Fundamental constants and electroweak phenomenology from the lattice

Lecture III: Chiral dynamics and light quark masses

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@ INT summer school 2007,

Seattle, August 2007.

III. Chiral dynamics and light quark masses

1. Chiral symmetry breaking and quark masses
   - GMOR relation
   - Chiral perturbation theory
   - Quark mass ratios
2. Lattice calculation of light quark masses
   - Basic strategy
   - Perturbative and non-perturbative matchings
3. Pion loop effects
   - Chiral log effects on chiral extrapolation
   - Pion form factor and general strategy

III. Chiral dynamics
1. Chiral symmetry breaking and quark masses

Chiral symmetry breaking

- In the QCD vacuum, chiral symmetry is broken.

  - Flavor $SU(3)_L xSU(3)_R \to SU(3)_V$
  - Non-zero chiral condensate $\langle \bar{q}q \rangle$
  - Nambu-Goldstone bosons (pion, kaon, η) nearly massless; in practice massive due to non-zero m_q.
    - Flavor-singlet axial U(1) is special, due to anomaly. η' is substantially heavier.
  - Other hadrons have a mass of $O(\Lambda_{QCD})$

  - Low energy effective theory for pions (and K, η) can be constructed = chiral perturbation theory (ChPT, χPT).

PCAC relation

- Partially Conserved Axial Current (PCAC)
  - From the QCD Lagrangian,

    $$A_\mu = \bar{u}\gamma_\mu\gamma_5 d,$$

    $$\partial_\mu A^\mu = (m_u + m_d)\bar{u}\gamma_5 d$$

  - The axial current may annihilate pion to the vacuum; Lorentz invariance restricts its form.

    $$\langle 0 | A_\mu(0) | \pi(p) \rangle = i f_\pi p_\mu,$$

    $$\langle 0 | \partial_\mu A^\mu(0) | \pi(p) \rangle = f_\pi m_\pi^2;$$

    $$\partial_\mu A^\mu(x) = f_\pi m_\pi^2 \phi_\pi(x)$$

- f_π is called the pion decay constant.
- Can be measured from the leptonic decay $\pi \to \mu\nu$.

  f_π = 131 MeV
- Its analog for kaon is f_K.

  f_K = 160 MeV

$\phi_\pi(x)$: operator to create a pion.

- Consider two-point functions

$$\Pi_5^{\mu\nu}(q) = i\int d^4x e^{iqx} \left\langle 0 \left| \mathrm{T}\left(A^\mu(x) A^\nu(0)^\dagger \right) \right| 0 \right\rangle,$$

$$\Psi_5(q) = i\int d^4x e^{iqx} \left\langle 0 \left| \mathrm{T}\left(\partial_\mu A^\mu(x) \partial_\nu A^\nu(0)^\dagger \right) \right| 0 \right\rangle$$

- Taking derivatives of T-products, we obtain

$$q_\mu q_\nu \Pi_5^{\mu\nu}(q) = \Psi_5(q) - q^\nu \int d^4x e^{iqx} \delta(x^0) \left\langle 0 \left| \left[A^0(x), A^\nu(0)^\dagger \right] \right| 0 \right\rangle$$
$$+ i\int d^4x e^{iqx} \delta(x^0) \left\langle 0 \left| \left[\partial_\mu A^\mu(x), A^0(0)^\dagger \right] \right| 0 \right\rangle$$

Spontaneous symmetry breaking $Q_5 \left| 0 \right\rangle \neq 0$

- In the limit of $q^\mu \to 0$, it leads to

$$(m_u + m_d)\left\langle \bar{u}u + \bar{d}d \right\rangle = -i f_\pi^2 m_\pi^4 \left\{ \left. \frac{-i}{m_\pi^2 - q^2} \right|_{q\to 0} + \text{other resonances} \right\}$$
$$= -f_\pi^2 m_\pi^2 \left\{ 1 + O(m_\pi^2) \right\}$$

- Gell-Mann-Oakes-Renner (GMOR) relation (1968)

$$(m_u + m_d)\langle \bar{u}u + \bar{d}d \rangle = -f_\pi^2 m_\pi^2 \{1 + O(m_\pi^2)\}$$

  - Chiral symmetry is broken = Non-zero chiral condensate $\langle \bar{q}q \rangle$
  - Pion mass squared proportional to quark mass

$$m_\pi^2 = B_0(m_u + m_d) + O(m_q^2)$$
$$= \frac{-2\langle \bar{q}q \rangle}{f_\pi^2}(m_u + m_d) + O(m_q^2)$$

    - Also for kaons, $m_{K^+}^2 = B_0(m_u + m_s) + O(m_q^2), m_{K^0}^2 = B_0(m_d + m_s) + O(m_q^2),$

$$m_\eta^2 = \frac{1}{3}B_0(m_u + m_d + 4m_s) + O(m_q^2),$$

    - Quark mass ratios can be predicted up to $O(m_q^2)$.

Chiral Lagrangian

- Low energy effective lagrangian is developed assuming
  - Spontaneous breaking of chiral symmetry
  - Pion (and kaon, eta) to be the Nambu-Goldston boson
- In the low energy regime, pions are only relevant dynamical degrees of freedom.

$$L_2 = \frac{f^2}{4}\mathrm{Tr}\left(D_\mu U D^\mu U^\dagger\right) + \frac{f^2}{4}\mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right),$$

$$U = \exp\left(\frac{i\tau^a\pi^a}{f}\right), \chi = 2B_0 m$$

  - Given by a non-linear sigma model.
  - Provides a systematic expansion in terms of m_π^2, p^2; the leading order is given above.

For full details, see Bernard's lectures

- Expansion in the pion field gives

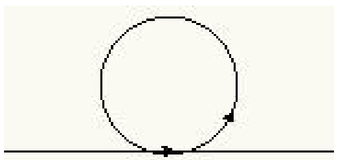
$$L_2 = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a - \frac{m_\pi^2}{2}\pi^a\pi^a + \frac{m_\pi^2}{24f^2}(\pi^a\pi^a)^2 + \frac{1}{6f^2}\left[(\pi^a\partial_\mu\pi^a)(\pi^b\partial^\mu\pi^b) - (\pi^a\pi^a)(\partial_\mu\pi^b\partial^\mu\pi^b)\right] + \ldots$$

  - Pion mass is obtained as $m_\pi^2 = 2B_0 m$
  - A chain of interaction terms: 4π, 6π, etc.
- Loop corrections are calculable.
  - Pick up a factor of $(m_\pi/4\pi f)^2$ or $(\boldsymbol{p}/4\pi f)^2$
  - Counter terms must also be added at order $(m_\pi/4\pi f)^2$ or $(\boldsymbol{p}/4\pi f)^2$
    - introduce the low energy constants (LECs): L_1~L_{10} at the one-loop level

For full details, see Bernard's lectures

One-loop example

- Pion self-energy



$$\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2} = \frac{1}{(4\pi)^2}\left[\Lambda^2 + m^2 \ln\frac{m^2}{\Lambda^2+m^2}\right]$$

