Summary lecture 2

- Showed how to twist SUSY theories in 2D: $Q^2 = 0$ and $S = Q\Lambda$.
- Appearance of KD fermions and relation to staggered.
- WZ model.
- Today. Gauge symmetry. 2D
- Twisting in 4D. $\mathcal{N} = 4$ SYM.
- Connection to orbifolding

$\mathcal{Q} = 4$ twisted SYM in 2D

The twisting argumnent applies to any 2D SUSY with 4 supersymmetries. eg.

$$S_{\rm YM} = \beta Q {\rm Tr} \int d^2x \left(\frac{1}{4}\eta[\phi,\overline{\phi}] + 2\chi_{12}F_{12} + \chi_{12}B_{12} + \psi_\mu D_\mu \overline{\phi} \right)$$

where

$$\begin{aligned}
QA_\mu &= \psi_\mu \\
Q\psi_\mu &= -D_\mu \phi \\
Q\overline{\phi} &= \eta \\
Q\eta &= [\phi,\overline{\phi}] \\
QB_{12} &= [\phi,\chi_{12}] \\
Q\chi_{12} &= B_{12} \\
Q\phi &= 0
\end{aligned}$$

SYM in 2D

Note

- All fields in adjoint $X = \sum_a^{N^2-1} X^a T^a$ for $SU(N)$ with antihermitian T^a.
- Covariant deriv $D_\mu f = \partial_\mu f + [A_\mu, f]$
- $Q^2 = \delta_\phi$ – SUSY generally accompanied by gauge transformation.
- Other SUSY's read off by transforming the Ψ matrix $\Psi \rightarrow \Psi\Gamma^i, i = 1 \ldots 4$ and then inserting new fermions in above expression.

Carrying on ...

Homework problem 5. Do Q-variation. Integrate out B field. Find usual SYM action:

$$S_b = \int d^2x \mathrm{Tr} - F_{12}^2 - D\phi D\overline{\phi} + \frac{1}{4}[\phi, \overline{\phi}]^2$$

$$S_F = S_{\mathrm{KD}} - \frac{1}{4}\eta[\phi, \eta] + \psi_\mu[\overline{\phi}, \psi_\mu]$$

Map KD-fields to spinors – equivalent conventional formulation.

Twisted form good for discretization.

(No) tuning in Q-exact SYM in 2d

Effective action Γ will pick up counterterms of form

$$\Gamma = Q\ (\mathrm{something})$$

$$\Gamma = \sum_\alpha \mathcal{O}^p_\alpha c^p_\alpha(ga)$$

where

$$c^p_\alpha = a^{p-7/2}\sum_{l=1}^{\infty} a_l (ga)^{2l}$$

(Kaplan et al.) and p is (mass) dimension of operator and l is no. loops.
Relevant operator requires:$l < 7/4 - p/2$ Divergences occur at 1-loop and for $p = 0$
These constraints prohibit any new terms except for a cosmological constant !

Discretization I

At least two possibilities (Sugino, Catterall)
Retaining geometrical character leads to G.T rules:

$$
\begin{aligned}
f(x) &\rightarrow G(x)f(x)G^\dagger(x) \\
f_\mu(x) &\rightarrow G(x)f_\mu(x)G^\dagger(x+\mu) \\
f_{\mu\nu}(x) &\rightarrow G(x)f_{\mu\nu}(x)G^\dagger(x+\mu+\nu)
\end{aligned}
$$

Compatible with differences:

$$
\begin{aligned}
D^+_\mu f(x) &= U_\mu(x)f(x+\mu) - f(x)U_\mu(x) \\
D^+_\mu f_\nu(x) &= U_\mu(x)f_\nu(x+\mu) - f_\nu(x)U_\mu(x+\nu)
\end{aligned}
$$

Discretization II

- Keep Q-symmetry same except for
  $QU_\mu = \psi_\mu \quad Q\psi_\mu = -D^+\phi$
- Point split commutators for G.I eg
  $[\phi, \psi_\mu] \rightarrow \phi(x)\psi_\mu(x) - \psi_\mu(x)\phi(x+\mu)$
- $F_{\mu\nu} = D^+_\mu U_\nu$
- Necessary to complexify fields. Allows us to construct G.I ops. and well-defined SUSY.
- $\int \mathrm{Tr}(A_\mu B_\mu) \rightarrow \sum \mathrm{Tr}(A^\dagger_\mu B_\mu + \mathrm{h.c}$
- $A^\dagger \neq A$ requires complex $A^a(x)$.

