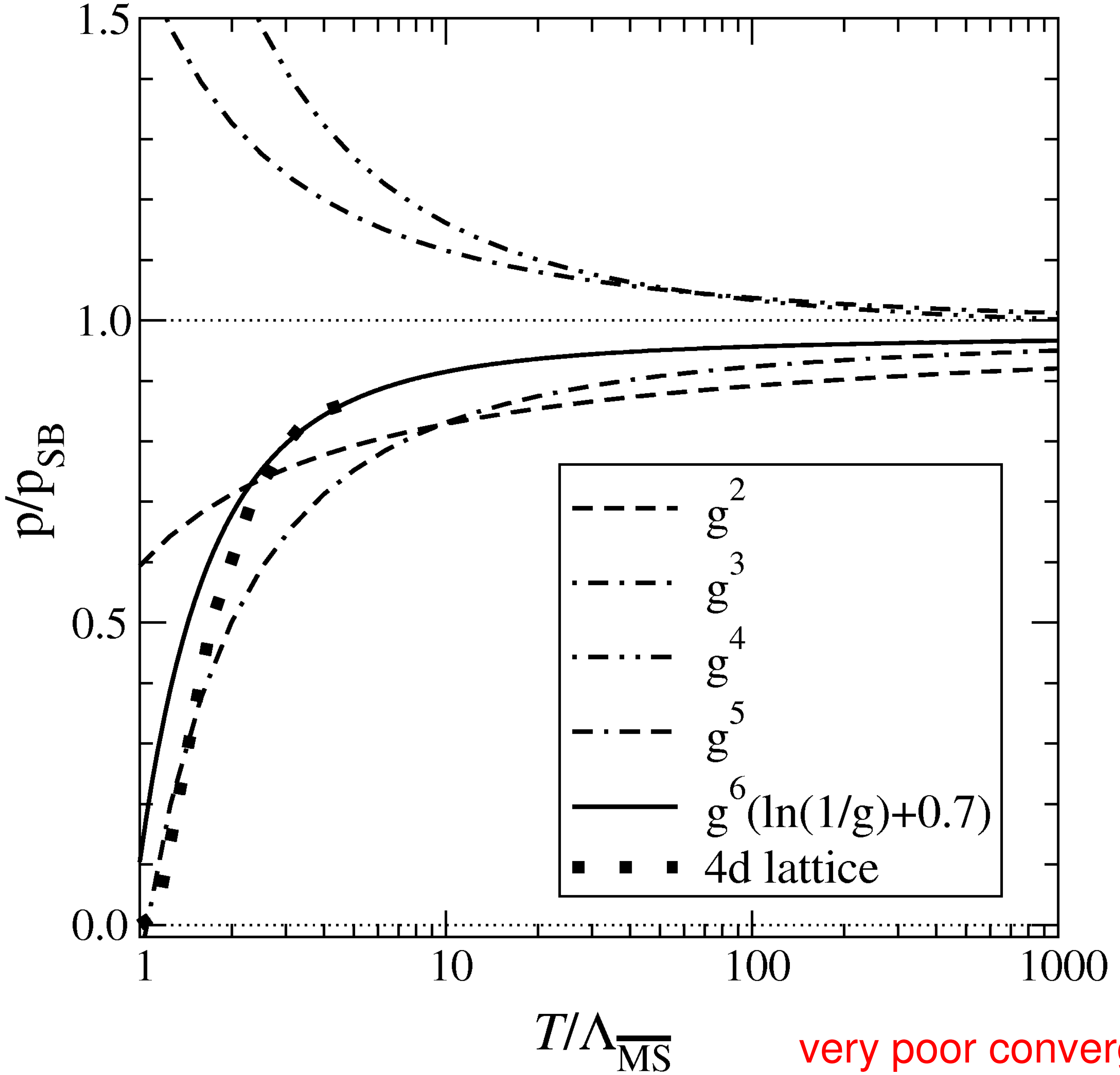
chromo-electric fields are screened but chromo-magnetic fields are not screened (at least in perturbation theory)