Cutoff regularization

$$= \frac{m^2}{(4\pi)^2}\left(\frac{2}{\varepsilon}+\gamma-\ln 4\pi+\ln\frac{m^2}{\mu^2}-1\right)$$

Dimensional reg

- Log dependence $m^2\ln(m^2)$: called the chiral logarithm.
- Comes from the infrared end of the integral = long distance effect of (nearly massless) pion loop.
- Counter terms are necessary in order to renormalize the UV divergence.
- After subtracting the UV divergences

$$m_\pi^2 = 2B_0 m_q\left[1+\frac{1}{2}\frac{m_\pi^2}{(4\pi f)^2}\ln\frac{m_\pi^2}{\mu^2}+(\mathrm{const})\times\frac{m_\pi^2}{(4\pi f)^2}+O(m_\pi^4)\right]$$

Counter terms

- At the order $(m_\pi/4\pi f)^2$ or $(p/4\pi f)^2$, there are 10 possible counter terms
  - 10 new parameters, $L_1 \sim L_{10}$ = low energy constant at NLO
    c.f. 2 parameters at LO: Σ and f.
  - Depends on how one renormalize the UV divergence, just as in the small coupling perturbation. $L_1 \sim L_{10}$ depends on the renormalization scale μ.
  - Once these parameters are determined (e.g. from pion scattering data), one can predict other quantities.
  - Lattice QCD may be used to *calculate* these parameters.

Quark mass ratio

- At NLO, the quark mass ratio is given as

$$\frac{m_K^2}{m_\pi^2} = \frac{m_s + m_{ud}}{2m_{ud}} \left[1 + \frac{1}{2} \frac{m_\pi^2}{(4\pi f)^2} \ln \frac{m_\pi^2}{\mu^2} - \frac{1}{2} \frac{m_\eta^2}{(4\pi f)^2} \ln \frac{m_\eta^2}{\mu^2} - \frac{8(m_K^2 - m_\pi^2)}{f^2} (2L_8 - L_5) \right]$$

  - Assumes that the isospin breaking $m_u \neq m_d$ is negligible.
  - Requires the knowledge of the NLO LEC $2L_8$-L_5.
  - Results in m_s/m_{ud}=25~30 (PDG 2006); large uncertainty due to the unknown LEC.

  - Comparison with the exp number gives LECs. But the predictive power is lost.
  - Instead, lattice calculation can be used to fix LECs.

Isospin breaking

- In the real world, $\pi^\pm$ and π^0 have different masses; two sources
  - Small mass difference between up and down quarks.
  - Electromagnetic effect: Q_u=+2/3, Q_d=-1/3.

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0+\frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0+\frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta \end{pmatrix}$$

- Quark mass difference
  - When $m_u \neq m_d$, π^0 and η can mix

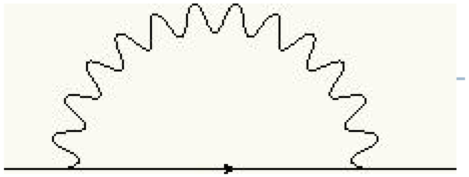
$$|\pi^0\rangle = |\pi^0\rangle_0 + \frac{\sqrt{3}}{4}\frac{m_d-m_u}{m_s-\hat{m}}|\eta\rangle_0 + O((m_d-m_u)^2) \qquad |\pi^0\rangle_0 = \frac{1}{\sqrt{2}}(u\bar{u}-d\bar{d})$$

$$|\eta\rangle = |\eta\rangle_0 - \frac{\sqrt{3}}{4}\frac{m_d-m_u}{m_s-\hat{m}}|\pi^0\rangle_0 + O((m_d-m_u)^2) \qquad |\eta\rangle_0 = \frac{1}{\sqrt{6}}(u\bar{u}+d\bar{d}-2s\bar{s})$$

  - Then, from $M_\pi^2 = \langle\pi|\left[m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s\right]|\pi\rangle$

$$M_{\pi^0}^2 = B_0(m_u+m_d) - \frac{B_0}{4}\frac{(m_d-m_u)^2}{(m_s-\hat{m})} \Rightarrow \begin{array}{l} \Delta M_\pi \sim 0.2\ \mathrm{MeV} \\ \text{c.f. } (\Delta M_\pi)^{(\mathrm{phys})} = 4.6\ \mathrm{MeV} \end{array}$$

Electromagnetic effect



- Self-energy with a photon propagator
  - Using the PCAC relation, related to a current two point function (VV-AA).
$$\Pi_{JJ}^{\mu\nu}(q) = i\int d^4x\, e^{iqx} \left\langle 0 \middle| \mathrm{T} J^{\mu}(x) J^{\nu}(0)^{\dagger} \middle| 0 \right\rangle$$
  - Das-Guralnick-Low-Mathur-Young sum rule (1967), a close relative of the Weinberg sum rules (1967)
$$\Delta M_{\pi}^{2} = \frac{3\alpha_{em}}{4\pi f^{2}} \int_0^{\infty} dQ^2 Q^2 \left[\Pi_{VV}(Q^2) - \Pi_{AA}(Q^2)\right]$$
  - Sum rule estimate gives ΔM_{π}~5 MeV, comparative to the exp value 4.59 MeV. Lattice calculation is also possible.
  - At the leading order, the same effect for kaon (Dashen's theorem)
$$M_{K^+}^{2} - M_{K^0}^{2} = M_{\pi^+}^{2} - M_{\pi^0}^{2}$$

Estimate of m_u/m_d

- At the leading order, Weinberg (1977)
  - Using the GMOR relation and the EM correction,

$$M_{\pi^\pm}^2 = B_0(m_u + m_d) + \Delta_{em}$$
$$M_{\pi^0}^2 = B_0(m_u + m_d)$$
$$M_{K^\pm}^2 = B_0(m_u + m_s) + \Delta_{em}$$
$$M_{K^0}^2 = B_0(m_d + m_s)$$

Dashen's theorem

  - Combine them to obtain

$$\frac{m_u}{m_d} = \frac{M_{\pi^0}^2 + (M_{K^\pm}^2 - M_{K^0}^2) - (M_{\pi^\pm}^2 - M_{\pi^0}^2)}{M_{\pi^\pm}^2 - (M_{K^\pm}^2 - M_{K^0}^2)} = 0.55,$$

$$\frac{m_s}{m_d} = \frac{(M_{K^\pm}^2 + M_{K^0}^2) - M_{\pi^\pm}^2}{M_{\pi^\pm}^2 - (M_{K^\pm}^2 - M_{K^0}^2)} = 20.1$$