Gauge action

$$F_{\mu\nu} = U_\mu(x)U_\nu(x+\mu) - U_\nu(x)U_\mu(x+\nu)$$

$$F^\dagger_{\mu\nu}F_{\mu\nu} = S^W_p + S^L$$

S^W_p Wilson plaquette term.

$$S^L = \mathrm{Tr}\left(U^\dagger_\nu(x+\mu)U^\dagger_\mu(x)U_\mu(x)U_\nu(x+\mu) - I\right)$$

$U_\mu(x)$ complex $U_\mu(x) = R(x)u_\mu(x)$ with R hermitian and u_μ unitary.

$$S^L = \sum_\mu \mathrm{Tr}(R_(x)^2 R^2(x+\mu) - I)$$

Consider $\beta \to \infty$. R driven to 1 and action is Wilson plaquette action!

Simulations I

Integrate twisted fermions – $\det(M(U,\phi))$. Target theory has real fields. G.I of scalar and gauge sector preserved if we do path integral "along real line"

Fermions: replace $\det(M)$ by $\mathrm{Pf}(M)=\det^{\frac{1}{2}}(M)$ up to sign. G.I

But what happens to SUSY Ward identities ?

Any Q-invariant observable can be computed as $\beta\to\infty$ exactly.

In this limit complex theory coincides with truncated theory

Prohibits fine tuning of truncated theory at large β.

Numerical simulations back this up

Simulations II

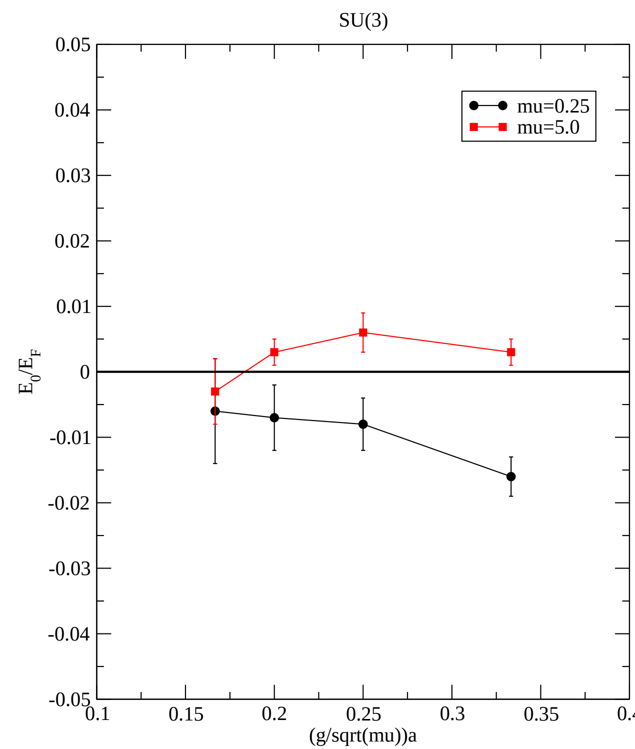
- Boson action=Wilson + scalar kinetic and $[\phi, \overline{\phi}]^2$ term.
- Realize $\mathrm{Pf}(M)$ via pseudofermions with action

$$S_{PF} = F^\dagger (M^\dagger M)^{-\frac{1}{4}} F$$

- RHMC alg. to handle fractional power (HMC with rational approx to power plus multimass CG solver)
- Measure phase and reweight ?
- Monitor $< S_B >$ and $< QO >$.
- Measure distribution of eigenvalues of scalars, phase of Pfaffian ...