$$
\begin{aligned}
F \;=\; & d_{\mathsf{A}} T^4 \frac{\pi^2}{9}\Big\{ -\frac{1}{5}\left(1+\frac{7d_{\mathsf{F}}}{4d_{\mathsf{A}}}\right) + \left(\frac{g}{4\pi}\right)^2 \left(C_{\mathsf{A}} + \tfrac{5}{2}S_{\mathsf{F}}\right) \\
& -48\left(\frac{g}{4\pi}\right)^3 \left(\frac{C_{\mathsf{A}}+S_{\mathsf{F}}}{3}\right)^{3/2} - 48\left(\frac{g}{4\pi}\right)^4 C_{\mathsf{A}}(C_{\mathsf{A}}+S_{\mathsf{F}}) \ln\left(\frac{g}{2\pi}\sqrt{\frac{C_{\mathsf{A}}+S_{\mathsf{F}}}{3}}\right) \\
& + \left(\frac{g}{4\pi}\right)^4 \Big[C_{\mathsf{A}}^2 \left(\frac{22}{3}\ln\frac{\bar{\mu}}{4\pi T} + \frac{38}{3}\frac{\zeta'(-3)}{\zeta(-3)} - \frac{148}{3}\frac{\zeta'(-1)}{\zeta(-1)} - 4\gamma_{\mathsf{E}} + \frac{64}{5}\right) \\
& \quad + C_{\mathsf{A}} S_{\mathsf{F}} \left(\frac{47}{3}\ln\frac{\bar{\mu}}{4\pi T} + \frac{1}{3}\frac{\zeta'(-3)}{\zeta(-3)} - \frac{74}{3}\frac{\zeta'(-1)}{\zeta(-1)} - 8\gamma_{\mathsf{E}} + \frac{1759}{60} + \frac{37}{5}\ln 2\right) \\
& \quad + S_{\mathsf{F}}^2 \left(-\frac{20}{3}\ln\frac{\bar{\mu}}{4\pi T} + \frac{8}{3}\frac{\zeta'(-3)}{\zeta(-3)} - \frac{16}{3}\frac{\zeta'(-1)}{\zeta(-1)} - 4\gamma_{\mathsf{E}} - \frac{1}{3} + \frac{88}{5}\ln 2\right) \\
& \quad + S_{2\mathsf{F}} \left(-\frac{105}{4} + 24\ln 2\right)\Big] \\
& - \left(\frac{g}{4\pi}\right)^5 \left(\frac{C_{\mathsf{A}}+S_{\mathsf{F}}}{3}\right)^{1/2} \Big[C_{\mathsf{A}}^2 \left(176\ln\frac{\bar{\mu}}{4\pi T} + 176\gamma_{\mathsf{E}} - 24\pi^2 - 494 + 264\ln 2\right) \\
& \qquad + C_{\mathsf{A}} S_{\mathsf{F}} \left(112\ln\frac{\bar{\mu}}{4\pi T} + 112\gamma_{\mathsf{E}} + 72 - 128\ln 2\right) \\
& \qquad + S_{\mathsf{F}}^2 \left(-64\ln\frac{\bar{\mu}}{4\pi T} - 64\gamma_{\mathsf{E}} + 32 - 128\ln 2\right) \\
& \qquad - 144 S_{2\mathsf{F}}\Big] + O(g^6)\Big\},
\end{aligned}
$$

$$
d_{\mathsf{A}} = N_c^2 - 1\,, \quad C_{\mathsf{A}} = N_c\,, \quad d_{\mathsf{F}} = N_c N_{\mathsf{f}}\,, \quad S_{\mathsf{F}} = \frac{1}{2}N_{\mathsf{f}}\,, \quad S_{2\mathsf{F}} = \frac{N_c^2-1}{4N_c}N_{\mathsf{f}}\,.
$$

Arnold, Zhai, Phys.Rev. D51 (1995) 1906, Kastening, Zhai, Phys.Rev. D52 (1995) 7232

Kajantie et al, Phys. Rev. D 67 (2004) 105008



very poor convergence !

Pressure at g^6 order ? Magnetic mass ? Infrated sensitive contribution to the partition function at $l+1$-loop order

$$g^{2l}\left(T\int d^3p\right)^{l+1} p^{2l}(p^2+m_{mag}^2)^{-3l}$$

$$g^{2l}T^4, \quad l=1,2$$

$$g^6T^4\ln(T/m_{mag}), \quad l=3$$

$$g^6T^4(g^2T/m_{mag})^{l-3}, \quad l>3$$

$m_{mag}\sim g^2T \Rightarrow$ infinitely many diagramls contribute at g^6 order !

Homework:

Prove the integral equation :

$$\Delta(\tau) = \int_0^\infty dk_0 \sigma(k_0) \frac{\cosh(k_0 \cdot (\tau - \beta/2))}{\sinh(\beta k_0/2)}$$

Show that:

$$\sigma(k_0) = \frac{1}{Z(\beta)} \sum_{n,m} e^{-\beta E_n} \left[\delta(k_0 + E_n - E_m) - \delta(k_0 + E_m - E_n)\right] |\langle n|\widehat{q}|m\rangle|^2$$

use relation of $\sigma(k_0)$ and $D^{>,<}(k_0)$ and insert a complete set of energy eigenstates into $D^{>,<}(t)$)

Prove the sum rule

$$\int_{-\infty}^{\infty} k_0 \sigma(k_0) dk_0 = 1$$

For questions see me in my office B468