Further estimate of m_u/m_d

- At NLO, those simple relations are lost.
  - NLO formula (Gasser-Leutwyler (1985))

$$\frac{M_K^2}{M_\pi^2} = \frac{m_s+\hat{m}}{m_d+m_u}\left\{1+\Delta_M+O(m^2)\right\}$$

$$\frac{M_{K^0}^2 - M_{K^\pm}^2}{M_K^2 - M_\pi^2} = \frac{m_d-m_u}{m_s-\hat{m}}\left\{1+\Delta_M+O(m^2)\right\}$$

$$\Delta_M = \frac{8(M_K^2-M_\pi^2)}{f^2}(2L_8-L_5)+\chi\text{logs}$$

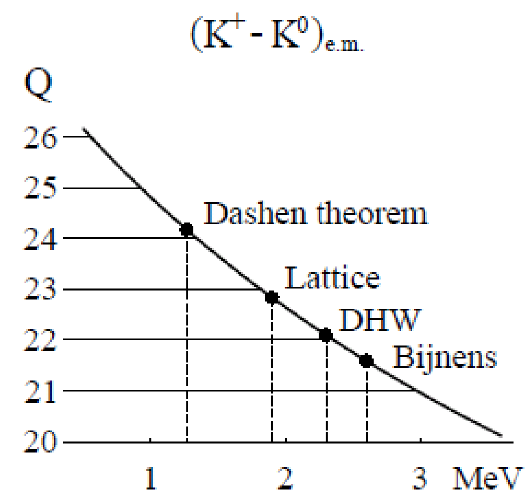
  - A double ratio is free from the NLO correction

$$Q^2 \equiv \frac{m_s^2-\hat{m}^2}{m_d^2-m_u^2} = \frac{M_K^2}{M_\pi^2}\frac{M_K^2-M_\pi^2}{M_{K^0}^2-M_{K^\pm}^2}\left\{1+O(m^2)\right\}$$

which can be written in a simple form

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1$$

  - A slight ambiguity comes from a violation of the Dashen's theorem.



from Leutwyler, PLB378 (1996) 313.

NLO constraints

- The NLO formula makes an ellipse.

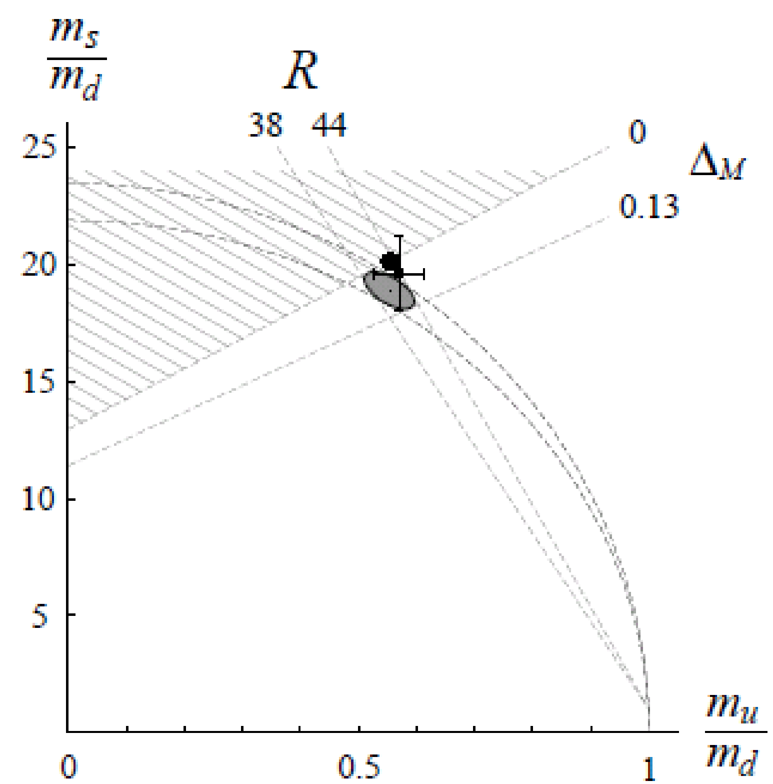
$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1$$

- Other constraints:
  - Δ_M>0, from large Nc.
  - Another constraint on

$$R = \frac{m_s - \hat{m}}{m_d - m_u}$$

from a charmonium decay

$$\frac{\Gamma(\psi' \to \psi\pi^0)}{\Gamma(\psi' \to \psi\eta^0)}$$



III. Chiral dynamics
2. Lattice calculation of light quark masses

Inputs

In general, lattice QCD simulation requires inputs for

- Lattice scale $1/a \Rightarrow$ determines $\alpha_s(1/a)$
  - Inputs discussed in Part I.
- Quark masses for each flavor
  - up and down quarks m_{ud} (often assumed to be degenerate)
    - from pseudo-scalar meson mass m_π, good sensitivity because $m_\pi^2 \sim m_{ud}$.
  - Strange quark m_s
    - from m_K, for the same reason.
  - Charm quark m_c
    - either from D (heavy-light) or J/ψ (heavy-heavy) mass
  - Bottom quark m_b
    - either from B (heavy-light) or Υ (heavy-heavy) mass

Chiral extrapolation

- Lattice simulation is harder for lighter sea quarks.
  - Computational cost grows as m_q^{-n} (n~2).
  - Finite volume effect becomes more important ~$\exp(-m_\pi L)$
- Practical calculation involves the *chiral extrapolation*. At the leading order, it is very simple:
  1. Fit the pseudo-scalar mass with $m_\pi^2 = B_0(m_u + m_d) + O(m_q^2)$
  2. Input the physical pion mass $m_{\pi 0}$=135 MeV to get $m_{ud}=(m_u+m_d)/2$. (Forget about the isospin breaking for the moment.)
  3. Renormalize it to the continuum scheme.
- Including higher orders is non-trivial…

NLO example

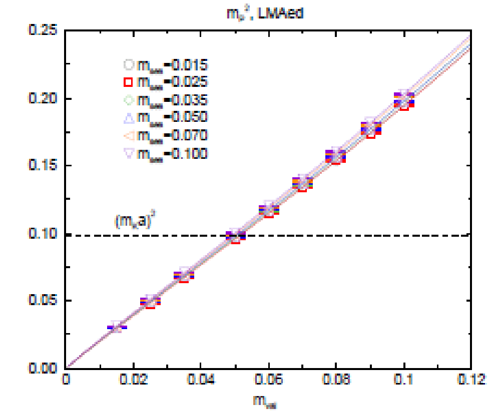
Chiral expansion

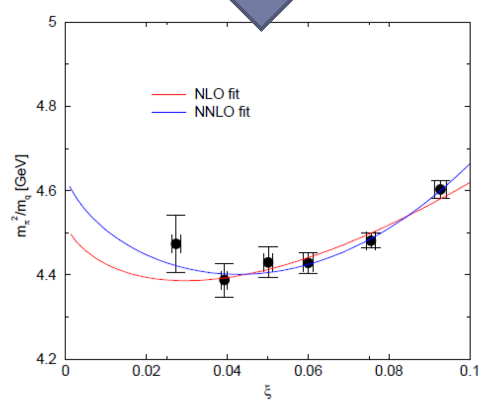
$$m_\pi^2 = 2B_0 m_q \left[1 + \frac{1}{2}\frac{m_\pi^2}{(4\pi f_\pi)^2}\ln\frac{m_\pi^2}{\mu^2} + c_3 \frac{m_\pi^2}{(4\pi f_\pi)^2} + \text{NNLO} \right]$$

- LO (linearity) looks very good, but if you look more carefully NLO is visible.