Simulations III

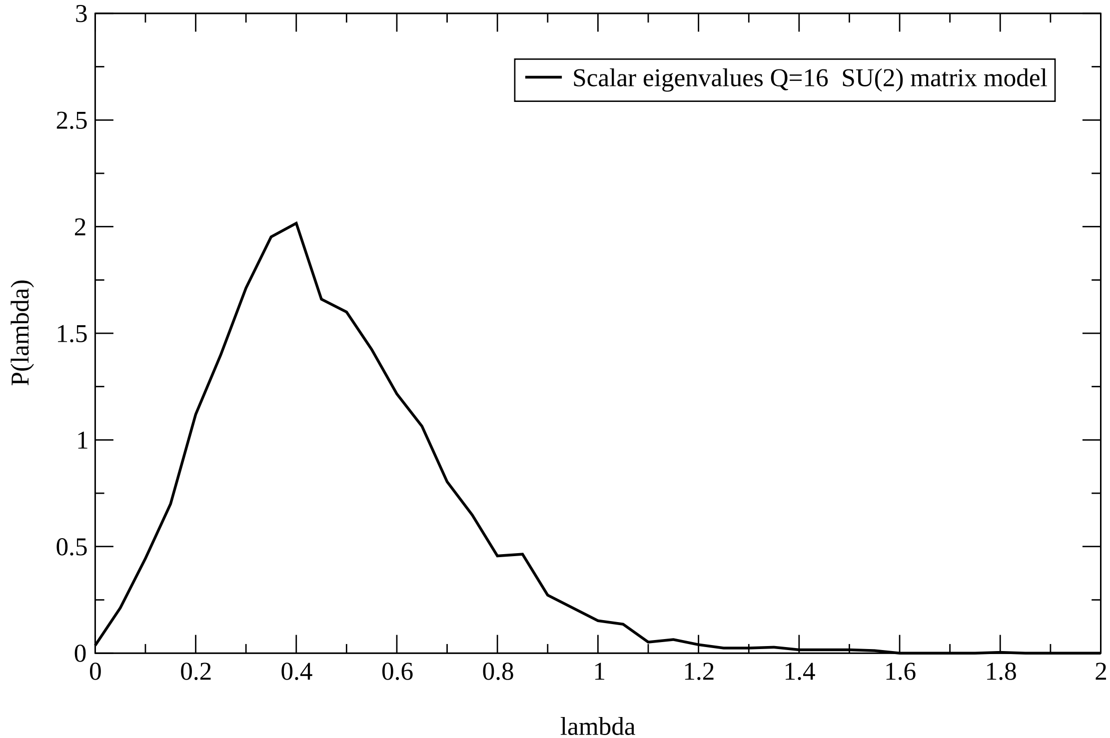
Continuum limit $L \to \infty$ where $\beta = \frac{L^2}{\mu}$



