Numerical methods in lattice field theory
Problem sheet 1

Mike Peardon (Trinity College Dublin) – mjp@maths.tcd.ie
Office C420, INT Summer School, Seattle

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My solutions (NB: many problems here don't have a unique solution!) will be posted at the weekend. Stuck? Feel free to contact me!

1 Transformation and rejection methods

Define efficient algorithms that will sample values of X, random numbers with the following probability density functions. You may assume you have an algorithm to generate uniform variate random numbers, $U \in [0, 1]$ available.

1. $f_X(x) = \lambda e^{-\lambda x}$, with $x \geq 0, \lambda > 0$
2. $f_X(x) = 3\lambda x^2 e^{-\lambda x^3}$, with $x \geq 0, \lambda > 0$
3. $f_X(x) = c \sin^2 x e^{-\lambda x}$, with $x \geq 0, \lambda > 0$ and c the appropriate normalisation constant.

2 Examples of Markov processes

2.1 The plastic souvenir Space Needle factory

A factory runs a machine to produce plastic souvenir Space Needles. Every day, there is a probability $\lambda_1, (0 \leq \lambda \leq 1)$ that the operating machine breaks down. The factory has a single spare machine and a workshop to repair their machines. Machines arrive at the workshop at the end of the day. The spare machine is put into service immediately the first machine breaks and there is a probability $\lambda_2, (0 \leq \lambda_2 \leq 1 - \lambda_1)$ that both machines break. Every day, one machine in the workshop is repaired and returned to the factory at the end of the day. Write a Markov matrix to describe the stochastic transitions this system makes during a day of operation. On what fraction of days is the factory incapable of making plastic souvenir Space Needles because both machines are broken at the start of the day?

2.2 The two-state Markov chain

For the two-state system undergoing a Markov process with transition matrix

$$M = \begin{pmatrix} 1-\kappa_1 & \kappa_2 \\ \kappa_1 & 1-\kappa_2 \end{pmatrix}$$

with $0 < \kappa_{1,2} < 1$, derive an expression for the transition probability after n applications of the process. Show that as $n \to \infty$ the stochastic state of the system collapses to a fixed point and find this fixed point.

3 Programming exercises

3.1 The Box-Muller algorithm

Write a function to use the Box-Muller algorithm to return a normally distributed random number, $X \in (-\infty, \infty)$ with probability density function

$$f_X(x) = \frac{1}{|\sigma|\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

3.2 Random phases

Write a function to generate a random number, $\Theta \in (-\pi, \pi]$ drawn from the distribution

$$f_\Theta(\theta) = \frac{1}{Z(\beta)} e^{\beta\cos(\theta-\delta)}$$

where β is a real parameter and $Z(\beta)$ normalises the distribution. For what values of β is your method efficient?

3.3 Free bosons in 1-dimension

Write a program that uses a Gibbs sampler to generate importance sampling configurations of a real bosonic field with mass m_0 on a circle, discretised onto an N point lattice. The action for this system is

$$S[\phi] = \sum_{i,j=1}^{N} \phi_i \left((2 + (am_0)^2)\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}\right)\phi_j$$

and the field ϕ has periodic boundary conditions, such that $\phi_{i+N} = \phi_i$. Use Monte Carlo to estimate the two-point correlation function for the system,

$$C(t) = \frac{1}{N}\sum_{i=1}^{N}\langle\phi_{i+t}\phi_i\rangle$$

and test your method by comparing this to the analytic result.