- m_π^2/m_q not constant.
- Chiral log term has a definite coefficient = curvature fixed.
- Analytic term has an unknown constant, to be fitted with lattice data = linear slope

JLQCD (2007)
dynamical overlap (N_f=2)





Strange quark

- Must consider 2+1-flavor theory.
  - If your simulation contains only 2-flavors (up and down quarks), then a possible choice is to use the *Partially Quenched* ChPT. Although it is not the correct theory after all, it will provide a consistent description of the lattice data.
- NLO effect is less pronounced for strange.

$$M_\pi^2 = 2\hat{m}B_0\left[1+\mu_\pi-\frac{1}{3}\mu_\eta+2\hat{m}K_3+(2\hat{m}+m_s)K_4\right] \quad , \quad \mu_\pi=\frac{1}{2}\frac{m_\pi^2}{(4\pi f)^2}\ln\frac{m_\pi^2}{\mu^2}$$

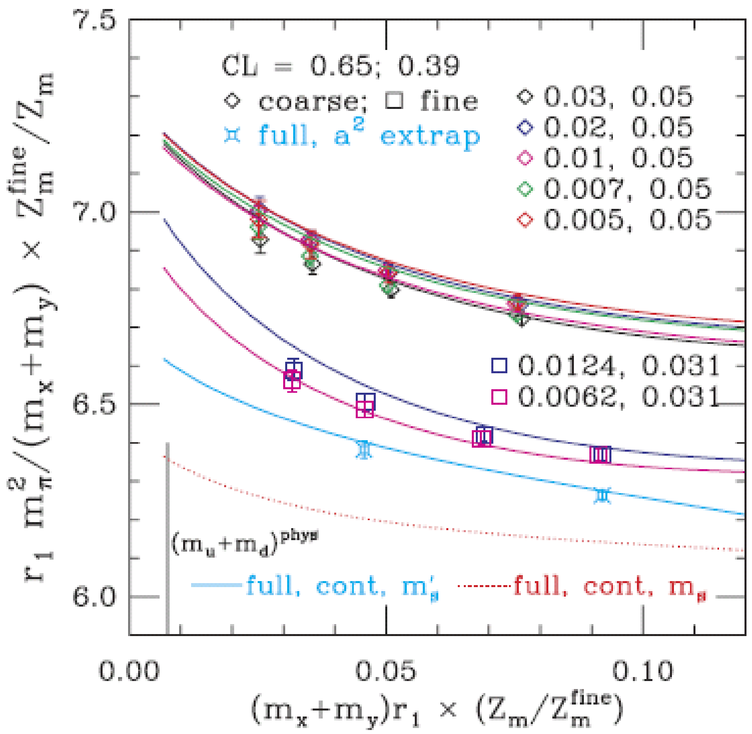
$$M_K^2 = (\hat{m}+m_s)B_0\left[1+\frac{2}{3}\mu_\eta+(\hat{m}+m_s)K_3+(2\hat{m}+m_s)K_4\right] \quad , \quad \mu_\eta=\frac{1}{2}\frac{m_\eta^2}{(4\pi f)^2}\ln\frac{m_\eta^2}{\mu^2}$$

  - No singularity in the chiral limit. (This may not be the case for other quantities like f_K.)
  - Numerical analysis will be more stable.

A case study: MILC+HPQCD 2+1

MILC+HPQCD, PRD70, 031504(R) (2004), MILC, PRD70, 114501 (2004).

- Bare quark masses taken from the MILC 2+1 asqtad simulations.
  - Two lattice spacings "coarse" (a=0.125 fm) and "fine" (a=0.090 fm).
  - Complicated fit including the taste-breaking effects of staggered fermion, vanishing at a=0.
  - NNLO analytic terms are included. Non-analytic (chiral log) terms are discarded.



Renormalization

convert:
$m^{MS}(\mu)=Z_m(\mu a)m^{lat}(a^{-1})$

- Once the bare quark mass is fixed on the lattice, it must be converted to the continuum definition, because the pole mass is not adequate.
  - Just like the conversion of the coupling constant.
  - May use the perturbation theory (Use the renormalized coupling!). But, in most cases, known only at the one-loop level. (Exceptions are HQET, Asqtad, stochastic PT(?).)

    Ex). O(a)-improved Wilson fermion

$$m^{MS}(\mu=a^{-1})=[1+2.05\alpha_s+...]m^{lat}(a^{-1})$$

  - Non-perturbative renormalization is desirable.

MILC+HPQCD 2+1

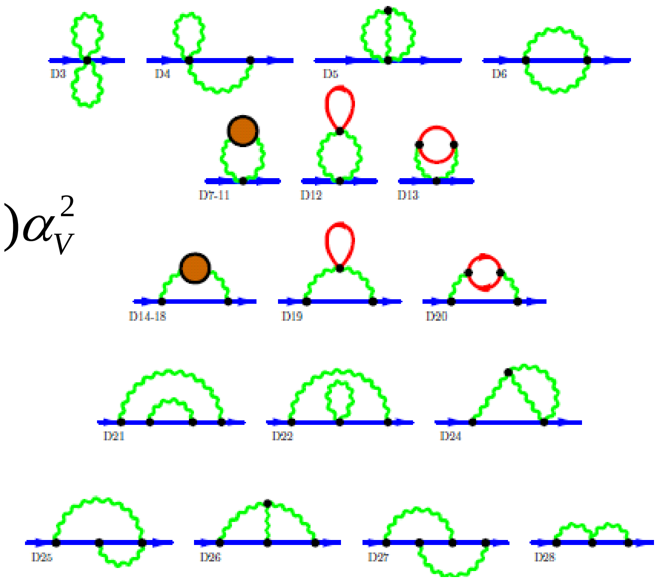
- Conversion is done perturbatively.
  - Calculated to two-loop. HPQCD (Mason et al.), PRD73, 114501 (2006).