$$< S_b >= \frac{3}{2} L^2 (N^2 - 1)$$

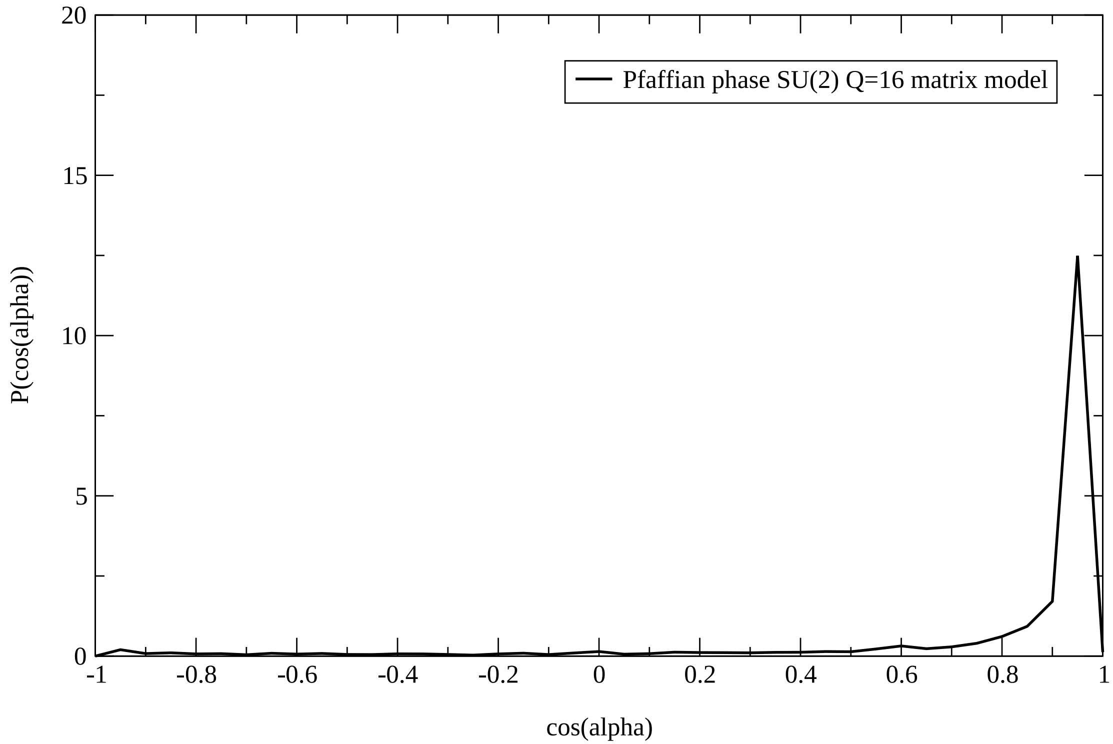
Simulations IV

Moduli space:



Simulations V

Pfaffian phase:



Q=16 SYM in 4D

Theory has 4 Majorana spinors with $SO(4)$ symmetry. Twist with rotational.

$$SO(4)' = \mathrm{diag}(SO(4) \times SO(4)_{\mathrm{rot}})$$

Supercharges/spinors matrices

$$\begin{aligned}\Psi &= \frac{\eta}{2} I + \psi_\mu \gamma_\mu + \frac{1}{2!} \chi_{\mu\nu} \gamma_\mu \gamma_\nu + \\ &+ \frac{1}{3!} \theta_{\mu\nu\lambda} \gamma_\mu \gamma_\nu \gamma_\lambda + \frac{1}{4!} \kappa_{\mu\nu\lambda\rho} \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho\end{aligned}$$

Superpartners

$$(\overline{\phi}, A_\mu, B_{\mu\nu}, W_{\mu\nu\lambda}, C_{\mu\nu\lambda\rho})$$

Q supersymmetry

Analog of 2D case:

$$
\begin{aligned}
Q\overline{\phi} &= \eta \;\; Q\eta = [\phi, \overline{\phi}] \\
QA_\mu &= \psi_\mu \;\; Q\psi_\mu = -D_\mu \phi \\
QB_{\mu\nu} &= [\phi, \chi_{\mu\nu}] \;\; Q\chi_{\mu\nu} = B_{\mu\nu} \\
QW_{\mu\nu\lambda} &= \theta_{\mu\nu\lambda} \;\; Q\theta_{\mu\nu\lambda} = [\phi, W_{\mu\nu\lambda}] \\
QC_{\mu\nu\lambda\rho} &= [\phi, \kappa_{\mu\nu\lambda\rho}] \;\; Q\kappa_{\mu\nu\lambda\rho} = C_{\mu\nu\lambda\rho} \\
Q\phi &= 0
\end{aligned}
$$

Again, $Q^2 = \delta_\phi$

Action $S = \beta Q \Lambda$ with ...

Q-exact action

$$
\begin{aligned}
\Lambda &= \int d^4x \mathrm{Tr}\left[\chi_{\mu\nu}\left(F_{\mu\nu} - \frac{1}{2}[V_\mu, V_\nu] + \frac{1}{\sqrt{2}}\epsilon_{\mu\nu\rho\lambda}D_\rho V_\lambda + \frac{1}{2}B_{\mu\nu}\right)\right.\\
&+ \left.\overline{\eta}D_\mu V_\mu + \frac{1}{2}\overline{\eta}\overline{C} + \psi_\mu D_\mu\overline{\phi} + \frac{1}{4}\eta[\phi,\overline{\phi}] + \frac{1}{2}\psi_\mu[V_\mu,\overline{\phi}]\right]
\end{aligned}
$$

where

$$
\begin{aligned}
W_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda\rho}V_\rho \\
\theta_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda\rho}\overline{\psi}_\rho \\
\kappa_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda\rho}\overline{\eta} \\
C_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda\rho}\overline{C}
\end{aligned}
$$