    $$Z_m(\mu a=1)=1+0.119\alpha_V(q^*)+(2.22-0.02n_f)\alpha_V^2$$

    with $q^*=1.88/a$.
  - With $\alpha_V(q^*)=0.27$, this series is

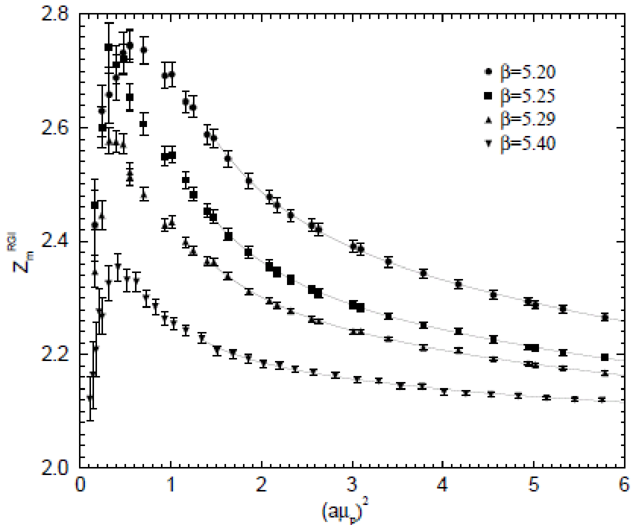
    Z_m = 1 + 0.03 + 0.16 + …
  - The uncertainty from higher orders is estimated as $2\alpha_V^3(q^*)$~5%.
  - After the continuum extrapolation with $\alpha_V a^2$, they quote

$$m_s^{\overline{MS}}(2\,\mathrm{GeV})=87(0)(4)(4)(0)\,\mathrm{MeV}$$



A case study: QCDSF-UKQCD N_f=2

- Another work by QCDSF-UKQCD with the O(a)-improved Wilson fermion, PRD73, 054508 (2006).
  - One may doubt about the convergence of the perturbative expansion. Non-perturbative renormalization is desirable if possible.
  - Non-perturbative renormalization is done using the RI/MOM scheme. It has its own subtlety: the renormalization *constant* is not really constant, due to SχSB.
  - NP results are about **20%** larger than the one-loop calculation.



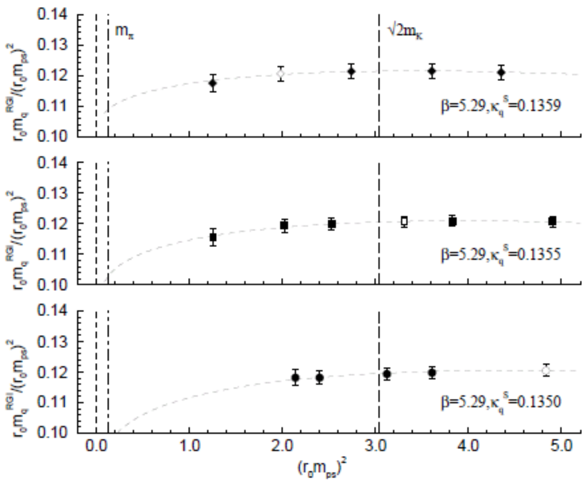
$$Z_m(\mu a) + c_1(\mu a)^2 + c_2 \frac{\langle \bar{\psi}\psi \rangle}{(\mu a)^2}$$

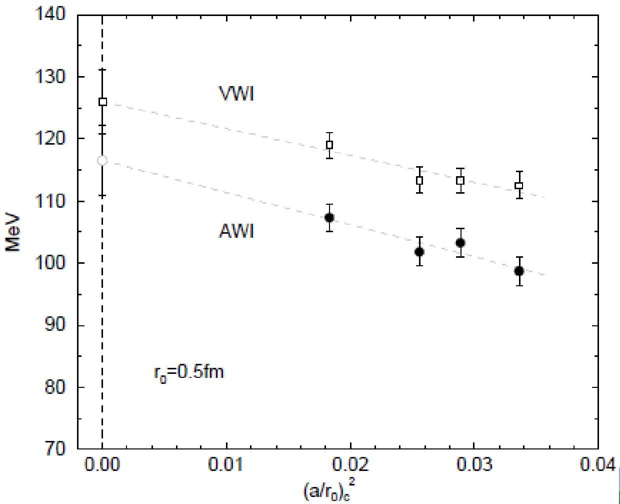
For more details, see the lectures by S. Sint.

QCDSF-UKQCD N_f=2

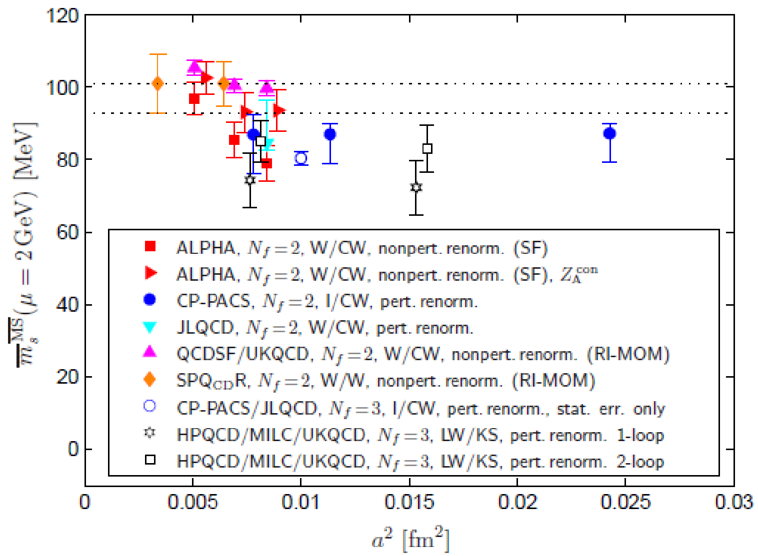
- Lattice data fit to the one-loop PQχPT formula (but the magnitude of the χlog is a free parameter).
- Conversion to MSbar is (partially) non-perturbative.
- Continuum extrapolation is carried out with 4 data points. Substantial rise in the continuum limit.

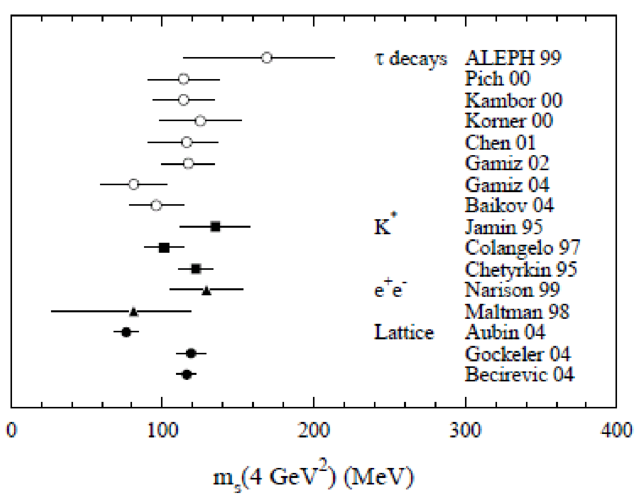
$$m_s^{\overline{MS}}(2\,\mathrm{GeV}) = 111(6)(4)(6)\,\mathrm{MeV}$$





Present status





- A recent compilation by Knechtli, hep-ph/0511033.
  - Non-perturbative renormalization yields higher m_s?

    or

  - Nf=3 gives lower m_s?
- Including other determinations (Plot from Davier, Hocker, Zhang, Rev. Mod. Phys.78,1043 (2006)).
  - Consistent within large errors.

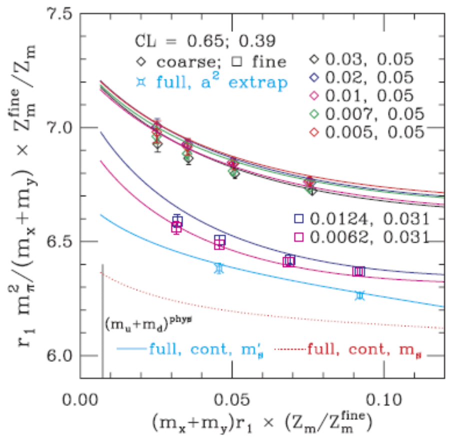
Up and down quarks

- Ratio to strange
  - Basically obtained by the ChPT formula.
  - At NLO, equivalent to the calculation of the LEC.

$$\frac{m_K^2}{m_\pi^2} = \frac{m_s + m_{ud}}{2m_{ud}} \left[1 + \frac{1}{2}\frac{m_\pi^2}{(4\pi f)^2} \ln\frac{m_\pi^2}{\mu^2} - \frac{1}{2}\frac{m_\eta^2}{(4\pi f)^2} \ln\frac{m_\eta^2}{\mu^2} - \frac{8(m_K^2 - m_\pi^2)}{f^2}(2L_8 - L_5) \right]$$

  - MILC obtained m_s/m_{ud}=27.4(1)(4)(1) from the SχPT fit. Much more delicate.
  - m_u/m_d can also be obtained (up to the EM uncertainty) from the relation

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1$$



III. Chiral dynamics
3. Pion loop effects

Chiral log effects

- We learned that the chiral extrapolation is non-trivial.
  - Especially so, if pions are involved as external states
    - pion mass, decay constant
    - form factors, $\pi\pi$-scattering...
  - Even when pion does not appear as external states, it could be there in the loop. May lead to the chiral log.
    - Kaon decay constant
    - Nucleon (masses, matrix elements)
    - Heavy-light meson (masses, decay constants, form factors)

  - Thus, could always be a delicate problem when one aims at good precision.

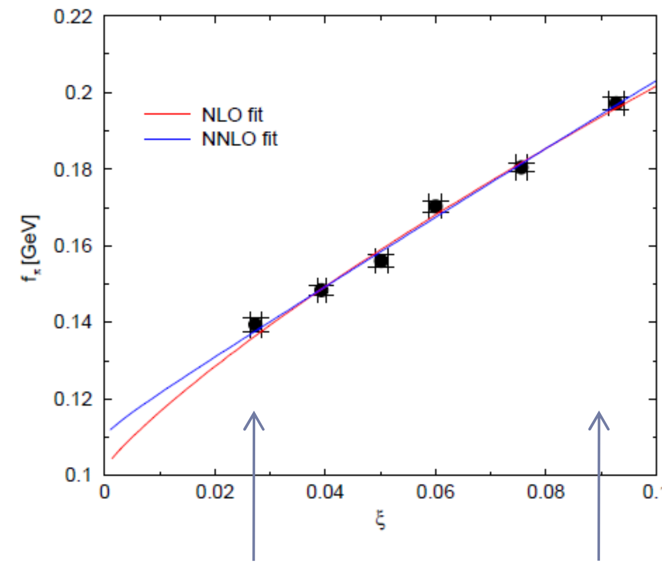
An example

- Pion decay constant
  - At NNLO, it has the form

$$(f_\pi)_{\rm NNLO} = f\left[1 - 2\xi\ln\xi + 5(\xi\ln\xi)^2 + \frac{3}{2}\left(\tilde{L}^{\rm phys} + \frac{53}{2}\right)\xi^2\ln\xi\right] + L_4(\xi - 10\xi^2\ln\xi) + \alpha_2\xi^2 + O(\xi^3),$$

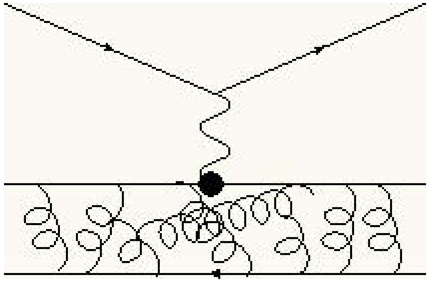
  with $\xi = m_\pi^2/(4\pi f_\pi)^2$.
  - Leading-log terms have known coefficients.
  - Free parameters in the analytic term (NLO, NNLO) and NLL term (NNLO).
  - Turns out that the chiral log effect is not substantial, but non-negligible.

JLQCD (2007)
dynamical overlap (N_f=2)
(talk by Noaki at lat07)



~m_s/6   ~m_s

Pion form factor



- The simplest form factor

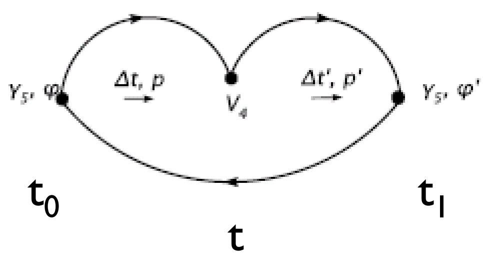
  $$\langle \pi(p')|V_\mu|\pi(p)\rangle = i(p_\mu + p_\mu')F_V(q^2), \quad q_\mu \equiv p_\mu' - p_\mu$$

  - Momentum transfer q_μ by a virtual photon. Space-like ($q^2<0$) in the $\pi e \to \pi e$ process.
  - Vector form factor $F_V(q^2)$ normalized as $F_V(0)=1$, because the vector current is conserved.

  $$F_V(q^2) = 1 + \frac{1}{6}\langle r^2\rangle_V^\pi q^2 + O(q^4),$$

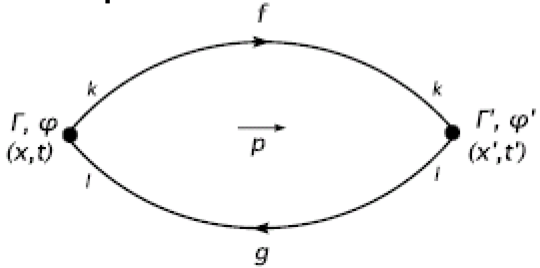
  - Vector (or EM) charge radius $\langle r^2\rangle_V^\pi$ is defined through the slope at $q^2=0$.

Lattice calculation of 3pt function



- $\pi(p) \to \pi(p')$
  - An interpolating operator for the initial state $\pi(p)$ at $t=t_0$
  - Another interpolating operator for the final state $\pi(p')$ at $t=t_1$
  - Current insertion V_μ in the middle t.
  - Spatial momentum inserted at two operators.