Orbifold constructions

Another way to derive lattice actions with exact SUSY (Kaplan et al.)
Intimate connections to the twisted approach (Damgaard et al., Unsal)
Here, show equivalence for $\mathcal{Q} = 4$ SYM model.
Starting point: $\mathcal{N} = 1$ SYM in 4D.
Reduce to zero dimensions (preserves $\mathcal{Q}$, mother theory)

$$S = \frac{1}{g^2}\mathrm{Tr}\left(-\frac{1}{4}[v_\alpha, v_\beta]^2 + \frac{i}{2}\overline{\Psi}\Gamma_\alpha[v_\alpha, \Psi]\right) \quad \alpha, \beta = 0 \ldots 3$$

with $\overline{\Psi} \equiv \Psi^T C$ and $C^{-1}\Gamma_\alpha C = -\Gamma^T$ Take

$$\Gamma = \begin{pmatrix} 0 & \sigma_\alpha \\ \bar{\sigma}_\alpha & 0 \end{pmatrix} \quad \text{with } \sigma_\alpha = (I, i\sigma_i)$$

Change variables

$$v_0 = A_1, \quad v_3 = -A_2 \quad v_1 + iv_2 = i\phi, \quad v_1 - iv_2 = -i\overline{\phi},$$

$$\Psi^{(1)} = \begin{pmatrix} -i\chi_{12} - \frac{1}{2}\eta \\ \psi_1 - i\psi_2 \end{pmatrix} \quad \Psi^{(2)} = \begin{pmatrix} -i\chi_{12} + \frac{1}{2}\eta \\ \psi_1 + i\psi_2 \end{pmatrix}$$

where $\Psi^{(1)}, \Psi^{(2)}$ chiral components. Action is:

$$\begin{aligned} S &= \frac{1}{g^2}\mathrm{Tr}\Big(-B_{\mu\nu}^2 + iB_{\mu\nu}[A_\mu, A_\nu] - \frac{1}{2}[A_\mu, \phi][A_\mu, \overline{\phi}] + \frac{1}{8}[\phi, \overline{\phi}]^2 - \\ &- i\eta[A_\mu, \psi_\mu] - \chi_{\mu\nu}\left([A_\mu, \psi_\nu] - [A_\nu, \psi_\mu]\right) \\ &- \frac{i}{4}\eta[\phi, \eta] + i\psi_\mu\left[\overline{\phi}, \psi_\mu\right] - \frac{i}{2}\chi_{\mu\nu}\left[\phi, \chi_{\mu\nu}\right]\Big) \end{aligned}$$

Q-exact mother

$$S = \frac{1}{g^2}\mathrm{Tr}Q\{-\chi_{\mu\nu}\left(B_{\mu\nu} - i[A_\mu, A_\nu]\right) + i\psi_\mu[A_\mu, \overline{\phi}] + \frac{i}{4}\eta[\phi, \overline{\phi}]\},$$

Q-variations fields are those given earlier (now hermitian basis for T^a)

Note: just dimensional reduction of continuum 2D twisted theory (as should be!)

Symmetries of mother:

- $G_R = SO(4) \times U(1)$ Dimensional reduction + chiral symmetry
- Global $U(kN^2)$ eg $\Phi \to U\Phi U^\dagger$ for any field Φ

Basic idea

Imagine matrices are composed of $k \times k$ blocks.
Imagine zeroing out most blocks leaving only blocks whose block indices can be thought of as labeling endpoints of links on a 2D lattice

$$U = U^{\alpha\beta}_{\mathbf{m},\mathbf{n}}, \alpha, \beta = 1 \ldots k;$$

and $\mathbf{m}, \mathbf{n}$ are vectors having components $(m_i, n_i) = 1 \ldots N$
Accomplished by assigning 2 charges $r_i, i = 1, 2$ to each field and requiring that the field be invariant under the orbifold action

$$\Phi^r_{\mathbf{m},\mathbf{n}} \to e^{\frac{2\pi i}{N}(r_i + m_i - n_i)} \Phi_{\mathbf{m},\mathbf{n}} \quad i = 1, 2$$

Clearly, the field lies on link between $\mathbf{m} \to \mathbf{n}$. Breaks $\mathcal{G}$ symmetry down to $U(k)^{N^2}$.

r-charges

Need

- integer components r_i
- $\sum_A r_i^A = 0$ where A labels fields. Needed for lattice interpretation (and G.I of target)
- Depend only on symmetries of mother.