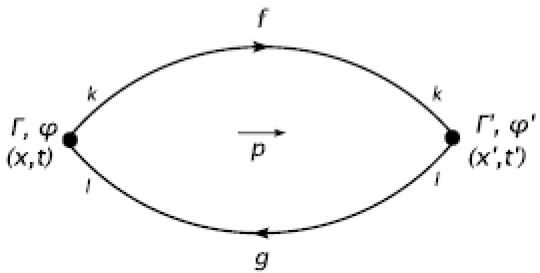
c.f. 2pt func

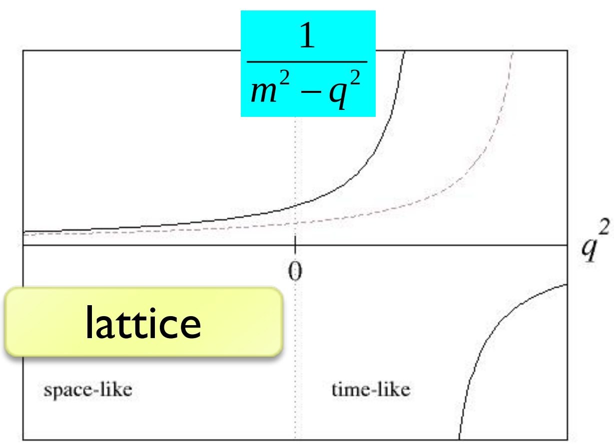


- (sequential) source method
  - Calculate a quark propagator starting from a previous quark propagator at t.

$$(\not{D}+m)S_2(x) = e^{i\mathbf{q}\cdot\mathbf{x}}\Gamma S_1(x)\delta(x_0 - t)$$

Ground state?





- Working on the Euclidean lattice
  - On-shell particle will never appear (except for the massless pion in the chiral limit).
  - Instead, one calculates two-point function

$$C^{(2)}(t_1, t_0) \sim Z\, e^{-m(t_1 - t_0)}$$

  - This is a Fourier transform of the two-point function in the space-like regime.

$$C^{(2)}(t) \sim \int_{-\pi/a}^{+\pi/a} \frac{dq_0}{2\pi} \Pi(q^2) e^{iq_0 t} = \int_{-\pi/a}^{+\pi/a} \frac{dq_0}{2\pi} \frac{e^{iq_0 t}}{m^2 + q_0^2 + \mathbf{q}^2}$$

  - All the information encoded in the space-like two-point function $\Pi(q^2)$.

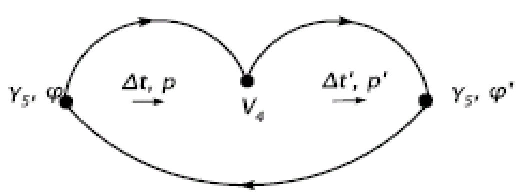
Lattice calculation of 3pt function

- At large enough time separations $\Delta t = t - t_0$, $\Delta t' = t_1 - t$, the ground state pions dominate.

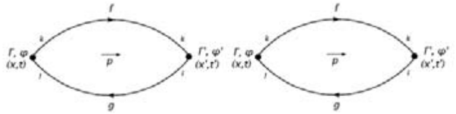
$$C_{4,\text{smr},\text{smr}}(\Delta t, \Delta t'; \mathbf{p}, \mathbf{p}') \quad \rightarrow \quad \frac{\sqrt{Z_{\pi,\text{smr}}(|\mathbf{p}|)\, Z_{\pi,\text{smr}}(|\mathbf{p}'|)}}{4E(p)E(p')\, Z_V}\, e^{-E(p)\Delta t}\, e^{-E(p')\Delta t'}\, \langle \pi(p')\, | V_4 |\, \pi(p) \rangle$$

- Extra factors can be taken off with 2pt functions.

$$\left(C_{\phi,\phi'}(\Delta t; \mathbf{p}) = \frac{\sqrt{Z_{\pi,\phi}(|\mathbf{p}|)\, Z_{\pi,\phi'}(|\mathbf{p}'|)}}{2E(p)}\, e^{-E(p)\Delta t}, \qquad \sqrt{Z_{\pi,\phi}(|\mathbf{p}|)} = \langle \pi(p) | O_{\pi,\phi}(\mathbf{p})^\dagger \rangle \right)$$



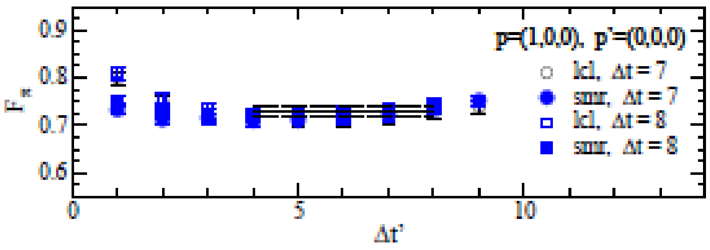
$$R(\Delta t, \Delta t'; \mathbf{p}, \mathbf{p}') \quad = \quad \frac{C^{\pi\pi}_{\mu,\text{smr},\text{smr}}(\Delta t, \Delta t'; \mathbf{p}, \mathbf{p}')}{C^{\pi}_{\text{smr},\text{lcl}}(\Delta t; \mathbf{p})\, C^{\pi}_{\text{lcl},\text{smr}}(\Delta t'; \mathbf{p}')}$$

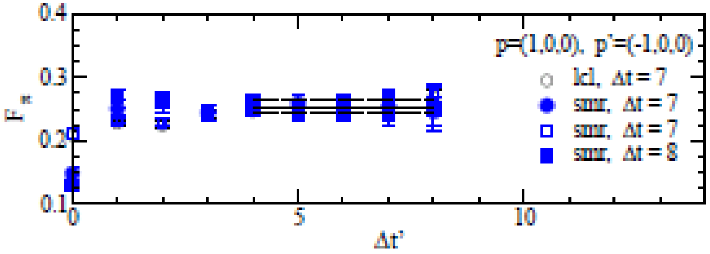


$$\boxed{F_\pi(q^2) \quad = \quad \frac{2M_\pi}{E(p) + E(p')}\, \frac{R^{\pi\pi}_{\mu,\phi,\phi'}(\Delta t, \Delta t'; \mathbf{p}, \mathbf{p}')}{R^{\pi\pi}_{4,\phi,\phi'}(\Delta t, \Delta t'; \mathbf{0}, \mathbf{0})}}$$

A recent calculation

JLQCD (2007)
dynamical overlap (Nf=2)
(talk by Kaneko at lat07)

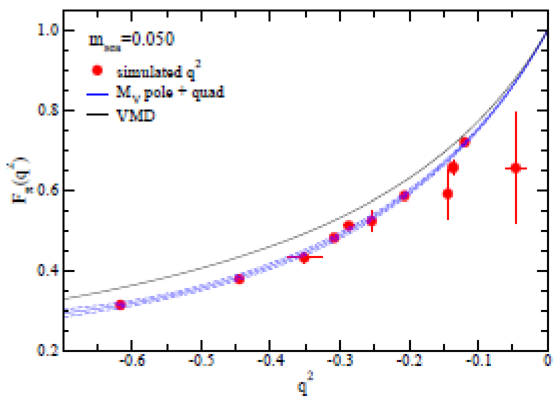




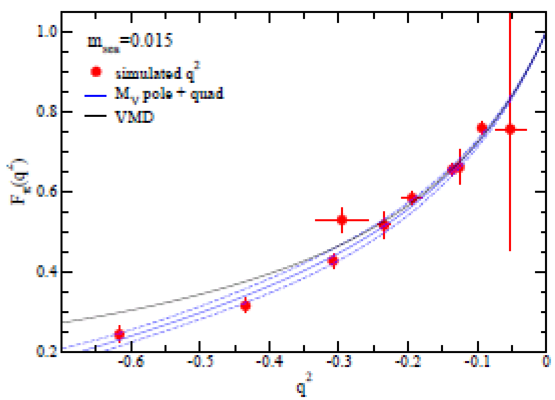
- **Lattice signal**
  - Look for a plateau, where the ground state pion dominates.
  - Noisier for larger pion momentum.
  - Note:
    - The actual data were obtained using the all-to-all technique, so that the data points at different t_0,t,t_1 and different momentum combinations can all be averaged.

A recent calculation

$m_q \sim m_s/2$



$m_q \sim m_s/6$



- Many points corresponds to many momentum combinations (p, p').
  - (1,0,0) $\rightarrow$ (0,1,0), … etc.
  - in units of $2\pi/L$.
  - Too large p's are contaminated by discretization effects $(ap)^2$.

- q^2 dependence well approximated by a vector meson pole

$$F_\pi(q^2) = \frac{1}{1 - q^2/m_V^2} + c_1 q^2 + \ldots$$

  - with the independently calculated m_V at the same quark mass.

Analyticity

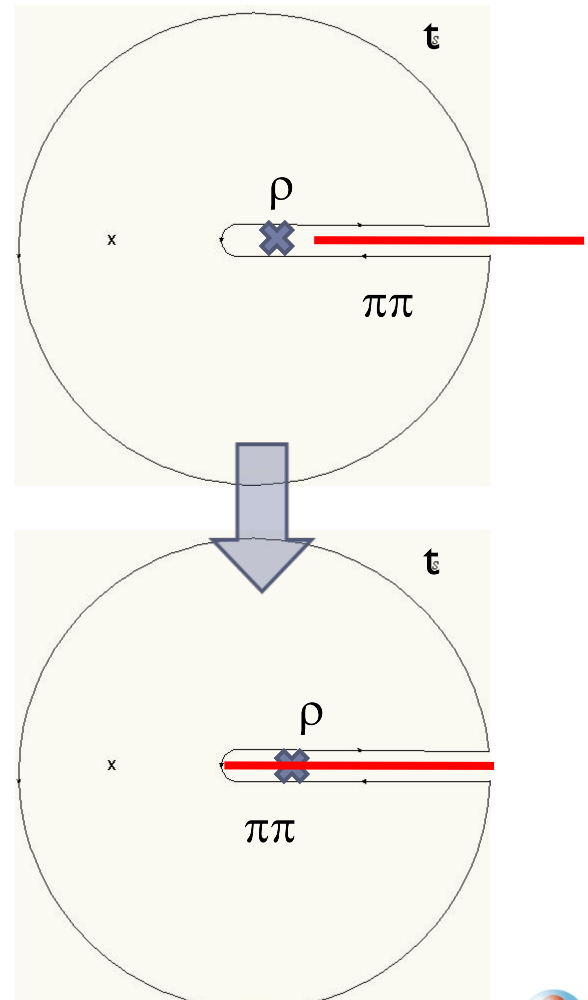
- Vector meson dominance is understood using analyticity.

$$F(q^2) = \frac{1}{2\pi i}\oint dt \frac{F(t)}{t-q^2} = \frac{1}{\pi}\int_{t_0}^{\infty} dt \frac{\operatorname{Im} F(t)}{t-q^2}$$

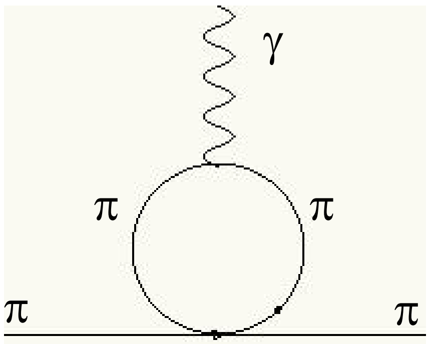
  - written in terms of the form factor in the time-like region t>0.

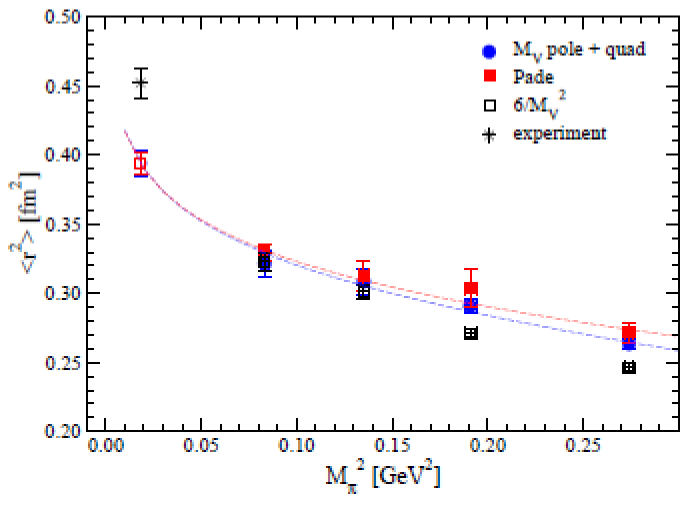
$$\langle \pi(p)\pi(p')|V_\mu|0\rangle$$

    - In the heavier quark mass region, ρ meson is a nearest isolated pole. ππ is subleading.
    - For the physical quark mass, ππ is nearest. ρ is a part of ππ (broad resonance).



Chiral extrapolation





- Charge radius has a chiral log contribution.

$$\left\langle r^2 \right\rangle_V^\pi = -\frac{1}{(4\pi f_\pi)^2}\left[\ln\frac{m_\pi^2}{\mu^2} + 12(4\pi)^2 L_9 + O(m_\pi^2)\right]$$

  - Must diverge in the chiral limit: pion cloud gets larger.
  - Valid only in the region where $2m_\pi < m_\rho$.
  - Lattice data actually increases towards the chiral limit. Chiral log further enhance its value.

$$\langle r^2\rangle_V^\pi = 0.388(9)(12)\ \mathrm{fm}^2$$

General problem

- Chiral extrapolation is a serious issue in current lattice QCD studies. Questions arise…
  - How closely one must approach the chiral limit?
  - One-loop enough? Two-loop needed?
  - Finite volume effect might become significant.

  - How big is the effect of chiral symmetry violation of Wilson, twisted-mass, staggered and domain-wall fermions?
  - Modified χPTs for these lattice actions contain many parameters. Possible to determine all of them to necessary precision?

- Answer depends on the process, action, …