Use linear combinations of 3 $U(1)$ charges associated with G_R.
Need to consider complexified theory to assign charges consistently.
Introduce both Φ and $\Phi^\dagger$.
Possible r-charges are:$r^{(1)} = (0, 1)$,$r^{(2)} = (0, 1)$ and $r^{(3)} = (-1, -1)$ plus negatives –note similarity to lattice tensors

Preserved SUSY

No. (SUSY's surviving projection) = No. ($\mathbf{r}=0$ fermions)
Term like:

$$(A^{r1}_{\mathbf{m},\mathbf{n}}B^{r2}_{\mathbf{n},\mathbf{m}})$$

(suppress $U(k)$ indices) becomes

$$(A^{r1}_{\mathbf{m},\mathbf{m}+\mathbf{r1}}A^{r2}_{\mathbf{m}+\mathbf{r1},\mathbf{m}+\mathbf{r1}+\mathbf{r2}})$$

since $\mathbf{r1}+\mathbf{r2}=\mathbf{0}$ we find

$$\sum_{\mathbf{m}} A^{r1}(\mathbf{m})\mathbf{B}^{\mathbf{r2}}(\mathbf{m}+\mathbf{r1})$$

Projected fields transform like bifundamentals

$$\Phi^r(\mathbf{m})\to\mathbf{g}(\mathbf{m})\boldsymbol{\Phi}^{\mathbf{r}}\mathbf{g}(\mathbf{m}+\mathbf{r})^\dagger$$

Deconstruction

Substituting projected fields back into the mother action and replacing A_μ by the (complex) matrix U_μ. Find twisted action with the discretization prescription given earlier !

$$
\begin{aligned}
S &= \frac{1}{2g^2}\mathrm{Tr}Q\sum_{\mathbf{n}}\Big(\\
&\quad \chi^\dagger_{\mu\nu}(\mathbf{n})\left[-\mathbf{B}_{\mu\nu}(\mathbf{n})-\mathbf{i}\left(\mathbf{U}_\mu(\mathbf{n})\mathbf{U}_\nu(\mathbf{n}+\mu)-\mathbf{U}_\nu(\mathbf{n})\mathbf{U}_\mu(\mathbf{n}+\nu)\right)\right] \\
&- \psi^\dagger_\mu(\mathbf{n})\left(\mathbf{U}_\mu(\mathbf{n})\overline{\phi}(\mathbf{n}+\mu)-\overline{\phi}(\mathbf{n})\mathbf{U}_\mu(\mathbf{n})\right) \\
&+ \frac{i}{4}\eta_+(\mathbf{n})[\phi(\mathbf{n}),\overline{\phi}(\mathbf{n})]+\frac{\mathbf{1}}{\mathbf{2}}\eta_-(\mathbf{n})\mathbf{d}(\mathbf{n})+\mathrm{h.c}\Big),
\end{aligned}
$$

What I didn't get to ...

- (Twisted) sigma models
- Gauge-gravity duality (AdSCFT like). Thermal gauge theories in $D < 4$ as probes of black hole physics ... SYMQM simple ex.
- Approaches attempting to preserve full supersymmetry (d'Adda et al.)
- Supersymmetry breaking.
- ...

Summary

- Lattice SUSY is a fascinating field with potential to play a role both in LHC physics and string theory.
- In low dimensions fine tuning is manageable (maybe also $\mathcal{N} = 1$ SYM in $D = 4$ using eg. DWF).
- Gauge-string dualities make strong coupling Yang-Mills systems useful tools to investigate eg black hole entropy.
- New algorithms eg RHMC make simulation feasible.
- In some cases a fraction of SUSY can be preserved on lattice – twisting, orbifold approaches.

References

See these recent reviews and references therein.

- A. Feo, Mod.Phys.Lett.A19:2387-2402,2004, hep-lat/0410012
- D. B. Kaplan, Nucl.Phys.Proc.Suppl.129:109-120,2004, hep-lat/0309099
- S. Catterall, PoS LAT2005:006,2006, hep-lat/0509136
- J. Giedt, Int.J.Mod.Phys.A21:3039-3094,2006, hep-lat/0602007. PoS LAT2006:008,2006, hep-lat/0701006